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Critical points for nondifferentiable functions in presence of splitting

R. Livrea [a], S.A. Marano [a,*], D. Motreanu [b]

[a] *Dipartimento di Patrimonio Architettonico e Urbanistico, Università degli Studi Mediterranea di Reggio Calabria, Salita Melissari, 89100 Reggio Calabria, Italy*
[b] *Département de Mathématiques, Université de Perpignan, Avenue de Villeneuve 52, 66860 Perpignan cedex, France*

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Abstract

A classical critical point theorem in presence of splitting established by Brézis–Nirenberg is extended to functionals which are the sum of a locally Lipschitz continuous term and of a convex, proper, lower semicontinuous function. The obtained result is then exploited to prove a multiplicity theorem for a family of elliptic variational–hemivariational eigenvalue problems.

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1. Introduction

A meaningful consequence of Ghoussoub's min–max principle (see, for instance, [11, Theorem 5.2]) is the critical point theorem in presence of splitting established by Brézis–Nirenberg in 1991, i.e., [5, Theorem 4]. Roughly speaking, it is assumed that

* Corresponding author.
E-mail addresses: roberto.livrea@unirc.it (R. Livrea), marano@unirc.it, samarano@mail.gte.it (S.A. Marano), motreanu@univ-perp.fr (D. Motreanu).

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there exist a Banach space X with a direct sum decomposition $X = X_1 \oplus X_2$, where $\dim(X_2) < +\infty$, and a bounded below function $f \in C^1(X, \mathbb{R})$ having a local linking at 0, namely

(f) $f|_{\overline{B}_r \cap X_2} \leqslant 0$ as well as $f|_{\overline{B}_r \cap X_1} \geqslant 0$ for some $r > 0$.

If $\inf_{x \in X} f(x) < 0$, $f(0) = 0$, and the Palais–Smale condition holds true, then f admits at least two nonzero critical points.

Very recently, in [14], Ghoussoub's result has been extended to functions f on a Banach space X fulfilling a structural hypothesis of the type

(H_f) $f(x) := \Phi(x) + \psi(x)$ for all $x \in X$, where $\Phi : X \to \mathbb{R}$ is locally Lipschitz continuous and $\psi : X \to \mathbb{R} \cup \{+\infty\}$ is convex, proper, and lower semicontinuous.

Critical points of f are defined as solutions to the problem:

Find $x \in X$ such that

$$\Phi^0(x; z - x) + \psi(z) - \psi(x) \geqslant 0 \quad \forall z \in X, \tag{1}$$

with $\Phi^0(x; z - x)$ being the generalized directional derivative [7, p. 25] of Φ in x along the direction $z - x$. The Palais–Smale condition for C^1 functions becomes here

$(\mathrm{PS})_f$ Every sequence $\{x_n\} \subseteq X$ such that $\{f(x_n)\}$ is bounded and

$$\Phi^0(x_n; z - x_n) + \psi(z) - \psi(x_n) \geqslant -\epsilon_n \|z - x_n\| \quad \forall n \in \mathbb{N},\ z \in X,$$

where $\epsilon_n \to 0^+$, possesses a convergent subsequence.

This abstract framework was previously introduced and developed by Motreanu and Panagiotopoulos [17]. Inequality (1) is usually called a *variational–hemivaritional inequality*. It has been exploited for mathematically formulating several engineering, besides mechanical, questions, and extensively studied from many points of view in the latest years [17–19]. If $\psi \equiv 0$, then (1) coincides with the problem treated by Chang [6], who also exploits various abstract results to study elliptic equations having discontinuous nonlinear terms. When $\Phi \in C^1(X, \mathbb{R})$, problem (1) reduces to a variational inequality, and significant applications as well as the relevant critical point theory are developed in [21]. Finally, if both $\Phi \in C^1(X, \mathbb{R})$ and $\psi \equiv 0$, then (1) simplifies to the Euler equation, which is classical.

In this paper we first extend the above-mentioned Brézis–Nirenberg critical point theorem to Motreanu–Panagiotopoulos' setting (see Theorem 3.1 below) by using the structural hypothesis, previously introduced in [14],

(H'_f) $f(x) := \Phi(x) + \psi(x)$ for all $x \in X$, where $\Phi : X \to \mathbb{R}$ is locally Lipschitz continuous and $\psi : X \to \mathbb{R} \cup \{+\infty\}$ is convex, proper, and lower semicontinuous. Moreover, ψ is continuous on any nonempty compact set $A \subseteq X$ such that $\sup_{x \in A} \psi(x) < +\infty$.

Although less general than (H_f), this condition still works in all the most important concrete situations. For instance, $\psi := I_K$, with I_K being the indicator function of some nonempty, convex, closed set $K \subseteq X$, represents a standard but meaningful case of ψ. The Banach space X is supposed to be reflexive and with a direct sum decomposition $X = X_1 \oplus X_2$, where $0 < \dim(X_2) < +\infty$, while assumption (f) is replaced by the more restrictive one

(f′) $f|_{\overline{B}_r \cap X_1} \geqslant 0$, $f|_{\overline{B}_r \cap X_2} \leqslant 0$, and $f|_{\partial B_r \cap X_2} < 0$ for some $r > 0$ small enough,

which arises from the different construction of the pseudo-gradient vector field in our abstract situation. We do not know at present whether (f′) can be weakened to (f). The locally Lipschitz continuous case, i.e., $\psi \equiv 0$, has been recently treated in [13].

One application to an elliptic variational–hemivariational inequality patterned after problem (38) in [5] (see also [15, problem (5.7)]) is then presented. Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, let $X := H_0^1(\Omega)$, and let

$$\mathcal{G}(u) := \int_\Omega G\big(u(x)\big)\,dx \quad \forall u \in X,$$

where $G(\xi) := \int_0^\xi -g(t)\,dt$, $\xi \in \mathbb{R}$, with $g : \mathbb{R} \to \mathbb{R}$ measurable. Given $\lambda > 0$ and a nonempty, convex, closed set $K_\lambda \subseteq X$ depending on λ, we prove that if g satisfies suitable growth conditions then the problem:

Find $u \in K_\lambda$ fulfilling

$$-\int_\Omega \nabla u(x) \cdot \nabla(v-u)(x)\,dx - \int_\Omega a(x)u(x)(v-u)(x)\,dx \leqslant \lambda \mathcal{G}^0(u; v-u)$$

for all $v \in K_\lambda$, where $a \in L^\infty(\Omega)$, possesses at least two nontrivial solutions provided λ is sufficiently large.

2. Basic definitions and preliminary results

Let $(X, \|\cdot\|)$ be a real Banach space. If V is a subset of X, we write $\mathrm{int}(V)$ for the interior of V, $\overline{V}$ for the closure of V, ∂V for the boundary of V. When V is nonempty, $x \in X$, and $\delta > 0$, we define $B(x,\delta) := \{z \in X\colon \|z - x\| < \delta\}$ as well as $B_\delta := B(0,\delta)$. Given $x, z \in X$, the symbol $[x,z]$ indicates the line segment joining x to z, namely

$$[x,z] := \big\{(1-t)x + tz\colon t \in [0,1]\big\}.$$

Moreover, $]x,z] := [x,z] \setminus \{x\}$. We denote by X^* the dual space of X, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X and X^*. A function $\Phi : X \to \mathbb{R}$ is called locally Lipschitz continuous when to every $x \in X$ there correspond a neighbourhood V_x of x and a constant $L_x \geqslant 0$ such that

$$\left|\Phi(z) - \Phi(w)\right| \leqslant L_x \|z - w\| \quad \forall z, w \in V_x.$$

If $x, z \in X$, we write $\Phi^0(x; z)$ for the generalized directional derivative of Φ at the point x along the direction z, i.e.,

$$\Phi^0(x; z) := \limsup_{w \to x,\ t \to 0^+} \frac{\Phi(w + tz) - \Phi(w)}{t}.$$

It is known [7, Proposition 2.1.1] that Φ^0 is upper semicontinuous on $X \times X$. The generalized gradient of the function Φ in x, denoted by $\partial\Phi(x)$, is the set

$$\partial\Phi(x) := \left\{x^* \in X^*\colon \left\langle x^*, z\right\rangle \leqslant \Phi^0(x; z)\ \forall z \in X\right\}.$$

Proposition 2.1.2 of [7] ensures that $\partial\Phi(x)$ turns out nonempty, convex, in addition to weak* compact.

