Proceedings of the Edinburgh Mathematical Society (2006) **49**, 267–275 ©
DOI:10.1017/S001309150400149X Printed in the United Kingdom

A MULTIPLICITY THEOREM FOR A PERTURBED SECOND-ORDER NON-AUTONOMOUS SYSTEM

FRANCESCA FARACI AND ANTONIO IANNIZZOTTO

Department of Mathematics and Computer Science, University of Catania, Viale A. Doria 6, 95125 Catania, Italy
(ffaraci@dmi.unict.it; iannizzotto@dmi.unict.it)

(Received 6 December 2004)

Abstract In this paper we establish a multiplicity result for a second-order non-autonomous system. Using a variational principle of Ricceri we prove that if the set of global minima of a certain function has at least k connected components, then our problem has at least k periodic solutions. Moreover, the existence of one more solution is investigated through a mountain-pass-like argument.

Keywords: multiple periodic solutions; second-order non-autonomous system; critical point theory

2000 *Mathematics subject classification:* Primary 34C25; 35A15

1. Introduction

In this paper we consider the second-order non-autonomous system

$$\begin{cases} \ddot{u} = \alpha(t)(Au - \nabla F(u)) + \lambda \nabla_x G(t,u), & \text{a.e. in } [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{P_λ}$$

where A is an $N \times N$ symmetric matrix satisfying

$$Ax \cdot x \geqslant c|x|^2 \quad \text{for all } x \in \mathbb{R}^N, \tag{1.1}$$

where c is some positive constant. Assume that $\lambda > 0$, $\alpha \in L^\infty([0,T])$, $a = \text{ess inf}_{[0,T]}\alpha > 0$, $F : \mathbb{R}^N \to \mathbb{R}$ is continuously differentiable, and that $G : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is measurable in t for all $x \in \mathbb{R}^N$ and continuously differentiable in $x \in \mathbb{R}^N$ for a.e. $t \in [0,T]$. Moreover, assume

$$\sup_{|x| \leqslant s} |\nabla_x G(\cdot\,,x)| \in L^1([0,T]) \quad \text{for every } s > 0, \quad G(\cdot\,,0) \in L^1([0,T]). \tag{1.2}$$

It is well known (see [**3**]) that a solution of (P_λ) is a function $u \in C^1([0,T],\mathbb{R}^N)$, with $\dot{u}$ absolutely continuous, such that

$$\begin{cases} \ddot{u}(t) = \alpha(t)(Au(t) - \nabla F(u(t))) + \lambda \nabla_x G(t,u(t)), & \text{a.e. in } [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases}$$

For the more general problem

$$\begin{cases} \ddot{u} = \nabla_x \phi(t,u), & \text{a.e. in } [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

the existence of at least three solutions has previously been studied in [**1**], [**6**], [**7**] and [**9**] under the following assumption, firstly introduced by Brezis and Nirenberg: there exist $r > 0$ and an integer $k \geqslant 0$ such that

$$-\tfrac{1}{2}(k+1)^2 w^2 |x|^2 \leqslant \phi(t,x) - \phi(t,0) \leqslant -\tfrac{1}{2}k^2 w^2 |x|^2 \tag{1.3}$$

for each $|x| \leqslant r$, a.e. $t \in [0,T]$, where $w = 2\pi/T$.

The perturbed problem

$$\begin{cases} \ddot{u} = \nabla_x \phi(t,u) + \lambda \psi(t), & \text{a.e. in } [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases}$$

was studied in [**8**], in which Tang proves the existence of at least three solutions, for $\lambda > 0$ small enough, under the stronger condition that there exist $r > 0$ and an integer $k \geqslant 0$ such that

$$-\mu |x|^2 \leqslant \phi(t,x) - \phi(t,0) \leqslant -\nu |x|^2 \tag{1.4}$$

for each $|x| \leqslant r$, a.e. $t \in [0,T]$, where $\nu > \frac{1}{2}k^2 w^2$, $\mu < \frac{1}{2}(k+1)^2 w^2$ and $w = 2\pi/T$.

In this paper we prove a multiplicity result of the following type: for each integer $k > 1$, (P_λ) has, for λ small enough, at least k solutions.

