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Vanishing of Tor modules and homological dimensions of unions of aCM schemes

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Abstract

We study the vanishing of some $\mathrm{Tor}_i(M, R/J)$ when R is a local Cohen–Macaulay ring, J any ideal of R with R/J Cohen–Macaulay and M a finitely generated R-module. We use this result to study the homological dimension of unions $X \cup Y$ of arithmetically Cohen–Macaulay closed subschemes of $\mathbb{P}^r$. In particular, we show that "generically" such a homological dimension is the expected one. We give some generalization when one of the two schemes has codimension 2 and we apply this result to the monomial case.

Keywords: Tor modules; Cohen–Macaulay rings; Free resolutions; Monomial schemes

Introduction

Let X, Y be two closed subschemes of the projective space $\mathbb{P}^r$. A very naive question is to relate properties of X and Y with properties of the union $X \cup Y$. For instance, in the book [MDP] and in the paper [Mi], respectively for disjoint aCM curves in $\mathbb{P}^3$ and for disjoint aCM schemes of codimension 2 and $r-1$ in $\mathbb{P}^r$, the authors are able to give precise results on the homological dimension, on the deficiency module, on the Hilbert function, on the Betti numbers of the scheme $X \cup Y$. In a more general context, even the simplest question of computing the homological dimension of $X \cup Y$ in terms of the homological dimensions of X and Y is not trivial and quite

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open, depending on the scheme $X \cap Y$. For instance, if X and Y are complete intersections of codimension c and d, respectively, with $c + d \leqslant r + 1$ and $X \cap Y$ is a complete intersection of codimension $c + d$ then $\operatorname{hd} R/I_X \cap I_Y \leqslant c + d - 1$. Our results on this paper generalize both the previous situations to aCM subschemes X, Y of any codimension c and d which meet properly, i.e. with $X \cap Y$ of codimension $c + d \leqslant r + 1$ (Corollary 2.6). This result says, in particular, that "generically" the homological dimension of the union of two aCM schemes is the aspected one (see Theorem 2.7). In order to get the mentioned results we make use of a general result on the vanishing of some $\operatorname{Tor}_i^R(M, R/J)$ where R is a local Cohen–Macaulay ring (see Theorems 1.3 and 2.1).

When only one of the two schemes X, Y is aCM, the question becomes more complicated and we can give some result in case of codimension 2 for the aCM scheme X. Sometimes, instead of working with $I_{X\cup Y} = I_X \cap I_Y$, it can be easier to give information on the homological dimension of $R/I_X I_Y$ (see Theorem 2.10). Since in the reduced case, $I_X \cap I_Y = \sqrt{I_X I_Y}$, the previous information can be used, essentially, whenever one can link the homological dimension of an ideal J with the homological dimension of its radical $\sqrt{J}$. This is used here to get a result on the homological dimension of union of monomial reduced 2-codimensional aCM schemes (Theorem 2.11). These results will be applied in a next paper for discussing the homological dimension of some special schemes which are union of linear varieties, in particular for studying their Cohen–Macaulayness. These special schemes also arise on studying fat point schemes of $\mathbb{P}^2$.

The generality of the result of Corollary 2.6 was obtained because of a lot of useful discussions that the authors had with Silvio Greco which therefore they would like to thank deeply.

1. Vanishing of some Tor modules

This section is devoted to prove an algebraic result which will be applied in a geometrical context in Section 2.

To start with we need the following lemma.

Lemma 1.1. *Let R be a commutative ring with unit and let $I, J, K \subset R$ be ideals. If $I \subseteq \sqrt{J}$ then $\sqrt{(I + H)/H} \subseteq \sqrt{(J + H)/H}$ in the ring R/H.*

Proof. The proof uses a standard argument of Commutative Algebra. □

Corollary 1.2. *If $\sqrt{I} = \sqrt{J}$ then $\sqrt{(I + H)/H} = \sqrt{(J + H)/H}$, in the ring R/H.*

Proof. Trivial consequence of the previous lemma. □

Now we prove the main theorem of this section.