Let f be a function on X satisfying the structural hypothesis (H_f) in Section 1. Put $D_\psi := \{x \in X\colon\ \psi(x) < +\infty\}$. Since ψ turns out continuous on $\mathrm{int}(D_\psi)$ (see, for instance, [8, Exercise 1, p. 296]) the same holds regarding f. To simplify notation, always denote by $\partial\psi(x)$ the subdifferential of ψ at x in the sense of convex analysis, while

$$D_{\partial\psi} := \left\{x \in X\colon\ \partial\psi(x) \neq \emptyset\right\}.$$

Theorem 23.5 of [8] gives $\mathrm{int}(D_\psi) = \mathrm{int}(D_{\partial\psi})$. Moreover, by [8, Theorems 23.5 and 23.3], $\partial\psi(x)$ is always convex and weak* closed. We say that $x \in D_\psi$ is a critical point of f when (1) holds true. The symbol $K(f)$ indicates the set of all critical points for f. Given a real number c, we write

$$f_c := \left\{x \in X\colon\ f(x) \leqslant c\right\}, \qquad f^c := \left\{x \in X\colon\ f(x) \geqslant c\right\},$$

and

$$K_c(f) := K(f) \cap f^{-1}(c).$$

If $K_c(f) \neq \emptyset$ then $c \in \mathbb{R}$ is called a critical value of f.

The following variant [10, pp. 444, 456] of the famous variational principle of Ekeland will be repeatedly employed.

Theorem 2.1. *Let (Z, d) be a complete metric space and let Π be a proper, lower semicontinuous, bounded below function from Z into $\mathbb{R} \cup \{+\infty\}$. Then to every $\epsilon, \delta > 0$ and every $\bar{z} \in Z$ satisfying $\Pi(\bar{z}) \leqslant \inf_{z \in Z} \Pi(z) + \epsilon$ there corresponds a point $z_0 \in Z$ such that*

$$\Pi(z_0) \leqslant \Pi(\bar{z}), \qquad d(z_0, \bar{z}) \leqslant \frac{1}{\delta}, \qquad \Pi(z) - \Pi(z_0) \geqslant -\epsilon\delta d(z, z_0) \quad \forall z \in Z.$$

Propositions 2.1 and 2.2 below are established via Theorem 2.1. The first of them represents a nonsmooth version of [5, Proposition 2].

Proposition 2.1. *Assume f is bounded below and satisfies* $(\mathrm{PS})_f$ *in addition to* (H_f). *Then each minimizing sequence for f possesses a convergent subsequence.*

Proof. Let $\{x_n\} \subseteq X$ fulfils $\lim_{n\to+\infty} f(x_n) = \inf_{x\in X} f(x)$. Passing to a subsequence if necessary, we may suppose $f(x_n) \leqslant \inf_{x\in X} f(x) + 1/n^2$, $n \in \mathbb{N}$. By Theorem 2.1, for every $n \in \mathbb{N}$ there exists a point $z_n \in X$ enjoying the following properties:

$$f(z_n) \leqslant f(x_n), \tag{2}$$

$$\|z_n - x_n\| \leqslant \frac{1}{n}, \tag{3}$$

$$f(z) - f(z_n) \geqslant -\frac{1}{n}\|z - z_n\| \quad \forall z \in X. \tag{4}$$

Through (2) we obtain that $\{f(z_n)\}$ is bounded, while (4) leads to

$$\Phi^0(z_n; x - z_n) + \psi(x) - \psi(z_n) \geqslant -\frac{1}{n}\|x - z_n\| \quad \forall x \in X. \tag{5}$$

Indeed, if $x \in X$ and $z := z_n + t(x - z_n)$, with $t \in]0, 1[$, then from (4), besides (H_f), it follows

$$\Phi\big(z_n + t(x - z_n)\big) - \Phi(z_n) + t\big[\psi(x) - \psi(z_n)\big] \geqslant -\frac{t}{n}\|x - z_n\|.$$

Dividing by t and letting $t \to 0^+$ we achieve (5). At this point, condition $(\mathrm{PS})_f$ forces $z_n \to x_0$ for suitable $x_0 \in X$, where a subsequence is considered when necessary, and thus, by (3), also $x_n \to x_0$. □

Remark 2.1. The preceding result guarantees that every function f which is bounded below and satisfies (H_f) as well as $(\mathrm{PS})_f$ attains its minimum at some $x_0 \in X$.

Proposition 2.2. *Let f be bounded below and fulfil* $(\mathrm{PS})_f$ *in addition to* (H_f). *Assume the global minimum point x_0 is unique. Then, for every $\rho_0 > 0$ there exists a $\rho > 0$ such that*

$$U_\rho := \big\{x \in X\colon\ f(x) < f(x_0) + \rho\big\} \subseteq B(x_0, \rho_0).$$

Proof. Arguing by contradiction one can find a $\rho_0 > 0$ and a sequence $\{x_n\} \subseteq X$ such that

$$f(x_n) < f(x_0) + \frac{1}{n^2}, \qquad \|x_n - x_0\| \geqslant \rho_0 \quad \forall n \in \mathbb{N}.$$

Now, Theorem 2.1 provides a point $z_n \in X$ satisfying (2)–(4). Set $z := z_n + t(x - z_n)$, with $x \in X$ and $t \in]0, 1[$. As in the proof of Proposition 2.1, inequality (4), besides the convexity of ψ, lead to (5). Thus, by condition $(\mathrm{PS})_f$, there exists a subsequence $\{z_{k_n}\}$ of $\{z_n\}$ strongly converging to some $z \in X$. Since $f(z_{k_n}) \to f(x_0)$ in view of (2), we have $f(z) = f(x_0)$, which forces $z = x_0$ taking into account the uniqueness of x_0. On the other hand, $x_{k_n} \to x_0$ due to (3). However, this is impossible because $\|x_{k_n} - x_0\| \geqslant \rho_0$ for all $n \in \mathbb{N}$, and the conclusion follows. □

Finally, the next result will play a basic role in establishing the abstract theorem of this paper. For its proof we refer to [14, Theorem 3.3]. Here, Q indicates a compact set in X, Q_0 is a nonempty closed subset of Q, γ_0 belongs to $C^0(Q_0, X)$, while

$$\Gamma := \{\gamma \in C^0(Q, X)\colon \gamma|_{Q_0} = \gamma_0\}.$$

Theorem 2.2. *Suppose the function f satisfies the assumptions below in addition to (H'_f) and $(\mathrm{PS})_f$.*

- (a_1) $\sup_{x \in Q} f(\hat{\gamma}(x)) < +\infty$ *for some* $\hat{\gamma} \in \Gamma$.
- (a_2) *There exists a closed subset S of X such that* $\sup_{x \in Q_0} f(\gamma_0(x)) \leqslant \inf_{x \in S} f(x)$ *and* $(\gamma(Q) \cap S) \setminus \gamma_0(Q_0) \neq \emptyset$ *for all* $\gamma \in \Gamma$.

Put $c := \inf_{\gamma \in \Gamma} \sup_{x \in Q} f(\gamma(x))$. *Then the set $K_c(f)$ is nonempty. If, moreover,* $\inf_{x \in S} f(x) = c$ *then* $K_c(f) \cap S \neq \emptyset$.