Our main tool is a recent theorem by Ricceri [**4**, Theorem 6], which, for the convenience of the reader, we state here.

Theorem A. *Let X be a reflexive and separable real Banach space and let $\Psi, \Phi : X \to \mathbb{R}$ be two functionals. Assume that there exists $r > \inf_X \Psi$ such that the set $\Psi^{-1}(]-\infty, r[)$ is bounded. Moreover, suppose that the functional Φ is bounded below in $(\overline{\Psi^{-1}(]-\infty, r[)})_w$ and the functional $\Psi + \lambda\Phi$ is sequentially weakly lower semicontinuous for each $\lambda \geqslant 0$ small enough. Finally, assume that the set $\Psi^{-1}(\inf_X \Psi)$ has at least k weakly connected components.*

Then, there exists $\lambda_r > 0$ such that, for each $\lambda \in]0, \lambda_r[$, the functional $\Psi + \lambda\Phi$ has at least k τ_Ψ-local minima lying in $\Psi^{-1}(]-\infty, r[)$, where τ_Ψ is the smallest topology on X which contains both the weak topology and the family of sets $\{\Psi^{-1}(]-\infty, \rho[)\}_{\rho \in \mathbb{R}}$.

2. A multiplicity theorem

Let us introduce the space

$$H^1_T = \{u : [0,T] \to \mathbb{R}^N \text{ absolutely continuous}, \ u(0) = u(T), \ \dot{u} \in L^2([0,T], \mathbb{R}^N)\}.$$

For all $u, v \in H^1_T$, define the scalar product as follows:

$$(u,v) = \int_0^T \dot{u}(t) \cdot \dot{v}(t) \, dt + \int_0^T \alpha(t) A u(t) \cdot v(t) \, dt.$$

The norm

$$\|u\| = \left(\int_0^T |\dot{u}(t)|^2 \,\mathrm{d}t + \int_0^T \alpha(t)Au(t)\cdot u(t)\,\mathrm{d}t\right)^{1/2}$$

in H_T^1 is equivalent to the usual one thanks to condition (1.1).

Let us observe that H_T^1 is compactly embedded in $C^0([0,T],\mathbb{R}^N)$. Define, for each $u \in H_T^1$,

$$\Psi(u) = \tfrac{1}{2}\|u\|^2 - \int_0^T \alpha(t)F(u(t))\,\mathrm{d}t,$$

$$\Phi(u) = \int_0^T G(t,u(t))\,\mathrm{d}t.$$

Clearly, Ψ is well defined, sequentially weakly continuous, and continuous together with its Gâteaux derivative. Moreover, from (1.2) we have

$$\sup_{|x|\leqslant s} |G(\cdot\,,x)| \in L^1([0,T]) \quad \text{for every } s>0.$$

Thus, it is easy to prove that Φ satisfies the same properties of Ψ.

We recall that u is a solution of (P_λ) if and only if $u \in H_T^1$ and it satisfies

$$\int_0^T [\dot{u}(t)\cdot\dot{v}(t) + \alpha(t)Au(t)\cdot v(t) - \alpha(t)\nabla F(u(t))\cdot v(t)]\,\mathrm{d}t + \lambda\int_0^T \nabla_x G(t,u(t))\cdot v(t)\,\mathrm{d}t = 0$$

for all $v \in H_T^1$, that is, if u is a critical point of $\Psi + \lambda\Phi$ in H_T^1.

Our result reads as follows.

Theorem 2.1. *Let α, A, F, G be as in § 1. Assume that*

(i) $\displaystyle\limsup_{|x|\to+\infty} \frac{F(x)}{|x|^2} < \tfrac{1}{2}c$;

(ii) *the set of global minima of the function $H(x) = \frac{1}{2}Ax\cdot x - F(x)$ has at least k connected components in $\mathbb{R}^N$ $(k \geqslant 2)$.*

Then, for every $r > \|\alpha\|_{L^1}\inf_{\mathbb{R}^N} H$, there exists $\lambda_r > 0$ such that, for every $\lambda \in\,]0,\lambda_r[$, (P_λ) has at least k solutions in $\Psi^{-1}(]-\infty, r[)$.