Theorem 1.3. *Let R be a local Cohen–Macaulay ring and let $J \subset R$ be an ideal such that R/J is Cohen–Macaulay too. Furthermore, let M be a finitely generated R-module of finite homological dimension such that $\sqrt{\operatorname{Ann}_R M} \not\subseteq \sqrt{J}$. Then*

$$\operatorname{Tor}_i^R(M, R/J) = 0 \quad \text{for } i \geqslant \operatorname{ht} J + 1 + \dim M/JM - \operatorname{depth} M.$$

Proof. Let

$$0 \to F_c \xrightarrow{\varphi_c} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0 \tag{1}$$

be a minimal free resolution of M. Of course, $\operatorname{Tor}_i^R(M, R/J) = 0$ for $i \geqslant c+1$. Tensoring by R/J we obtain the complex

$$0 \to F_c \otimes R/J \xrightarrow{\varphi'_c} \cdots \xrightarrow{\varphi'_2} F_1 \otimes R/J \xrightarrow{\varphi'_1} F_0 \otimes R/J \to M/JM \to 0;$$

if we set $F'_i = F_i \otimes R/J$, we get the following complex of free $R' = R/J$-modules

$$0 \to F'_c \xrightarrow{\varphi'_c} \cdots \xrightarrow{\varphi'_2} F'_1 \xrightarrow{\varphi'_1} F'_0.$$

Now we call $F'_\bullet$ the complex of free R'-modules

$$0 \to F'_c \xrightarrow{\psi_u} \cdots \xrightarrow{\psi_2} F'_{c-u+1} \xrightarrow{\psi_1} F'_{c-u},$$

where $u = \operatorname{ht}(\operatorname{Ann}_R M + J) - \operatorname{ht} J$ and $\psi_i = \varphi'_{i+c-u}$ for $1 \leqslant i \leqslant u$. We would like to show that $F'_\bullet$ is exact, using the Buchsbaum–Eisenbud criterion, see [BE].

At first we observe that $I(\varphi_i) \nsubseteq J$. Namely, $I(\varphi_i) \subseteq J$ implies that $\sqrt{I(\varphi_i)} \subseteq \sqrt{J}$; but $\sqrt{\operatorname{Ann}_R M} \subseteq \sqrt{I(\varphi_1)} \subseteq \sqrt{I(\varphi_i)}$ for every $i \geqslant 1$ (cf. [E, Corollary 20.12 and Proposition 20.7]), i.e. $\sqrt{\operatorname{Ann}_R M} \subseteq \sqrt{J}$ and this contradicts our hypotheses. Consequently $(I(\varphi_i) + J)/J \neq 0$, hence $\operatorname{rank} \varphi'_i = \operatorname{rank} \varphi_i$. Trivially $\operatorname{rank} F'_i = \operatorname{rank} F_i$ so the first condition of the criterion is satisfied. Moreover we have that $I(\psi_i) = (I(\varphi_{i+c-u}) + J)/J$. To conclude that $F'_\bullet$ is exact it is enough to show that $\operatorname{depth} I(\psi_i) \geqslant i$ for $1 \leqslant i \leqslant u$. Now

$$\begin{aligned}
&\sqrt{(\operatorname{Ann}_R M + J)/J} \subseteq \sqrt{\big(I(\varphi_i) + J\big)/J} \quad \text{for } 1 \leqslant i \leqslant c \\
&\quad\Rightarrow \quad \sqrt{(\operatorname{Ann}_R M + J)/J} \subseteq \sqrt{I(\psi_i)} \quad \text{for } 1 \leqslant i \leqslant u \\
&\quad\Rightarrow \quad \operatorname{ht}_{R'}(\operatorname{Ann}_R M + J)/J \leqslant \operatorname{ht}_{R'} I(\psi_i) \quad \text{for } 1 \leqslant i \leqslant u \\
&\quad\Rightarrow \quad \operatorname{depth}_{R'}(\operatorname{Ann}_R M + J)/J \leqslant \operatorname{depth}_{R'} I(\psi_i) \quad \text{for } 1 \leqslant i \leqslant u,
\end{aligned}$$