3. Critical points in presence of splitting

Throughout this section, $(X, \|\cdot\|)$ is a real reflexive Banach space while f denotes a function from X into $\mathbb{R} \cup \{+\infty\}$. The following hypotheses will be posited in the sequel:

- (f_1) f is bounded below and fulfils $(\mathrm{PS})_f$ besides (H_f).
- (f_2) $x_0 \in X$ is a global minimum point of the function f.

Observe that if (f_1) holds then f attains its minimum; see Remark 2.1. We shall further assume:

- (f_3) $x_0 \neq 0$. Moreover, x_0 and eventually 0 are the only critical points for f.
- (f_4) There exist two disjoint open neighbourhoods U_0 and N_0 of x_0 and 0, respectively, as well as a constant $b > \inf_{x \in X} f(x)$, satisfying $f_b \setminus (U_0 \cup N_0) \subseteq D_{\partial\psi}$.
- (f_5) If $\{x_n\} \subseteq f_b \setminus (U_0 \cup N_0)$, $x_n \to x$ in X, and $x_n^* \in \partial\psi(x_n)$ for all $n \in \mathbb{N}$, then to each $z \in X$ there corresponds an $x^* \in \partial\psi(x)$ such that $\langle x^*, z\rangle \leqslant \limsup_{n \to +\infty} \langle x_n^*, z\rangle$.

Proposition 3.1. *Suppose* (f_1)–(f_4) *hold true. Then there exists a constant $\sigma > 0$ such that for every $x \in f_b \setminus (U_0 \cup N_0)$, $x^* \in \partial\Phi(x)$, $z^* \in \partial\psi(x)$ one has $\|x^* + z^*\|_{X^*} \geqslant \sigma$.*

Proof. Arguing by contradiction one could construct three sequences $\{x_n\} \subseteq X$, $\{x_n^*\}$, $\{z_n^*\} \subseteq X^*$ with the following properties:

$$x_n \in f_b \setminus (U_0 \cup N_0), \quad n \in \mathbb{N}, \tag{6}$$

$$x_n^* \in \partial\Phi(x_n) \quad \text{and} \quad z_n^* \in \partial\psi(x_n) \quad \forall n \in \mathbb{N}, \tag{7}$$

$$\|x_n^* + z_n^*\|_{X^*} \to 0 \quad \text{as } n \to +\infty. \tag{8}$$

From (7) we obtain easily

$$\begin{aligned} \Phi^0(x_n; x - x_n) + \psi(x) - \psi(x_n) &\geqslant \langle x_n^*, x - x_n\rangle + \langle z_n^*, x - x_n\rangle \\ &\geqslant -\|x_n^* + z_n^*\|_{X^*}\|x - x_n\| \end{aligned}$$

for all $n \in \mathbb{N}$, $x \in X$. Thanks to $(\mathrm{PS})_f$, setting $\epsilon_n := \|x_n^* + z_n^*\|_{X^*}$ and using (8) produces $x_n \to \bar{x}$ in X, where a subsequence is considered when necessary. Moreover, by (6), the point $\bar{x}$ lies in $f_b \setminus (U_0 \cup N_0)$. Since Φ^0 and $-\psi$ are upper semicontinuous, this forces both $\bar{x} \in D_\psi$ and

$$\Phi^0(\bar{x}; x - \bar{x}) + \psi(x) - \psi(\bar{x}) \geqslant 0 \quad \forall x \in X,$$

namely $\bar{x}$ turns out a critical point of f different from x_0 and 0, against hypothesis (f_3). □

Proposition 3.2. *Let* (f_1)–(f_5) *be satisfied and let* σ *be as in Proposition* 3.1. *Then there exists a locally Lipschitz continuous function* $F : f_b \setminus (U_0 \cup N_0) \to X$ *such that, for every* $x \in f_b \setminus (U_0 \cup N_0)$, $\|F(x)\| \leqslant 1$ *and*

$$\langle x^* + z^*, F(x)\rangle > \frac{\sigma}{2} \quad \forall x^* \in \partial\Phi(x),\ z^* \in \partial\psi(x). \tag{9}$$

Proof. From now on, W denotes the set $f_b \setminus (U_0 \cup N_0)$. Pick $x \in W$. We first claim that the infimum

$$\delta(x) := \inf\{\|x^* + z^*\|_{X^*}\colon x^* \in \partial\Phi(x),\ z^* \in \partial\psi(x)\} \tag{10}$$

is attained. To show this, fix $\{x_n^*\} \subseteq \partial\Phi(x)$ and $\{z_n^*\} \subseteq \partial\psi(x)$ fulfilling

$$\lim_{n\to+\infty} \|x_n^* + z_n^*\|_{X^*} = \delta(x). \tag{11}$$

Since X is reflexive while $\partial\Phi(x)$ is weak* compact, we can find an $\bar{x}^* \in \partial\Phi(x)$ such that, along a subsequence if necessary, $x_n^* \rightharpoonup \bar{x}^*$. By (11) the sequence $\{z_n^*\}$ turns out bounded. So, as before, $z_n^* \rightharpoonup \bar{z}^*$ for some $\bar{z}^* \in \partial\psi(x)$. One clearly has

$$\|\bar{x}^* + \bar{z}^*\|_{X^*} \leqslant \liminf_{n\to+\infty} \|x_n^* + z_n^*\|_{X^*},$$

which implies $\|\bar{x}^* + \bar{z}^*\|_{X^*} = \delta(x)$.

Proposition 3.1 ensures that $\delta(x) \geqslant \sigma > 0$. Hence, $B_{\delta(x)}$ is nonempty and, on account of (10),

$$B_{\delta(x)} \cap \big(\partial\Phi(x) + \partial\psi(x)\big) = \emptyset.$$

Now, the Hahn–Banach Theorem [4, Theorem I.6] provides a point $\xi_x \in X$ with the properties $\|\xi_x\| = 1$ and, whenever $x^* \in \partial\Phi(x)$, $z^* \in \partial\psi(x)$,

$$\langle x^* + z^*, \xi_x\rangle \geqslant \langle w^*, \xi_x\rangle \quad \forall w^* \in B_{\delta(x)}.$$

This inequality and Proposition 3.1 lead to

$$\langle x^* + z^*, \xi_x\rangle \geqslant \delta(x) \geqslant \sigma \tag{12}$$

for all $x^* \in \partial\Phi(x)$, $z^* \in \partial\psi(x)$.

We next show that to each $x \in W$ there corresponds an open neighbourhood V_x of x such that as soon as $v \in V_x \cap W$ one has

$$\langle x^* + z^*, \xi_x\rangle > \frac{\sigma}{2} \quad \forall x^* \in \partial\Phi(v),\ z^* \in \partial\psi(v). \tag{13}$$

Indeed, if the assertion were false then we could find $x \in W$, $\{x_n\} \subseteq W$, and $\{x_n^*\}$, $\{z_n^*\} \subseteq X^*$ satisfying the following conditions:

$$x_n \to x, \qquad x_n^* \in \partial\Phi(x_n), \qquad z_n^* \in \partial\psi(x_n), \quad n \in \mathbb{N}, \tag{14}$$

$$\langle x_n^* + z_n^*, \xi_x\rangle \leqslant \frac{\sigma}{2} \quad \forall n \in \mathbb{N}. \tag{15}$$

Due to the reflexivity of X and (14), Proposition 2.1.2 in [7] yields an $x^* \in X^*$ such that $x_n^* \rightharpoonup x^*$ in X^*, where a subsequence is considered when necessary, while Proposition 2.1.5 of the same reference forces $x^* \in \partial\Phi(x)$. From (15) we thus get

$$\limsup_{n\to+\infty} \langle z_n^*, \xi_x\rangle \leqslant \frac{\sigma}{2} - \langle x^*, \xi_x\rangle.$$

Now, exploiting (f_5) provides a point $z^* \in \partial\psi(x)$ such that

$$\langle z^*, \xi_x\rangle \leqslant \frac{\sigma}{2} - \langle x^*, \xi_x\rangle,$$

which contradicts (12).

