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Multiple periodic solutions for second order systems with changing sign potential

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Abstract

This paper deals with the multiplicity of solutions of a second order nonautonomous system. We extend a previous result of the author relaxing the assumptions on the sign of the potential.

Keywords: Periodic solutions; Second order nonautonomous system; Variational methods

1. Introduction

In the present paper we consider the following second order nonautonomous system:

$$(S)\qquad \begin{cases} \ddot{u} - A(t)u = \nabla_u F(t,u) & \text{a.e. in } [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

where $A(t)$ is a $N \times N$ positive definite matrix, $F(t,u) : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is measurable in t and continuously differentiable in u.

We extend a multiplicity result obtained in [4] where we proved the existence of at least three periodic solutions for system (S) when $F(t,u) = b(t)V(u)$ with b nonnegative in

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$[0, T]$. In the present paper we allow the potential F to have a more general expression, and when $F(t,u) = b(t)V(u)$, we do not require any sign condition on b.

It is worth to mention some recent results related to the topic.

The existence of at least three solutions for the problem

$$(S^\star)\quad \begin{cases} \ddot{u} = \nabla_u G(t,u) & \text{a.e. in } [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

was already studied in [2,10–12]. As already noticed in [4], in these papers the main assumption, first introduced by Brezis and Nirenberg is:

(BN) there exist $r > 0$ and an integer $k \geqslant 0$ such that

$$-\frac{1}{2}(k+1)^2 w^2 |u|^2 \leqslant G(t,u) - G(t,0) \leqslant -\frac{1}{2}k^2 w^2 |u|^2$$

for all $|u| \leqslant r$, a.e. in $[0, T]$, where $w = \frac{2\pi}{T}$.

In [9] the author proves, for the problem with a nonnegative parameter λ,

$$(S_\lambda)\quad \begin{cases} \ddot{u} - A(t)u = \lambda \nabla_u F(t,u) & \text{a.e. in } [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases}$$

the existence of three solutions assuming, among the other hypotheses, that

(Sh) there exists $\gamma \in]0, T[$ such that $F(t,u) \leqslant 0$ for all $(t,u) \in [\gamma, T] \times \mathbb{R}^N$.

We mention finally another interesting result on the topic recently obtained by Cordaro in [3] where the author proves the existence of at least three periodic solutions for system (S_λ). We notice that in the previous results it is not known whether λ can be taken equal to 1.

Our aim is to provide a new contribution to the subject, under a set of hypotheses rather different to those of the quoted papers.

Our approach is variational and it is similar to the one used in [4]: the existence of three periodic solutions is proved by applying a suitable version of a local minimum principle by B. Ricceri [8] and a well-known three critical points theorem by Pucci and Serrin [7]. In the next section we describe the variational setting of the problem, while Section 3 is devoted to the proof of our results. Finally in the last section we present examples and comparison with the results cited above.

2. The variational setting

Throughout the sequel T is a positive number, $A : [0, T] \to \mathbb{R}^{N\times N}$ is a symmetric matrix valued function with bounded coefficients a_{ij} and $\|A\| = \sum_{i,j} \|a_{ij}\|_\infty$, $F(t,u) : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is measurable in t for all $u \in \mathbb{R}^N$ and continuously differentiable in u a.e. in $[0, T]$.

Let suppose that A is positive definite, i.e., there exists a positive constant α such that

$$A(t)u \cdot u \geqslant \alpha |u|^2$$

for every $u \in \mathbb{R}^N$ and a.e. in $[0, T]$.

Let us recall that a solution of (S) is a function $u \in C^1([0,T], \mathbb{R}^N)$ with $\dot{u}$ absolutely continuous, such that

$$\begin{cases} \ddot{u}(t) - A(t)u(t) = \nabla_u F(t,u) & \text{a.e. in } [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases}$$

That is, introduced the Sobolev space $\mathbb{H}^1_T$ of the functions $u \in \mathbb{L}^2([0,T], \mathbb{R}^N)$ having a weak derivative $\dot{u} \in \mathbb{L}^2([0,T], \mathbb{R}^N)$ and such that $u(0) = u(T)$ (see [6]), we are looking for functions $u \in \mathbb{H}^1_T$ such that

$$\int_0^T \dot{u}(t) \cdot \dot{v}(t)\, dt + \int_0^T A(t)u(t) \cdot v(t)\, dt + \int_0^T \nabla_u F\big(t, \big(u(t)\big) \cdot v(t)\big)\, dt = 0$$

for all $v \in \mathbb{H}^1_T$.