since R' is a Cohen–Macaulay ring. On the other hand,

$$\begin{aligned}
&\operatorname{depth}_{R'}(\operatorname{Ann}_R M + J)/J = \operatorname{ht}_{R'}(\operatorname{Ann}_R M + J)/J = \operatorname{ht}_R(\operatorname{Ann}_R M + J) - \operatorname{ht}_R J = u \\
&\quad\Rightarrow \quad \operatorname{depth}_{R'} I(\psi_i) \geqslant u \geqslant i.
\end{aligned}$$

This implies the exactness of $F'_\bullet$. Consequently $\operatorname{Tor}_i^R(M, R/J) = 0$ for $i \geqslant c-u+1$. But

$$u = \operatorname{ht}_R(\operatorname{Ann}_R M + J) - \operatorname{ht}_R J = \operatorname{ht}_R\big(\operatorname{Ann}_R(M/JM)\big) - \operatorname{ht}_R J,$$

since $\sqrt{\operatorname{Ann}_R(M/JM)} = \sqrt{\operatorname{Ann}_R M + J}$ (cf. [E, Proposition 10.8]), so

$$\begin{aligned} c-u+1 &= \operatorname{hd} M - \operatorname{ht}_R\bigl(\operatorname{Ann}_R(M/JM)\bigr) + \operatorname{ht}_R J + 1 \\ &= (\dim R - \operatorname{depth} M) - \dim R + \dim(M/JM) + \operatorname{ht}_R J + 1 \\ &= -\operatorname{depth} M + \dim(M/JM) + \operatorname{ht}_R J + 1, \end{aligned}$$

and consequently the requested vanishing of the Tor_i^R's. □

2. Results on homological dimension

We would like to apply the previous result to the case of projective schemes.

Let k be an algebraically closed field, $R := k[x_0, \dots, x_n]$ the polynomial ring and $\mathbb{P}^r = \operatorname{Proj} R$ the r-dimensional projective space. When X is a subscheme of $\mathbb{P}^r$ we denote by I_X its defining ideal. We are interested in studying the homological dimension of $X \cup Y$, i.e. $\operatorname{hd} R/I_X \cap I_Y$, in terms of the homological dimensions of the given subschemes X and Y of $\mathbb{P}^r$.

Of course, in order to treat the previous subject, we need to have information about the two schemes and about their intersection. For instance, if X and Y are complete intersections of codimension c and d, respectively, such that $X \cap Y$ is a complete intersection of codimension $c+d$ and $r \geqslant c+d-1$ then $\operatorname{hd} R/I_X \cap I_Y = c+d-1$. The easy proof uses a mapping cone computation.

On the other hand, it is well known that if C and D are two disjoint aCM curves in $\mathbb{P}^3$, then $I_C I_D = I_C \cap I_D = I_C \otimes I_D$, therefore one can deduce a minimal free resolution of $I_{C \cup D}$ from the minimal free resolutions of I_C and I_D (see [MDP]). This result is generalized in [Mi] to the case of two disjoint aCM subschemes C and D of $\mathbb{P}^r$ of codimensions 2 and $r-1$, respectively.

In some sense we would like to generalize these results to the case of aCM schemes in $\mathbb{P}^r$ of any codimensions.