The family $\mathcal{V} := \{V_x\colon x \in W\}$ represents an open covering of W. Since, by [9, Theorem VIII.2.4], this set is paracompact, $\mathcal{V}$ possesses an open locally finite refinement $\{V_i\colon i \in I\}$. Moreover, on account of (13), to each $i \in I$ there corresponds a $\xi_i \in X$ fulfilling $\|\xi_i\| = 1$ as well as, whenever $x \in V_i \cap W$,

$$\langle x^* + z^*, \xi_i\rangle > \frac{\sigma}{2} \quad \forall x^* \in \partial\Phi(x),\ z^* \in \partial\psi(x). \tag{16}$$

Let $\{\rho_i \colon i \in I\}$ be a partition of unity subordinated to $\{V_i \colon i \in I\}$ such that each ρ_i turns out locally Lipschitz continuous; for a possible construction we refer to [16, p. 145]. Define

$$F(x) := \sum_{i \in I} \rho_i(x)\xi_i, \quad x \in W. \tag{17}$$

The function F is evidently locally Lipschitz continuous and one has $\|F(x)\| \leqslant 1$ because $\sum_{i \in I} \rho_i(x) = 1$ in W. Exploiting (16) we then see at once that

$$\langle x^* + z^*, F(x) \rangle > \frac{\sigma}{2} \quad \forall x^* \in \partial\Phi(x),\ z^* \in \partial\psi(x),$$

which completes the proof. □

We are in a position now to establish the main result of this paper. It can be regarded as a nonsmooth version of the famous Brézis–Nirenberg critical point theorem [5, Theorem 4]; vide also [11, Theorem 5.18] and [15, Theorem 1]. Suppose

$$X := X_1 \oplus X_2,$$

where $\dim(X_1) > 0$, while $0 < \dim(X_2) < \infty$. The symbol (f_1') will denote (f_1) with (H_f) replaced by (H_f').

Theorem 3.1. *Assume* (f_1') *and* (f_2) *are satisfied,* $\inf_{x \in X} f(x) < f(0)$, $f(0) = 0$, *and, moreover,*

(f_6) *the set* $\{x \in X \colon f(x) < a\}$ *is open for some constant* $a > 0$,
(f_7) *there exists an* $r \in \left]0, \frac{\|x_0\|}{2}\right[$ *such that* $f|_{\overline{B}_r \cap X_1} \geqslant 0$, $f|_{\overline{B}_r \cap X_2} \leqslant 0$, *and* $f|_{\partial B_r \cap X_2} < 0$.

Then the function f *possesses at least two nontrivial critical points.*

Proof. One clearly has $x_0 \neq 0$ because $f(x_0) = \inf_{x \in X} f(x) < f(0)$. Suppose ($\mathrm{f}_3$) holds true, since otherwise we are done. It is not restrictive to write (f_7) for $r = 1$. Let us first note that

$$f(x_0) < \inf_{x \in \overline{B}_1 \cap X_2} f(x). \tag{18}$$

Indeed, $\overline{B}_1 \cap X_2$ turns out a compact subset of $\{x \in X \colon f(x) < a\}$ while, due to (f_6), f is locally Lipschitz continuous on this set. Thus, one can find an $x_1 \in \overline{B}_1 \cap X_2$ fulfilling $f(x_1) = \inf_{x \in \overline{B}_1 \cap X_2} f(x)$. If $f(x_1) = f(x_0)$ then x_1 would be a global minimum point for f. Since $\|x_1\| \leqslant 1 < \|x_0\|$ and $f(x_0) < f(0)$, we would get $x_1 \in K(f) \setminus \{0, x_0\}$, which contradicts ($\mathrm{f}_3$).

Now, from $f(x_0) < 0 < a$ it easily follows

$$f(x) < 0 \quad \text{in } \overline{B(x_0, \rho_0)} \tag{19}$$

for some $\rho_0 \in]0, \frac{\|x_0\|}{2}[$. Let $\rho > 0$ be as in Proposition 2.2. On account of (18), we may assume that

$$2\rho < \min\{\rho_0, 1\} \quad \text{and} \quad f(x_0) + \rho < \inf_{x \in \overline{B}_1 \cap X_2} f(x). \tag{20}$$

Moreover, by decreasing ρ when necessary, hypothesis (f_7) leads to

$$\sup_{x \in \partial B_1 \cap X_2} f(x) < -2\rho L < 0, \tag{21}$$

where L denotes a Lipschitz constant for f on a suitable closed ball centered at 0, which contains $\overline{B}_{2\rho}$. Pick any $b \in]0, a[$. Through (f_7) and (20) we obtain

$$\partial B_1 \cap X_2 \subseteq \{x \in X\colon\ f(x) < b\} \setminus \overline{U_\rho \cup B_{2\rho}}. \tag{22}$$

Observe next that (f_4) is satisfied with $U_0 := U_\rho$, $N_0 := B_{2\rho}$ because, in view of Proposition 2.2,

$$U_\rho \cap B_{2\rho} \subseteq B(x_0, \rho_0) \cap B_{2\rho} = \emptyset \tag{23}$$

while, due to the choice of b besides (f_6),

$$f_b \subseteq \{x \in X\colon\ f(x) < a\} \subseteq \operatorname{int}(D_\psi) = \operatorname{int}(D_{\partial\psi}). \tag{24}$$

Exploiting this inclusion we also see that (f_5) holds true. Indeed, since the set $\{x \in X\colon\ f(x) < a\}$ is open, ψ turns out locally Lipschitz continuous on a neighbourhood of f_b. If $\{x_n\} \subseteq f_b \setminus (U_0 \cup N_0)$, $x_n \to x$ in X, $x_n^* \in \partial\psi(x_n)$ for all $n \in \mathbb{N}$, and $z \in X$ then, by [6, Proposition 6], there exists a relabelled sequence $\{w_n^*\} \subseteq \partial\psi(x)$ fulfilling

$$\lim_{n\to+\infty} \langle x_n^* - w_n^*, z\rangle = 0. \tag{25}$$

We may suppose $w_n^* \rightharpoonup x^*$ for some $x^* \in \partial\psi(x)$, where a subsequence is considered when necessary. Consequently, owing to (25),

$$\langle x^*, z\rangle = \lim_{n\to+\infty} \langle w_n^*, z\rangle = \lim_{n\to+\infty} \langle w_n^* - x_n^* + x_n^*, z\rangle \leqslant \limsup_{n\to+\infty} \langle x_n^*, z\rangle,$$

as desired. At this point, Proposition 3.2 can be applied, and we get a locally Lipschitz continuous function $F: f_b \setminus (U_0 \cup N_0) \to X$ enjoying property (9), besides $\|F(x)\| \leqslant 1$. In particular, (9) evidently forces $F(x) \neq 0$ for all $x \in f_b \setminus (U_0 \cup N_0)$.