Proof. Set $X = H_T^1$. Let us show that the functionals Ψ and Φ defined above satisfy the hypotheses of Theorem A. The functional $\Psi + \lambda\Phi$ is sequentially weakly continuous for each $\lambda \geqslant 0$. We now prove that Ψ is coercive: let σ be a positive number such that

$$\limsup_{|x|\to+\infty} \frac{F(x)}{|x|^2} < \sigma < \tfrac{1}{2}c.$$

Then $F(x) < \sigma|x|^2 + m$ for all $x \in \mathbb{R}^N$, for some constant m, and

$$\begin{aligned}\Psi(u) &\geqslant \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + \frac{1}{2}\int_0^T \alpha(t)Au(t)\cdot u(t)\,\mathrm{d}t - \int_0^T \alpha(t)(\sigma|u(t)|^2 + m)\,\mathrm{d}t \\ &\geqslant \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + \left(\frac{1}{2} - \frac{\sigma}{c}\right)\int_0^T \alpha(t)Au(t)\cdot u(t)\,\mathrm{d}t - m\|\alpha\|_{L^1} \\ &\geqslant \left(\frac{1}{2} - \frac{\sigma}{c}\right)\|u\|^2 - m\|\alpha\|_{L^1},\end{aligned}$$

which implies that $\Psi(u)$ tends to infinity as $\|u\|$ goes to infinity.

Specifically, from the coercivity of Ψ it follows that for every $r > \|\alpha\|_{L^1}\inf_{\mathbb{R}^N} H$ the set $\Psi^{-1}(]-\infty, r[)$ is bounded. Moreover, we note that the restriction of Φ to the sequentially weakly compact set $(\overline{\Psi^{-1}(]-\infty, r[)})_w$ has a global minimum.

We claim that

$$\inf_X \Psi = \|\alpha\|_{L^1}\inf_{\mathbb{R}^N} H.$$

In fact, for all $u \in X$ we have

$$\begin{aligned}\Psi(u) &\geqslant \frac{1}{2}\int_0^T \alpha(t)Au(t)\cdot u(t)\,\mathrm{d}t - \int_0^T \alpha(t)F(u(t))\,\mathrm{d}t \\ &= \int_0^T \alpha(t)H(u(t))\,\mathrm{d}t \geqslant \|\alpha\|_{L^1}\inf_{\mathbb{R}^N} H.\end{aligned}$$

Let us denote by M the set of global minima of H in $\mathbb{R}^N$. If $x_0 \in M$, then the function defined by putting $u_0(t) = x_0$ belongs to X and

$$\Psi(u_0) = \int_0^T \alpha(t)H(u_0(t))\,\mathrm{d}t = \|\alpha\|_{L^1}\inf_{\mathbb{R}^N} H.$$

Thus, our claim is proved.

We note that, if $u \in X$ is not constant, then $|\dot{u}| > 0$ on some set of positive measure, hence it cannot be a global minimum of Ψ, and the same is true for constant functions whose value does not belong to M.

Let $\gamma : \mathbb{R}^N \to X$ be the function that maps $x \in \mathbb{R}^N$ into the constant function $u(t) = x$ in X: γ is then a homeomorphism between $\mathbb{R}^N$ and $\gamma(\mathbb{R}^N)$ (endowed with the relativization of the weak topology). The set of global minima of Ψ is equal to $\gamma(M)$; hence it has at least k weakly connected components.

By applying Theorem A we deduce for every $r > \|\alpha\|_{L^1}\inf_{\mathbb{R}^N} H$ the existence of $\lambda_r > 0$ such that, for every $\lambda \in]0, \lambda_r[$, the functional $\Psi + \lambda\Phi$ has at least k τ_Ψ-local minima lying in $\Psi^{-1}(]-\infty, r[)$.

Since Ψ is continuous, the topology τ_Ψ is weaker than the strong topology in X, and every τ_Ψ-local minimum is also a strong local minimum, and so a critical point of $\Psi + \lambda\Phi$. The proof is now complete. □

Theorem 2.2. *Let α, A, F, G be as in § 1, and let assumptions (i) and (ii) of Theorem 2.1 be satisfied. Moreover, assume that*

(iii) $\displaystyle\liminf_{|x|\to+\infty} \frac{\operatorname{ess\,inf}_{[0,T]} G(t,x)}{|x|^2} > -\infty.$

Then, for every $r > \inf_{\mathbb{R}^N} H\|\alpha\|_{L^1}$ there exists $\lambda_r^\star > 0$ such that, for every $\lambda \in]0, \lambda_r^\star[$, (P_λ) has at least $k+1$ solutions, k of which lie in $\Psi^{-1}(]-\infty, r[)$.