Theorem 2.1. *Let X, Y be closed subschemes of $\mathbb{P}^r$, with Y aCM and such that $Y_{\mathrm{red}} \nsubseteq X_{\mathrm{red}}$. Then*

$$\operatorname{Tor}_i^R(R/I_X, R/I_Y) = 0 \quad \textit{for } i \geqslant \operatorname{hd} R/I_X + \operatorname{codim} Y - \operatorname{codim}(X \cap Y) + 1.$$

Proof. Apply Theorem 1.3 with $M = R/I_X$ and $J = I_Y$ after localization of $R = k[x_0, \dots, x_n]$ at the irrelevant maximal ideal. □

Corollary 2.2. *Let X, Y be aCM closed subschemes of $\mathbb{P}^r$, such that $Y_{\mathrm{red}} \nsubseteq X_{\mathrm{red}}$. Then*

$$\operatorname{Tor}_i^R(R/I_X, R/I_Y) = 0 \quad \textit{for } i \geqslant \operatorname{codim} X + \operatorname{codim} Y - \operatorname{codim}(X \cap Y) + 1.$$

Proof. A direct consequence of the previous result when X is an aCM scheme. □

Now would like to apply the previous results to give information on the homological dimension of unions of schemes.

Corollary 2.3. *In the same hypotheses of Theorem* 2.1

(1) *if* $\operatorname{codim}(X \cap Y) \geqslant \operatorname{hd} R/I_X + \operatorname{codim} Y - 1$ *then* $I_X \otimes_R I_Y \cong I_X I_Y$;
(2) *if* $\operatorname{codim}(X \cap Y) = \operatorname{hd} R/I_X + \operatorname{codim} Y$ *then* $I_X \otimes_R I_Y \cong I_X I_Y \cong I_X \cap I_Y$.

Proof. Applying Theorem 2.1, if $\operatorname{codim}(X \cap Y) \geqslant \operatorname{hd} R/I_X + \operatorname{codim} Y - 1$ then $\operatorname{Tor}_i^R(R/I_X, R/I_Y) = 0$ for $i \geqslant 2$ and if $\operatorname{codim}(X \cap Y) = \operatorname{hd} R/I_X + \operatorname{codim} Y$ then $\operatorname{Tor}_i^R(R/I_X, R/I_Y) = 0$ for $i \geqslant 1$; so it is enough to remember that $\operatorname{Tor}_2^R(R/I_X, R/I_Y)$ is the kernel of the natural map $I_X \otimes_R I_Y \to I_X I_Y$ and $\operatorname{Tor}_1^R(R/I_X, R/I_Y) \cong (I_X \cap I_Y)/I_X I_Y$. $\square$

Corollary 2.4. *In the same hypotheses of Theorem* 1.3, *if* $\operatorname{ht}(\operatorname{Ann}_R M + J) = \operatorname{hd} M + \operatorname{ht} J$ *and* $F_\bullet$ *and* $G_\bullet$ *are graded minimal free resolution of* M *and* R/J, *respectively, then* $\operatorname{Tot} F_\bullet \otimes_R G_\bullet$ *is a graded minimal free resolution of* M/JM. *In particular* $\operatorname{hd} M/JM = \operatorname{hd} M + \operatorname{ht} J$.

Proof. The homology of the total complex

$$H_i(\operatorname{Tot} F_\bullet \otimes_R G_\bullet) \cong \operatorname{Tor}_i^R(M, R/J) = 0, \quad \text{for } i \geqslant 1,$$

by Theorem 1.3. So $\operatorname{Tot} F_\bullet \otimes_R G_\bullet$ is a graded resolution of $M \otimes_R R/J \cong M/JM$. Since $F_\bullet$ and $G_\bullet$ are minimal resolutions, then $\operatorname{Tot} F_\bullet \otimes_R G_\bullet$ is a minimal resolution too, since the entries in the matrices of $\operatorname{Tot} F_\bullet \otimes_R G_\bullet$ are entries of the matrices of $F_\bullet$ and $G_\bullet$, i.e. they are non-units. $\square$

Remark 2.5. If I, J are homogeneous ideals of the polynomial ring R, then the condition $\operatorname{ht}(I + J) = \operatorname{hd} R/I + \operatorname{ht} J$ is equivalent to the conditions that $\operatorname{ht}(I + J) = \operatorname{ht} I + \operatorname{ht} J$ and R/I is Cohen–Macaulay. In particular, Corollary 2.4 implies the classical result that the non-empty proper intersection of two arithmetically Cohen–Macaulay subschemes X and Y is arithmetically Cohen–Macaulay and that its minimal free resolution is the total complex of the tensor product of the resolutions of X and Y. In this case, we also obtain the minimal free resolution of the union $X \cup Y$.