Fix any $z \in \partial B_1 \cap X_2$. On account of (22) it makes sense to consider the Cauchy problem

$$\begin{cases} \frac{d\eta_z(t)}{dt} = -F(\eta_z(t)), \\ \eta_z(0) = z. \end{cases} \tag{26}$$

By the basic existence-uniqueness theorem for ordinary differential equations in Banach spaces it possesses a unique local solution η_z. Let T_z be the maximum of $\{T \in]0,+\infty]$: η_z is defined on $[0,T[\}$. We claim that $T_z < +\infty$. In fact, since f turns out locally Lipschitz continuous on a neighbourhood of f_b, Proposition 9 in [6] yields

$$\frac{d}{dt} f\big(\eta_z(t)\big) \leqslant \max_{x^* \in \partial\Phi(\eta_z(t)),\ z^* \in \partial\psi(\eta_z(t))} \big\langle x^* + z^*, -F\big(\eta_z(t)\big)\big\rangle$$

for almost every $t \in [0, T_z[$. Thanks to (9) we thus have

$$\frac{d}{dt} f\big(\eta_z(t)\big) \leqslant -\frac{\sigma}{2}. \tag{27}$$

Integrating over $[0,t]$, $t \in]0, T_z[$, provides

$$f\big(\eta_z(t)\big) - f(z) \leqslant -\frac{\sigma}{2} t, \tag{28}$$

which clearly leads to

$$T_z \leqslant \frac{2}{\sigma}\Big(f(z) - \inf_{x \in X} f(x)\Big) < +\infty.$$

Observe next that

$$\eta_z(t) = z - \int_0^t F\big(\eta_z(\tau)\big)\, d\tau \quad \forall t \in [0, T_z[. \tag{29}$$

Consequently, due to the boundedness of F, $\eta_z(t)$ converges as $t \to T_z$. Setting

$$w_z := \lim_{t \to T_z} \eta_z(t) \tag{30}$$

it results in $w_z \in \partial(f_b \setminus (U_0 \cup N_0))$, because $[0, T_z[$ is maximal. By (28) and (22), the point w_z cannot belong to the boundary of f_b. Therefore, $w_z \in \partial(U_0 \cup N_0)$. If $w_z \in \partial N_0$ then $\|w_z\| = 2\rho$. Using (28) again, (30), besides (21), one has

$$f(w_z) < f(z) < -2\rho L. \tag{31}$$

Since f is Lipschitz continuous on $\overline{B}_{2\rho}$, we also obtain

$$f(w_z) = f(w_z) - f(0) \geqslant -L\|w_z\| = -2\rho L,$$

which contradicts (31). Hence,

$$w_z \in \partial U_0 \quad \forall z \in \partial B_1 \cap X_2. \tag{32}$$

Now, pick an $e \in \partial B_1 \cap X_1$ and define

$$Q := \big([0, e] \oplus (\overline{B}_1 \cap X_2)\big) \cap \overline{B}_1. \tag{33}$$

The boundary Q_0 of Q relative to $\operatorname{span}\{e\} \oplus X_2$ is given by

$$Q_0 = \{e\} \cup (\overline{B}_1 \cap X_2) \cup \big(\partial B_1 \cap \big(]0, e] \oplus \{\mu z\colon\ \mu \in]0, 1], z \in \partial B_1 \cap X_2\}\big)\big).$$

Write $\gamma_0(e) := x_0$, $\gamma_0(x) := x$ for all $x \in \overline{B}_1 \cap X_2$, as well as

$$\gamma_0(x) := \begin{cases} \eta_z(2\lambda T_z) & \text{if } 0 < \lambda < \frac{1}{2}, \\ w_z & \text{if } \lambda = \frac{1}{2}, \\ (2\lambda - 1)x_0 + (2 - 2\lambda)w_z & \text{if } \frac{1}{2} < \lambda \leqslant 1, \end{cases} \tag{34}$$

provided $x := \lambda e + \mu z$, with $\lambda, \mu \in]0, 1]$, $z \in \partial B_1 \cap X_2$, and $\|x\| = 1$. A simple computation ensures that $\gamma_0 : Q_0 \to X$ turns out continuous. Moreover,

$$f\big(\gamma_0(x)\big) \leqslant 0 \quad \forall x \in Q_0. \tag{35}$$

Indeed, we evidently have $f(\gamma_0(e)) = f(x_0) < 0$ while, in view of (f_7), $f(\gamma_0(x)) = f(x) \leqslant 0$ for any $x \in \overline{B}_1 \cap X_2$. Put $x := \lambda e + \mu z$, where $\lambda, \mu \in]0, 1]$, $z \in \partial B_1 \cap X_2$. If $\lambda < 1/2$ then, thanks to (27),

$$f\big(\gamma_0(x)\big) = f\big(\eta_z(2\lambda T_z)\big) \leqslant f\big(\eta_z(0)\big) = f(z) \leqslant 0.$$

The same reasoning yields (35) for $\lambda = 1/2$. So, suppose $\lambda > 1/2$. Because of (32), besides Proposition 2.2, it results in

$$\big\|(2\lambda - 1)x_0 + (2 - 2\lambda)w_z - x_0\big\| \leqslant \|x_0 - w_z\| \leqslant \rho_0. \tag{36}$$

From (19) we thus achieve

$$f\big(\gamma_0(x)\big) = f\big((2\lambda - 1)x_0 + (2 - 2\lambda)w_z\big) < 0,$$

and (35) is proved. Let us next verify that

$$\big\|\gamma_0(x)\big\| \geqslant 2\rho \quad \forall x \in \partial B_1 \cap Q. \tag{37}$$

When $x := e$ or $x \in \overline{B}_1 \cap X_2$, this inequality is an immediate consequence of the choice of ρ_0 and (20). Pick $x := \lambda e + \mu z$, with $\|x\| = 1$, $\lambda, \mu \in]0, 1]$, $z \in \overline{B}_1 \cap X_2$. Since $\eta_z(t)$, $t \in]0, T_z[$, does not belong to N_0, (37) holds true for $0 < \lambda < 1/2$. If $1/2 \leqslant \lambda \leqslant 1$ then exploiting (36), besides (20), we infer

$$\gamma_0(x) \in \overline{B(x_0, \rho_0)} \subseteq X \setminus B_{2\rho},$$

namely, $\|\gamma_0(x)\| \geqslant 2\rho$, as desired.

Now, define

$$\Gamma := \{\gamma \in C^0(Q, X):\ \gamma|_{Q_0} = \gamma_0\}, \qquad c := \inf_{\gamma \in \Gamma} \sup_{x \in Q} f(\gamma(x)),$$

in addition to $S := \partial B_\rho \cap X_1$. Gathering (35), ($f_7$), the inequality $\rho < 1$ together one has

$$\sup_{x \in Q_0} f(\gamma_0(x)) \leqslant 0 \leqslant \inf_{x \in S} f(x).$$

Through (37) and [5, Lemma 3] we then get $(\gamma(Q) \cap S) \setminus \gamma_0(Q_0) \neq \emptyset$ for all $\gamma \in \Gamma$. Hence, assumption (a_2) of Theorem 2.2 is satisfied. To verify (a_1), observe at first that the set $\mathrm{conv}(\gamma_0(Q_0))$ turns out compact, because so is Q_0, while from (35), (24) it follows $\mathrm{conv}(\gamma_0(Q_0)) \subseteq \mathrm{int}(D_\psi)$. Thus, by the Generalized Theorem of Tietze [3, p. 77], there exists a $\hat{\gamma} \in \Gamma$ such that $\hat{\gamma}(Q) \subseteq \mathrm{conv}(\gamma_0(Q_0))$. Since f is continuous on $\mathrm{int}(D_\psi)$, this implies $\sup_{x \in Q} f(\hat{\gamma}(x)) < +\infty$, i.e., hypothesis ($a_1$) holds true too. Therefore, thanks to Theorem 2.2, $K_c(f) \neq \emptyset$. One clearly has $\inf_{x \in S} f(x) \leqslant c$. If $\inf_{x \in S} f(x) < c$ then the function f possesses a critical point different from x_0 and 0. Otherwise, $K_c(f) \cap S \neq \emptyset$, which again leads to the same conclusion. However, this contradicts condition (f_3). $\square$

Remark 3.1. Hypothesis (f_7) is obviously fulfilled in the meaningful special case:

(f_7') For some $r > 0$ one has $f|_{\overline{B}_r \cap X_1} \geqslant 0$ as well as $f|_{\overline{B}_r \cap X_2 \setminus \{0\}} < 0$,

namely, 0 turns out a local minimum of $f|_{X_1}$ and a proper local maximum for $f|_{X_2}$.