Proof. Let us show that, for $\lambda > 0$ small enough, $\Psi + \lambda\Phi$ is coercive. From the proof of Theorem 2.1 we already know that

$$\Psi(u) \geqslant \left(\frac{1}{2} - \frac{\sigma}{c}\right)\|u\|^2 - m\|\alpha\|_{L^1}.$$

Then, there exist $b < 0$ and $s > 0$ such that

$$G(t,x) > b|x|^2$$

for $|x| > s$ and a.e. $t \in [0,T]$, while

$$g = \sup_{|x|\leqslant s} |G(\cdot\,, x)| \in L^1([0,T]).$$

Summarizing, for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$ we have

$$G(t,x) \geqslant b|x|^2 - g(t),$$

which implies that

$$\begin{aligned}
\Phi(u) &\geqslant \int_0^T (b|u(t)|^2 - g(t))\,\mathrm{d}t \\
&\geqslant \frac{b}{ac}\int_0^T \alpha(t)Au(t)\cdot u(t)\,\mathrm{d}t - \|g\|_{L^1} \\
&\geqslant \frac{b}{ac}\|u\|^2 - \|g\|_{L^1}
\end{aligned}$$

and so

$$\Psi(u) + \lambda\Phi(u) \geqslant \left[\left(\frac{1}{2} - \frac{\sigma}{c}\right) + \lambda\frac{b}{ac}\right]\|u\|^2 - m\|\alpha\|_{L^1} - \lambda\|g\|_{L^1}.$$

Set

$$\lambda_r^\star = \min\left\{\lambda_r, -\frac{ac}{b}\left(\frac{1}{2} - \frac{\sigma}{c}\right)\right\},$$

where λ_r is as in Theorem 2.1. Then, for all $\lambda \in]0, \lambda_r^\star[$, the functional $\Psi + \lambda\Phi$ admits at least k local minima and is coercive. Thus, $\Psi + \lambda\Phi$ satisfies the Palais–Smale condition, as it is the sum of $\frac{1}{2}\|u\|^2$, whose derivative is a homeomorphism between H_T^1 and its dual, and of a functional with compact derivative. From [**2**] it follows that $\Psi + \lambda\Phi$ admits one more critical point. □

Remark 2.3. As seen in the proof of Theorem 2.2, the existence of $k+1$ solutions follows essentially from the Palais–Smale condition, and the latter is proved through the coercivity of the functional. By using another standard argument, we could assume that there exist $q > 2$ and $R > 0$ such that, for λ small enough,

$$0 < q[\alpha(t)F(x) - \lambda G(t,x)] \leqslant [\alpha(t)\nabla F(x) - \lambda \nabla_x G(t,x)] \cdot x \tag{2.1}$$

for a.e. $t \in [0,T]$ and for every x with $|x| > R$. This implies, as λ tends to zero, that

$$0 \leqslant qF(x) \leqslant \nabla F(x) \cdot x.$$

Now, if there is some x_1 such that $|x_1| > R$, $F(x_1) > 0$ and

$$F(x) < \tfrac{1}{2}c|x|^2 \tag{2.2}$$

for all $|x| > |x_1|$, then it is easy to prove that the function

$$\mu \to |\mu x_1|^{-q} F(\mu x_1)$$

is non-decreasing for $\mu \geqslant 1$ and so

$$F(\mu x_1) \geqslant |x_1|^{-q} F(x_1)|\mu x_1|^q,$$

which together with (2.2) gives a contradiction.

No contradiction arises, however, if we assume that $F(x) = 0$ for all $x \in \mathbb{R}^N$, $|x| > R$ (so condition (i) is obviously satisfied), together with (ii) and condition (2.1), which becomes

$$\nabla_x G(t,x) \cdot x \leqslant qG(t,x) < 0.$$

In this case we get $k+1$ solutions for $\lambda > 0$ small enough.