Corollary 2.6. *Let* $X, Y \subseteq \mathbb{P}^r$ *be arithmetically Cohen–Macaulay subschemes such that* $\operatorname{ht}(I_X + I_Y) = \operatorname{ht} I_X + \operatorname{ht} I_Y$. *Let* $F_\bullet$ *and* $G_\bullet$ *be graded minimal free resolution of* R/I_X *and* R/I_Y, *respectively, and we denote by* $\widetilde{F}_\bullet$ *and* $\widetilde{G}_\bullet$ *the graded minimal free resolutions of* I_X *and* I_Y *obtained from* $F_\bullet$ *and* $G_\bullet$ *by deleting the first module. Then* $\operatorname{Tot} \widetilde{F}_\bullet \otimes_R \widetilde{G}_\bullet$ *is a graded minimal free resolution of* $I_X \cap I_Y$. *In particular* $\operatorname{hd} R/(I_X \cap I_Y) = \operatorname{hd} R/I_X + \operatorname{hd} R/I_Y - 1$.

Proof. The homology of the total complex

$$\begin{aligned} H_i(\operatorname{Tot} \widetilde{F}_\bullet \otimes_R \widetilde{G}_\bullet) &\cong \operatorname{Tor}_i^R(I_X, I_Y) = H_i(\widetilde{F}_\bullet \otimes_R I_Y) \\ &= H_{i+1}(F_\bullet \otimes_R I_Y) = \operatorname{Tor}_{i+1}^R(R/I_X, I_Y) = H_{i+1}(R/I_X \otimes_R \widetilde{G}_\bullet) \\ &= H_{i+2}(R/I_X \otimes_R G_\bullet) = \operatorname{Tor}_{i+2}^R(R/I_X, R/I_Y) = 0 \quad \text{for } i \geqslant 1. \end{aligned}$$

So $\operatorname{Tot} \widetilde{F}_\bullet \otimes_R \widetilde{G}_\bullet$ is a graded resolution of $I_X \otimes_R I_Y \cong I_X \cap I_Y$, by Corollary 2.3. Since $F_\bullet$ and $G_\bullet$ are minimal resolutions, then $\operatorname{Tot} \widetilde{F}_\bullet \otimes_R \widetilde{G}_\bullet$ is a minimal resolution too, since the entries in the matrices of $\operatorname{Tot} \widetilde{F}_\bullet \otimes_R \widetilde{G}_\bullet$ are entries of the matrices of $F_\bullet$ and $G_\bullet$, i.e. they are non-units. $\square$

The next proposition shows that “generically” the union of two aCM schemes of codimension c and d in $\mathbb{P}^r$ $(c + d \leqslant r + 1)$ has homological dimension $c + d - 1$.

Theorem 2.7. *Let P and Q two Hilbert polynomials admissible for aCM subscheme of $\mathbb{P}^r$ of codimension c and d, respectively, with $c+d \leqslant r+1$. Let $S_P \subseteq \mathcal{H}_P$ and $S_Q \subseteq \mathcal{H}_Q$ be the subschemes of the Hilbert schemes parametrizing the aCM subschemes. Let R be the homogeneous coordinate ring of $\mathbb{P}^r$. Then there exists a non-empty open subset $U \subseteq S_P \times S_Q$, such that for any $(s,t) \in U$, $\operatorname{hd} R/I_{X_s} \cap I_{Y_t} = c+d-1$.*

Proof. Let $U = \{(s,t) \in S_P \times S_Q \mid \operatorname{codim} X_s \cap X_t = c+d\}$. By the theorem on the semicontinuity of fiber dimension it is an open set, trivially non-empty. The conclusion is a consequence of Corollary 2.6. $\square$

The next example shows that when the codimension of $X \cap Y$ is smaller than $c+d$, even if the two schemes have no common component, it can happen that the homological dimension of $R/I_X \cap I_Y$ is greater than $c+d-1$.