Remark 3.2. When $\dim(X_2) \geqslant 2$ assumption (f_7) can be replaced by the one below, which is more general:

(f_7'') There exists an $r \in]0, \frac{\|x_0\|}{2}[$ such that $f|_{\overline{B}_r \cap X_1} \geqslant 0$, $f|_{\overline{B}_r \cap X_2} \leqslant 0$, and $f|_{\overline{B}_r \cap X_2} \not\equiv 0$.

Indeed, in such a case, $f(\bar{z}) < 0$ for some $\bar{z} \in \overline{B}_r \cap X_2$. It is not restrictive to suppose both $\bar{z} \in \partial B_r \cap X_2$ and $r = 1$. Thus, inequality (21) becomes

$$f(\bar{z}) < -2\rho L < 0.$$

Arguing exactly as in the proof of Theorem 3.1 we get

$$w_z \in \partial(U_0 \cup N_0) \quad \forall z \in \partial B_1 \cap X_2,$$

besides $w_{\bar{z}} \in \partial U_0$. Define

$$A := \{z \in \partial B_1 \cap X_2:\ w_z \in \partial U_0\}, \qquad B := \{z \in \partial B_1 \cap X_2:\ w_z \in \partial N_0\}.$$

One clearly has $A \neq \emptyset$, $A \cup B = \partial B_1 \cap X_2$, and $A \cap B = \emptyset$ because, due to (23), $\overline{U}_0 \cap \overline{N}_0 = \emptyset$. Let us next verify that the sets A, B turn out closed. Pick a sequence $\{z_n\} \subseteq A$

satisfying $z_n \to z$. By continuous dependence on the initial data it follows $T_{z_n} \to T_z$. Hence, to any $\epsilon > 0$ sufficiently small there corresponds a $\nu \in \mathbb{N}$ such that

$$\|z_n - z\| < \epsilon, \qquad 0 < T_z - \epsilon < T_{z_n} < T_z + \epsilon \quad \forall n > \nu.$$

Exploiting (29), the pointwise convergence of $\{F(\eta_{z_n}(t))\}$ to $F(\eta_z(t))$ in $[0, T_z - \epsilon]$, and the inequality $\|F(x)\| \leqslant 1$, we achieve

$$\|w_{z_n} - w_z\| \leqslant \|z_n - z\| + \left\| \int_0^{T_z-\epsilon} \left[F\left(\eta_{z_n}(t)\right) - F\left(\eta_z(t)\right)\right] dt \right\| + \left\| \int_{T_z-\epsilon}^{T_{z_n}} F\left(\eta_{z_n}(t)\right) dt \right\| + \left\| \int_{T_z-\epsilon}^{T_z} F\left(\eta_z(t)\right) dt \right\| < 5\epsilon$$

provided $n > \nu$ is large enough. Consequently, $w_{z_n} \to w_z$, which implies $w_z \in \partial U_0$, i.e., $z \in A$. A similar reasoning ensures that B turns out closed. Since $\partial B_1 \cap X_2$ is connected, we must have $\partial B_1 \cap X_2 = A$, and (32) holds true. At this point, the proof goes on exactly as the one of Theorem 3.1.

Let x_1 be the critical point of f different from x_0 and 0 given by Theorem 3.1. Write

$$\hat{c} := \inf_{\gamma \in \widehat{\Gamma}} \sup_{x \in [x_0, x_1]} f\left(\gamma(x)\right),$$

where

$$\widehat{\Gamma} := \left\{\gamma \in C^0\left([x_0, x_1], X\right) \colon \gamma(x_i) = x_i,\ i = 0, 1\right\},$$

and observe that $\hat{c} < +\infty$ because $x_0, x_1 \in D_\psi$. Combining the above result with [14, Theorem 4.2] yields the following:

Theorem 3.2. *Suppose the assumptions of Theorem* 3.1 *are fulfilled,* $f(x_1) \geqslant 0$ *whenever* x_1 *is a local minimum, while* $f^{\hat{c}}$ *turns out closed. Then either* f *possesses a nonzero critical point, which is not a local minimum, or* $\hat{c} = f(x_1)$ *and* f *admits a continuum of local minima at the level* $\hat{c}$.

4. An application

In this section we shall exploit Theorem 3.1 to solve an elliptic variational–hemivariational inequality, in the sense of Panagiotopoulos [19], patterned after problem (38) in [5]; see besides [11, Theorem 5.22] and [15, Theorem 6].

Let Ω be a nonempty, bounded, open subset of the real Euclidean N-space $(\mathbb{R}^N, |\cdot|)$, $N \geqslant 3$, having a smooth boundary $\partial\Omega$. The symbol $H_0^1(\Omega)$ indicates the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$. On $H_0^1(\Omega)$ we introduce the norm

$$\|u\| := \left(\int_\Omega \left|\nabla u(x)\right|^2 dx\right)^{1/2}.$$

Denote by 2^* the critical exponent for the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. Recall that $2^* = 2N/(N-2)$, if $p \in [1, 2^*]$ then there exists a positive constant c_p such that

$$\|u\|_{L^p(\Omega)} \leqslant c_p\|u\|, \quad u \in H_0^1(\Omega), \tag{38}$$

and, in particular, the embedding is compact whenever $p \in [1, 2^*[$; see, e.g., [20, Proposition B.7].

Given a function $a \in L^\infty(\Omega)$, consider the eigenvalue problem

$$\begin{cases} -\Delta u + a(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{39}$$

It is well known [12, Section 8.12] that (39) possesses a sequence $\{\lambda_n\}$ of eigenvalues fulfilling $\lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$ (the number of times an eigenvalue appears in the sequence equals its multiplicity) and, moreover, that (vide [1, p. 14])

$$\lambda_1 > \operatorname*{ess\,inf}_{x\in\Omega} a(x). \tag{40}$$

Let $\{\phi_n\}$ be a corresponding sequence of eigenfunctions normalized as follows:

$$\int_\Omega \left(\left|\nabla\phi_n(x)\right|^2 + a(x)\phi_n(x)^2\right) dx = \lambda_n \int_\Omega \phi_n(x)^2\, dx = \lambda_n \tag{41}$$

for every $n \in \mathbb{N}$;

$$\int_\Omega \left(\nabla\phi_m(x)\cdot\nabla\phi_n(x) + a(x)\phi_m(x)\phi_n(x)\right) dx = \int_\Omega \phi_m(x)\phi_n(x)\, dx = 0 \tag{42}$$

provided $m, n \in \mathbb{N}$ and $m \neq n$.

To avoid technicalities, we shall examine below only the case when

$$\lambda_s < 0 < \lambda_{s+1} \quad \text{for some } s \in \mathbb{N}. \tag{43}$$

If $g : \mathbb{R} \to \mathbb{R}$ satisfies the conditions:

(g_1) g is measurable,
(g_2) there exist $a_1 > 0$, $p \in]2, 2^*[$ such that $|g(t)| \leqslant a_1(1 + |t|^{p-1})$ for every $t \in \mathbb{R}$,

then the functions $G:\mathbb{R}\to\mathbb{R}$ and $\mathcal{G}:H_0^1(\Omega)\to\mathbb{R}$ given by

$$G(\xi):=\int_0^\xi -g(t)\,dt\quad\forall\xi\in\mathbb{R},\qquad \mathcal{G}(u):=\int_\Omega G\big(u(x)\big)\,dx\quad\forall u\in H_0^1(\Omega),$$

respectively, are well defined and locally Lipschitz continuous. So, it makes sense to consider their generalized directional derivatives G^0 and $\mathcal{G}^0$. On account of [7, formula (9), p. 84] one has

$$\mathcal{G}^0(u;v)\leqslant\int_\Omega G^0\big(u(x);v(x)\big)\,dx,\quad u,v\in H_0^1(\Omega).\tag{44}$$

For our application, we will further assume

(g$_3$) $\lim_{t\to 0}\frac{g(t)}{t}=0$,
(g$_4$) $\limsup_{|t|\to+\infty}\frac{g(t)}{t}<0$, and
(g$_5$) there exists a $\xi_0\in\mathbb{R}$ such that $G(\xi_0)<0$.