In the case $N = 1$, (P_λ) becomes

$$\begin{cases} u'' = \alpha(t)(u - F'(u)) + \lambda G_x(t,u), & \text{a.e. in } [0,T], \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{P'_λ}$$

The following result, whose proof is analogous (with minor changes) to that of Theorem 1 in [**5**], yields the existence of $k+1$ solutions with no additional hypotheses on G.

Theorem 2.4. *Let α, F, G be as in § 1 (with $N = 1$). Assume that*

(iv) $\displaystyle\lim_{|x| \to +\infty} \frac{F'(x)}{x} = 0$;

(v) *the set of global minima of the function $H(x) = \frac{1}{2}x^2 - F(x)$ has at least k connected components in $\mathbb{R}$ ($k \geqslant 2$).*

Then, for every $r > \|\alpha\|_{L^1} \inf_{\mathbb{R}} H$ there exists $\lambda_r > 0$ such that, for every $\lambda \in\,]0, \lambda_r[$, (P'_λ) has at least $k+1$ solutions, k of which satisfy

$$\frac{1}{2}\int_0^T |u'(t)|^2 \,\mathrm{d}t + \int_0^T \alpha(t)(\tfrac{1}{2}|u(t)|^2 - F(u(t)))\,\mathrm{d}t < r. \tag{2.3}$$

3. Examples

In the following examples α and G are as in § 1, while the function F is chosen in order to satisfy assumptions (i) and (ii).

Example 3.1. Let A be the identity matrix ($c = 1$), let $f \in C^1([0, +\infty[)$ be a periodic function such that $f(0) > b = \inf_{\mathbb{R}} f$, and let $q \in]0, 1[$ and $p \in [2, +\infty[$. Define F as follows:

$$F(x) = \begin{cases} \frac{1}{2}|x|^2 - (f(|x|^{-q}) - b)|x|^p & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Clearly, $F \in C^1(\mathbb{R}^N)$. If $p = 2$, we get

$$\limsup_{|x| \to +\infty} \frac{F(x)}{|x|^2} = \tfrac{1}{2} - f(0) + b < \tfrac{1}{2},$$

while, if $p > 2$,

$$\limsup_{|x| \to +\infty} \frac{F(x)}{|x|^2} = -\infty.$$

So (i) is satisfied.

The function H, here, is given by

$$H(x) = (f(|x|^{-q}) - b)|x|^p.$$

H is non-negative and the set of its global minima is $\{x \in \mathbb{R}^N : f(|x|^{-q}) = b\} \cup \{0\}$, which has infinitely many connected components.

Then, for every $k \geqslant 2$, (P_λ) has at least k solutions for λ small enough.

Remark 3.2. The function F in Example 3.1 depends only on the norm of vector x. The thesis holds if we replace $|x|^p$ with $(|x_1| + |x_2| + \cdots + |x_N|)^p$ in the definition of F.

Remark 3.3. It is also clear that the problem considered in Example 3.1 does not satisfy condition (1.4). In fact, in this case, $\phi(t, x) = \alpha(t)(\frac{1}{2}|x|^2 - F(x))$. So, specifically, for all $x \in \mathbb{R}^N$, $x \neq 0$,

$$\frac{\phi(t, x) - \phi(t, 0)}{|x|^2} \geqslant a(f(|x|^{-q}) - b)|x|^{p-2} \geqslant 0.$$

Example 3.4. Let $\varphi_1, \varphi_2, \ldots, \varphi_k \in C^1([0, +\infty[, \mathbb{R})$ be functions such that

(1) for all $i \in \{1, 2, \ldots k\}$, $\varphi_i^{-1}(0) \neq \emptyset$;

(2) for all $i, j \in \{1, 2, \ldots k\}$, $i \neq j$, $\varphi_i^{-1}(0) \bigcap \varphi_j^{-1}(0) = \emptyset$;

(3) $\displaystyle \lim_{\rho \to \infty} \frac{1}{\rho} \prod_{i=1}^{k} (\varphi_i(\rho))^2 = +\infty.$