Example 2.8. Let $R = k[x,y,z,w,t]$ be the coordinate ring of $\mathbb{P}^4$. Let us consider the ideals $I = (x,z) \cap (x,w) \cap (y,z) = (xy, xz, zw)$ and $J = (t^2, x^2 - yz)$. I is the ideal of a cubic surface, union of three planes and J is a complete intersection. Then a computer computation shows that $I \cap J$ has graded minimal free resolution

$$0 \to R(-8) \to R(-6) \oplus R(-7)^4 \to R(-5)^6 \oplus R(-6)^4 \to R(-4)^7 \to I \cap J \to 0,$$

i.e. $\operatorname{hd} R/I \cap J = 4$.

In order to investigate the case in which the codimension of $X \cap Y$ is smaller than $c+d$, we will treat the case X an aCM scheme of codimension 2 and Y any scheme, not necessarily aCM.

In many questions about $I_X \cap I_Y$ it is useful to have information on $I_X I_Y$. Now since, in this context, $I_X \otimes I_Y$ is more easy to handle than $I_X I_Y$, one can be interested in the case when $I_X I_Y \cong I_X \otimes I_Y$.

Theorem 2.9. *Let $X, Y \subset \mathbb{P}^r$ be two subschemes with X aCM of codimension 2 not containing the support of any component of Y. Then $I_X \otimes I_Y \cong I_X I_Y$.*

Proof. Since $\operatorname{Tor}_2^R(R/I_X, R/I_Y)$ is the kernel of the natural map $I_X \otimes_R I_Y \to I_X I_Y$, it is enough to prove the vanishing of $\operatorname{Tor}_2^R(R/I_X, R/I_Y)$. Consider a minimal free resolution of R/I_X

$$0 \to F_2 \to F_1 \to R \to R/I_X \to 0;$$

tensoring by R/I_Y we get the complex

$$0 \to F_2 \otimes R/I_Y \xrightarrow{f} F_1 \otimes R/I_Y \to R \otimes R/I_Y \to 0$$

from which we deduce that $\mathrm{Tor}_2^R(R/I_X, R/I_Y) \cong \mathrm{Ker}(f)$. If we denote $n = \mathrm{rank}(F_1)$ we see that $f : (R/I_Y)^{n-1} \to (R/I_Y)^n$ is the map induced by the Hilbert–Burch matrix A which defines I_X. Take $(\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_{n-1}) \in \mathrm{Ker}(f)$ (here $\bar{x}$ means working in R/I_Y); then, we have

$$\overline{A}\begin{pmatrix} \overline{f}_1 \\ \overline{f}_2 \\ \vdots \\ \overline{f}_{n-1} \end{pmatrix} = \bar{0}$$

from which we have $\overline{A}_i \overline{f}_j = \bar{0}$, for all $i = 1, \ldots, n$, $j = 1, \ldots, n-1$ and $\overline{A}_i$, $1 \leqslant i \leqslant n$, denote the maximal minors of $\overline{A}$. Lifting in R we have $A_i f_j \in I_Y$, for all i, j. This implies that $f_j I_X \subseteq I_Y$, i.e. $f_j \in I_Y : I_X = I_Y$ since by the assumptions I_X is not contained in any associated prime ideal of I_Y. Hence $\overline{f}_j = \bar{0}$ for all j. Thus, $\mathrm{Ker}(f) = \bar{0}$ and therefore $\mathrm{Tor}_2^R(R/I_X, R/I_Y) = 0$. □

Theorem 2.10. *Let $X, Y \subset \mathbb{P}^r$ be two subschemes with X aCM of codimension 2 not containing the support of any component of Y and* $\mathrm{hd}\, R/I_Y = s$. *Then* $\mathrm{hd}\, R/I_X I_Y \leqslant s + 1$.