Through (g$_4$) one can easily find two positive constants β,γ satisfying

$$g(t)\geqslant-\beta t-\gamma\quad\forall t\leqslant 0,\qquad g(t)\leqslant-\beta t+\gamma\quad\forall t\geqslant 0.\tag{45}$$

Now, let $\lambda,\mu>0$. Define

$$r_{\lambda,\mu}:=\lambda\gamma c_1+\sqrt{(\lambda\gamma c_1)^2+2\mu},\tag{46}$$

with c_1 as in (38) written for $p=1$. A set $K_\lambda\subseteq H_0^1(\Omega)$ is called of type (K$_\lambda^g$) provided

(K$_\lambda^g$) K_λ is convex and closed in $H_0^1(\Omega)$. Moreover, there exists a $\mu>0$ such that $\overline{B}_{r_{\lambda,\mu}}\subseteq K_\lambda$.

Given $\lambda>0$ and K_λ satisfying (K$_\lambda^g$), denote by (P$_\lambda$) the elliptic variational–hemivariational inequality problem:

Find $u\in K_\lambda$ such that

$$-\int_\Omega\nabla u(x)\cdot\nabla(v-u)(x)\,dx-\int_\Omega a(x)u(x)(v-u)(x)\,dx\leqslant\lambda\mathcal{G}^0(u;v-u)$$

for all $v\in K_\lambda$.

Due to (44), any solution u of (P_λ) also fulfils the inequality

$$-\int_\Omega \nabla u(x)\cdot\nabla(v-u)(x)\,dx - \int_\Omega a(x)u(x)(v-u)(x)\,dx$$
$$\leqslant \lambda \int_\Omega G^0\big(u(x);(v-u)(x)\big)\,dx \quad \forall v \in K_\lambda.$$

When g is continuous, while $K_\lambda := H_0^1(\Omega)$, the function $u \in H_0^1(\Omega)$ turns out a weak solution to the Dirichlet problem

$$-\Delta u + a(x)u = \lambda g(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

which has been previously investigated in [5] under more restrictive conditions; see also [15, Theorem 6].

Theorem 4.1. *Suppose* (g_1)–(g_5) *hold true. Then, for every* λ *sufficiently large, problem* (P_λ) *possesses at least two nontrivial solutions.*

Proof. Write $X := H_0^1(\Omega)$ and define, whenever $u \in X$,

$$\Phi(u) := \frac{1}{2}\int_\Omega \big(\big|\nabla u(x)\big|^2 + a(x)u(x)^2\big)\,dx + \lambda\mathcal{G}(u)$$

as well as

$$\psi(u) := \begin{cases} 0 & \text{if } u \in K_\lambda, \\ +\infty & \text{otherwise}, \end{cases} \qquad f(u) := \Phi(u) + \psi(u),$$

where $\lambda > 0$ while $K_\lambda \subseteq H_0^1(\Omega)$ is of type (K_λ^g). Owing to (g_1), (g_2) the function $\Phi : X \to \mathbb{R}$ turns out locally Lipschitz continuous. Consequently, f satisfies condition (H'_f). We shall prove that

$$f \text{ is bounded below and coercive} \quad \text{for any } \lambda > -\frac{\alpha}{\beta}, \tag{47}$$

with $\alpha := \operatorname{ess\,inf}_{x\in\Omega} a(x)$. Fix $\lambda > -\alpha/\beta$. If $u \in X$ then from (45) it follows that

$$\int_{\Omega(u(x)\geqslant 0)} dx \int_0^{u(x)} g(t)\,dt \leqslant \int_{\Omega(u(x)\geqslant 0)} \left(-\frac{\beta}{2}u(x)^2 + \gamma u(x)\right)dx,$$

besides

$$\int_{\Omega(u(x)\leqslant 0)} dx \int_0^{u(x)} g(t)\,dt \leqslant \int_{\Omega(u(x)\leqslant 0)} \int_{u(x)}^{0} (\beta t+\gamma)\,dt$$
$$= \int_{\Omega(u(x)\leqslant 0)} \left(-\frac{\beta}{2}u(x)^2 - \gamma u(x)\right) dx.$$

Gathering these inequalities together yields

$$\int_\Omega dx \int_0^{u(x)} g(t)\,dt \leqslant -\frac{\beta}{2}\|u\|^2_{L^2(\Omega)} + \gamma\|u\|_{L^1(\Omega)},$$

which clearly means

$$\mathcal{G}(u) \geqslant \frac{\beta}{2}\|u\|^2_{L^2(\Omega)} - \gamma\|u\|_{L^1(\Omega)} \quad \forall u \in X. \tag{48}$$

Now, through (38) and (48) we obtain

$$f(u) \geqslant \Phi(u) \geqslant \frac{1}{2}\|u\|^2 + \frac{1}{2}(\alpha+\lambda\beta)\|u\|^2_{L^2(\Omega)} - \lambda\gamma\|u\|_{L^1(\Omega)}$$
$$\geqslant \frac{1}{2}\|u\|^2 + \frac{1}{2}(\alpha+\lambda\beta)\|u\|^2_{L^2(\Omega)} - \lambda\gamma c_1\|u\|,$$

i.e., due to the choice of λ,

$$f(u) \geqslant \frac{1}{2}\|u\|^2 - \lambda\gamma c_1\|u\|, \quad u \in X. \tag{49}$$

Therefore, (47) holds true. Let us next show that the function f satisfies condition $(\mathrm{PS})_f$ provided $\lambda > -\alpha/\beta$. So, pick a sequence $\{u_n\} \subseteq X$ such that $\{f(u_n)\}$ is bounded and

$$\Phi^0(u_n; v-u_n) + \psi(v) - \psi(u_n) \geqslant -\epsilon_n\|v-u_n\| \tag{50}$$

for all $n \in \mathbb{N}$, $v \in X$, where $\epsilon_n \to 0^+$. By (50) one evidently has $\{u_n\} \subseteq K_\lambda$. Since f is coercive, the sequence $\{u_n\}$ turns out bounded. Thus, passing to a subsequence if necessary, we may suppose both $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^2(\Omega)$. The point u belongs to K_λ because this set is weakly closed. Exploiting (50) with $v := u$ we then get

$$\int_\Omega \nabla u_n(x)\cdot\nabla(u-u_n)(x)\,dx + \int_\Omega a(x)u_n(x)(u-u_n)(x)\,dx$$
$$+ \lambda\mathcal{G}^0(u_n; u-u_n) \geqslant -\epsilon_n\|u-u_n\| \quad \forall n \in \mathbb{N}. \tag{51}$$

From $u_n \to u$ in $L^2(\Omega)$ it follows

$$\lim_{n\to+\infty}\int_\Omega a(x)u_n(x)(u-u_n)(x)\,dx = 0. \tag{52}$$

The upper semicontinuity of $\mathcal{G}^0$ on $L^2(\Omega)\times L^2(\Omega)$ forces

$$\limsup_{n\to+\infty}\mathcal{G}^0(u_n; u-u_n) \leqslant \mathcal{G}^0(u;0) = 0. \tag{53}$$

Taking account of (52), (53), besides the weak convergence of $\{u_n\}$ to u, and letting $n \to +\infty$ in (51) yields

$$\limsup_{n\to\infty}\int_\Omega \left|\nabla u_n(x)\right|^2 dx \leqslant \int_\Omega \left|\nabla u(x)\right|^2 dx,$$

namely, by [4, Proposition III.30], $u_n \to u$ in X. Hence, hypothesis (f_1') in Theorem 3.1 is fulfilled.