Then the problem

$$\begin{cases} \ddot{u} = 4\alpha(t)\Big(\sum_{i=1}^{k}\varphi_i(|u|^2)\varphi_i'(|u|^2)\prod_{j\neq i}(\varphi_j(|u|^2))^2\Big)u + \lambda\nabla_x G(t,u), & \text{a.e. in } [0,T], \\ u(0)-u(T) = \dot{u}(0)-\dot{u}(T) = 0 \end{cases} \qquad (P_\lambda)$$

has at least k solutions for $\lambda > 0$ small enough. It can immediately be seen that parts (i) and (ii) of Theorem 2.1 are satisfied: specifically, $x \in \mathbb{R}^N$ is a global minimum of the function $H(x) = \prod_{i=1}^{k}(\varphi_i(|x|^2))^2$ if and only if $\varphi_i(|x|^2) = 0$ for some $i \in \{1, 2, \dots k\}$.

We would like to emphasize that, for λ not sufficiently small, with all the other assumptions of our theorem fulfilled, the thesis may fail, as the following counterexample shows.

Example 3.5. Let us consider the one-dimensional problem

$$\begin{cases} u'' = u - F'(u) + \lambda G'(u), & \text{a.e. in } [0,T], \\ u(0)-u(T) = u'(0)-u'(T) = 0, \end{cases} \qquad (Q_\lambda)$$

where the function F is defined by

$$F(x) = \tfrac{1}{2}x^2 - \tfrac{1}{4}d_1x^4 - \tfrac{1}{2}d_2x^2$$

and $d_1 > 0$, $d_2 < 0$. Clearly, F satisfies (i), as

$$\lim_{|x|\to+\infty}\frac{F(x)}{x^2} = -\infty.$$

The function

$$H(x) = \tfrac{1}{4}d_1x^4 + \tfrac{1}{2}d_2x^2$$

admits two global minima. Choose

$$G(x) = \tfrac{1}{2}x^2.$$

The problem (Q_λ) reads as follows:

$$\begin{cases} u'' = d_1u^3 + (d_2+\lambda)u, & \text{a.e. in } [0,T], \\ u(0)-u(T) = u'(0)-u'(T) = 0. \end{cases}$$

Theorem 2.2 provides the existence of at least three solutions for λ small enough. Specifically, if $\lambda < -d_2$, (Q_λ) has at least three solutions (the constant functions corresponding to the critical points of $H + \lambda G$). If $\lambda \geqslant -d_2$, (Q_λ) has only the trivial solution $u = 0$, since no non-constant function u can be a solution. In fact, if there exists $t_1 \in]0,T[$ such that $u(t_1) > 0$, then we get

$$\max_{[0,T]} u = u(t^*) > 0,$$

hence

$$0 \geqslant u''(t^*) = u(t^*)[d_1u(t^*)^2 + (d_2+\lambda)] > 0,$$

a contradiction (analogously, a contradiction is reached if $u(t_1) < 0$).

Acknowledgements. The authors thank Professor Ricceri for useful suggestions.

References

1. H. Brezis and L. Nirenberg, Remarks on finding critical points, *Commun. Pure Appl. Math.* **44** (1991), 939–963.
2. N. Ghoussoub and D. Preiss, A general mountain pass principle for locating and classifying critical points, *Annls Inst. H. Poincaré Analyse Non Linéaire* **6** (1989), 321–330.
3. J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems* (Springer, 1989).
4. B. Ricceri, Sublevel sets and global minima of coercive functionals and local minima of their perturbations, *J. Nonlin. Convex Analysis* **5**(2) (2004), 157–168.
5. B. Ricceri, A multiplicty theorem for the Neumann problem, *Proc. Am. Math. Soc.*, in press.
6. C. L. Tang, Existence and multiplicity of periodic solutions of nonautonomous second order systems, *Nonlin. Analysis* **32** (1998), 299–304.
7. C. L. Tang, Periodic solutions of nonautonomous second order systems with sublinear nonlinearity, *Proc. Am. Math. Soc.* **126**(11) (1998), 3263–3270.
8. C. L. Tang, Multiplicity of periodic solutions for second order systems with a small forcing term, *Nonlin. Analysis* **38**(4) (1999), 471–479.
9. C. L. Tang and X. P. Wu, Periodic solutions for second order systems with not uniformly coercive potential, *J. Math. Analysis Applic.* **259**(2) (2001), 386–397.