Proof. By previous theorem we have $I_X I_Y \cong I_X \otimes_R I_Y$. Now, by hypothesis, R/I_Y has a minimal free resolution

$$F_\bullet \quad 0 \to F_s \to F_{s-1} \to \cdots \to F_1 \to R.$$

Let

$$G_\bullet \quad 0 \to G_2 \to G_1 \to R$$

be a minimal free resolution of R/I_X and consider the tensor product of the corresponding minimal free resolutions $\widetilde{F}_\bullet$ and $\widetilde{G}_\bullet$ of I_Y and I_X, i.e. the complex $C_\bullet = \mathrm{Tot}\, \widetilde{F}_\bullet \otimes_R \widetilde{G}_\bullet$. Since $H_i(C_\bullet) \cong \mathrm{Tor}_i^R(I_X, I_Y)$, and $\mathrm{Tor}_i^R(I_X, I_Y)$ is the ith homology module of the complex

$$0 \to G_2 \otimes I_Y \xrightarrow{\alpha} G_1 \otimes I_Y \to 0$$

we get $H_i(C_\bullet) = 0$ for $i \geqslant 2$. On the other hand, from the following exact diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] \\
0 \arrow[r] & I_Y \otimes G_2 \arrow[r, "f"] \arrow[d, "\alpha"'] & G_2 \arrow[d, "\beta"] \\
0 \arrow[r] & I_Y \otimes G_1 \arrow[r, "g"] & G_1
\end{tikzcd}$$

we see that α is injective, i.e. $H_1(C_\bullet) \cong \mathrm{Tor}_1^R(I_X, I_Y) = 0$ (see also [Mi, Lemma 1.1]). In conclusion, $C_\bullet$ is an exact complex, hence a free resolution for $I_X \otimes I_Y$; then $\mathrm{hd}(I_X \otimes I_Y) \leqslant s$, hence $\mathrm{hd}(R/I_X I_Y) \leqslant s + 1$. □

We now apply the previous results to give a bound to the homological dimension of some monomial schemes.

Theorem 2.11. *Let $X_1, X_2, \dots, X_n$ be reduced aCM subschemes of codimension 2 in $\mathbb{P}^r$ with X_i and X_j having no common components for $i \neq j$ and such that their defining ideals I_{X_i} are monomial ideals for all i. Set R the homogeneous coordinate ring of $\mathbb{P}^r$ and $Y = X_1 \cup X_2 \cup \dots \cup X_n$. Then*

$$\operatorname{hd}(R/I_Y) \leqslant n+1.$$

Proof. We use induction on n. For $n=1$, $Y = X_1$ is aCM of codimension 2, so $\operatorname{hd}(R/I_Y) = 2$. Suppose now $n > 1$ and denote $Z = X_1 \cup X_2 \cup \dots \cup X_{n-1}$. Then $Y = Z \cup X_n$ and $I_Y = I_Z \cap I_{X_n}$ and $\operatorname{hd} R/I_Z \leqslant n$. By previous theorem we have that $\operatorname{hd} R/I_Z I_{X_n} \leqslant n+1$. Since X_i are reduced we have that $I_Z \cap I_{Xn} = \sqrt{I_Z I_{X_n}}$; using the monomial hypothesis for the first part of the proof of Theorem 2.6 [HTT] we obtain $\operatorname{hd}(R/I_Z \cap I_{X_n}) \leqslant \operatorname{hd}(R/I_Z I_{X_n}) \leqslant n+1$. $\square$

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