Through (g_5) we can construct an $u_0 \in X$ such that $\mathcal{G}(u_0) < 0$. Moreover, $u_0 \in \overline{B}_{r_{\lambda,\mu}}$ for any $\lambda \geqslant \frac{1}{2\gamma c_1}\|u_0\|$. Therefore, $\inf_{u\in X} f(u) < 0$ provided

$$\lambda > \max\left\{\frac{1}{2\gamma c_1}\|u_0\|, -\frac{1}{2\mathcal{G}(u_0)}\int_\Omega \left(\left|\nabla u_0(x)\right|^2 + a(x)u_0(x)^2\right)dx\right\},$$

while $f(0) = \lambda\mathcal{G}(0) = 0$.

Our next objective is to verify (f_6). Since K_λ is of type (K_λ^g), the set

$$\left\{u\in X\colon\ f(u) < \mu\right\} \quad \text{is open.} \tag{54}$$

Indeed, inequality (49) ensures that

$$\left\{u\in X\colon\ f(u) < \mu\right\} \subseteq B_{r_{\lambda,\mu}} \subseteq K_\lambda.$$

Consequently,

$$\left\{u\in X\colon\ f(u) < \mu\right\} = \left\{u\in K_\lambda\colon\ \Phi(u) < \mu\right\} = \left\{u\in X\colon\ \Phi(u) < \mu\right\}$$

which leads to (54).

Finally, reasoning as in [2, p. 137] we obtain

$$\lim_{u\to 0}\frac{\mathcal{G}(u)}{\|u\|^2} = 0 \tag{55}$$

while to any $\epsilon > 0$ there corresponds a $\delta \in]0,1[$ such that

$$\mathcal{G}(u) \geqslant -\|u\|^2\left(\frac{\epsilon}{2}c_2^2 + \frac{2a_1c_p^p}{\delta^p}\|u\|^{p-2}\right) \quad \forall u \in X, \tag{56}$$

with c_2, c_p given by (38). Write $X_2 := \mathrm{span}\{\phi_1, \dots, \phi_s\}$ and $X_1 := X_2^{\perp}$, where the orthogonal complement is taken in X. One clearly has $X = X_1 \oplus X_2$, $\dim(X_1) > 0$, besides $0 < \dim(X_2) < +\infty$. Moreover, if $u \in X_2$ then $u = \sum_{i=1}^{s} t_i\phi_i$ for some $t_1, \dots, t_s \in \mathbb{R}$. A simple computation shows that

$$\|u\|^2 \leqslant (\lambda_s - \alpha)\|u\|^2_{L^2(\Omega)}, \quad u \in X_2, \tag{57}$$

with $\lambda_s - \alpha \geqslant \lambda_1 - \alpha > 0$ because of (40). Thanks to (K_λ^g) and (41)–(43) we get

$$f(u) = \Phi(u) = \frac{1}{2}\sum_{i=1}^{s} t_i^2\lambda_i + \lambda\mathcal{G}(u) \leqslant \frac{1}{2}\lambda_s\|u\|^2_{L^2(\Omega)} + \lambda\mathcal{G}(u)$$

whenever $\|u\| \leqslant r_{\lambda,\mu}$. By (55), the above inequality, and (57), for every $\sigma > 0$ there exists a $\rho \in]0, r_{\lambda,\mu}[$ satisfying

$$f(u) \leqslant \left[\frac{\lambda_s}{2} + \lambda\sigma(\lambda_s - \alpha)\right]\|u\|^2_{L^2(\Omega)} \quad \forall u \in \overline{B}_\rho \cap X_2.$$

At this point, choose $\sigma > 0$ so small that $\lambda_s/2 + \lambda\sigma(\lambda_s - \alpha) < 0$, which is possible on account of (43). Bearing in mind (57) we have

$$f(u) < 0 \quad \forall u \in \overline{B}_\rho \cap X_2 \setminus \{0\}. \tag{58}$$

Let us next prove that

$$\int_\Omega \left(\left|\nabla u(x)\right|^2 + a(x)u(x)^2\right)dx \geqslant \theta\|u\|^2 \quad \text{in } X_1 \tag{59}$$

for a suitable constant $\theta > 0$. Indeed, if the assertion were false then there would exist a sequence $\{u_n\} \subseteq X_1$ enjoying the properties

$$\|u_n\| = 1, \quad n \in \mathbb{N}, \tag{60}$$

$$\int_\Omega \left(\left|\nabla u_n(x)\right|^2 + a(x)u_n(x)^2\right)dx < \frac{1}{n} \quad \forall n \in \mathbb{N}. \tag{61}$$

Passing to a subsequence when necessary, we may suppose $u_n \rightharpoonup u$ in X as well as $u_n \to u$ in $L^2(\Omega)$, with $u \in X_1$. Thus, letting $n \to +\infty$ in (61) yields

$$\int_\Omega \left(\left|\nabla u(x)\right|^2 + a(x)u(x)^2\right) dx \leqslant 0. \tag{62}$$

From $u \in X_1$ it follows $u = \sum_{i=s+1}^{+\infty} t_i \phi_i$, where $t_i \in \mathbb{R}$, $i \geqslant s+1$. Through (41)–(43) we obtain

$$\lambda_{s+1} \|u\|^2_{L^2(\Omega)} \leqslant \int_\Omega \left(\left|\nabla u(x)\right|^2 + a(x)u(x)^2\right) dx. \tag{63}$$

Gathering (62) and (63) together leads to $u = 0$. By (61) this forces $u_n \to 0$ in X, against (60). Combining (59) with (56) provides

$$f(u) \geqslant \|u\|^2 \left[\frac{\theta}{2} - \lambda\left(\frac{\varepsilon}{2} c_2^2 + \frac{2a_1 c_p^p}{\delta^p} \|u\|^{p-2}\right)\right] \tag{64}$$

for all $u \in X_1$. Pick $\epsilon > 0$ and $r \in]0, \rho[$ such that

$$\frac{\theta}{2} - \lambda\left(\frac{\epsilon}{2} c_2^2 + \frac{2a_1 c_p^p}{\delta^p} r^{p-2}\right) > 0.$$

Then, thanks to (64) we have

$$f(u) \geqslant 0 \quad \forall u \in \overline{B}_r \cap X_1. \tag{65}$$

Finally, taking account of Remark 3.1, (58) and (65) immediately yield condition (f_7).

We are now in a position to apply Theorem 3.1. So, there exist at least two points $u_1, u_2 \in X \setminus \{0\}$ such that

$$\Phi^0(u_i; v - u_i) + \psi(v) - \psi(u_i) \geqslant 0$$

for all $v \in X$, $i = 1, 2$. The choice of ψ gives both $u_i \in K_\lambda$ and $\Phi^0(u_i; v - u_i) \geqslant 0$, $v \in K_\lambda$, $i = 1, 2$, namely u_1, u_2 turn out nontrivial solutions to problem (P_λ), which completes the proof. □

Remark 4.1. Reading the above arguments we realize that the conclusion of Theorem 4.1 holds true as soon as

$$\lambda > \max\left\{-\frac{\alpha}{\beta}, \frac{1}{2\gamma c_1}\|u_0\|, -\frac{1}{2\mathcal{G}(u_0)} \int_\Omega \left(\left|\nabla u_0(x)\right|^2 + a(x)u_0(x)^2\right) dx\right\},$$

where $\alpha := \operatorname{ess\,inf}_{x \in \Omega} a(x)$, β and γ are given by (45), c_1 comes from (38) written for $p = 1$, while $u_0 \in X$ fulfils $\mathcal{G}(u_0) < 0$.

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