Consider $\mathbb{H}^1_T$ equipped with the norm

$$\|u\| = \left(\int_0^T \big|\dot{u}(t)\big|^2\, dt + \int_0^T A(t)u(t) \cdot u(t)\, dt \right)^{1/2}$$

that is equivalent to the usual one, that is

$$\|u\|_* = \left(\int_0^T \big|\dot{u}(t)\big|^2\, dt + \int_0^T \big|u(t)\big|^2\, dt \right)^{1/2}.$$

It is well known that $\mathbb{H}^1_T$, endowed with the norm $\|\cdot\|_*$, is compactly embedded in $C^0([0,T], \mathbb{R}^N)$ and so, since the norms $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent, the constant

$$c = \sup_{u \in \mathbb{H}^1_T \setminus \{0\}} \frac{\|u\|_{C^0}}{\|u\|}$$

is finite.

Our main tool is a variational principle of B. Ricceri [8] that can be stated as follows:

Theorem R. [8, Theorem 2.5] *Let X be a Hilbert space, $\Phi, \Psi : X \to \mathbb{R}$ two sequentially weakly lower semicontinuous functionals. Assume that Ψ is strongly continuous and coercive. For each $\rho > \inf_X \Psi$, set*

$$\varphi(\rho) := \inf_{\Psi^\rho} \frac{\Phi(u) - \inf_{\mathrm{cl}_w \Psi^\rho} \Phi}{\rho - \Psi(u)}, \tag{1}$$

where $\Psi^\rho := \{u \in X\colon \Psi(u) < \rho\}$ and $\mathrm{cl}_w \Psi^\rho$ is the closure of Ψ^ρ in the weak topology of X. Then, for each $\rho > \inf_X \Psi$ and each $\mu > \varphi(\rho)$, the restriction of the functional $\Phi + \mu\Psi$ to Ψ^ρ has a global minimum point in Ψ^ρ.

Theorem R implies in particular that if there exists $\rho > 0$, such that $\varphi(\rho) < \frac{1}{2}$, then the functional $J = \frac{1}{2}\Psi + \Phi$ has a local minimum point in Ψ^{ρ}.

Throughout the sequel we make the following assumptions on the potential:

(A) there exist functions $a, \bar{a} \in C^0(\mathbb{R}_0^+, \mathbb{R}_0^+)$ and $b, \bar{b} \in \mathbb{L}^1([0,T], \mathbb{R}_0^+)$ such that

$$\left|F(t,u)\right| \leqslant a\big(|u|\big)b(t), \qquad \left|\nabla_u F(t,u)\right| \leqslant \bar{a}\big(|u|\big)\bar{b}(t)$$

for all $u \in \mathbb{R}^N$ and a.e. in $[0,T]$.

Define on the space $\mathbb{H}_T^1$ the functionals

$$\Psi(u) = \|u\|^2 \quad \text{and} \quad \Phi(u) = \int_0^T F(t,u)\,dt.$$

Lemma 1. *Assume* (A). *Then, the functional* Φ *is well defined on* $\mathbb{H}_T^1$ *and sequentially weakly continuous. Moreover, it is Gâteaux differentiable and its derivative is given by*

$$\Phi'(u)v = \int_0^T \nabla_u F\big(t, u(t)\big) \cdot v(t)\,dt \quad \textit{for all } v \in \mathbb{H}_T^1.$$

We deduce that any critical point of J is a solution of (S).

3. Results

Theorem 2. *Assume* (A) *and*

(A_1) *there exist* $\rho > 0$ *and* $v_0 \in \mathbb{R}^N$ *such that*
(i) $\max_{[0,c\rho]} \bar{a} < \frac{\rho}{c\|\bar{b}\|_1}$;
(ii) $\int_0^T F(t, v_0)\,dt < -\{(\max_{[0,c\rho]} a)\|b\|_1 + \frac{1}{2}|v_0|^2 T \|A\|\}$;

(A_2) *there exist* $M > 0$ *and a function* $B \in \mathbb{L}^1([0,T], \mathbb{R}_0^+)$ *with* $\|B\|_1 < \frac{1}{2c^2}$ *such that*

$$F(t,u) \geqslant -B(t)|u|^2 \quad \textit{for all } u\colon |u| \geqslant M \textit{ and a.e. in } [0,T].$$

Then, system (S) *has at least three solutions.*

Proof. *Step 1. Existence of a local minimum for* J. We are going to apply Theorem R to the functionals Ψ and Φ introduced in the previous section. Rewriting (1), we deduce that, if there exists $\rho > 0$ such that

$$\varphi\big(\rho^2\big) = \inf_{\Psi^{\rho^2}} \frac{\Phi(u) - \inf_{\mathrm{cl}_w \Psi^{\rho^2}} \Phi}{\rho^2 - \Psi(u)} = \inf_{\|u\|<\rho} \frac{\Phi(u) - \inf_{\|u\| \leqslant \rho} \Phi}{\rho^2 - \|u\|^2} < \frac{1}{2}, \tag{2}$$

then the energy functional $J(u) = \frac{1}{2}\|u\|^2 + \Phi(u)$ has a global minimum in $\mathbb{H}_T^1$ whose norm is less than ρ. For $\rho > 0$ define

$$\phi(\rho) := \inf_{\|v\| \leqslant \rho} \int_0^T F\big(t, v(t)\big)\, dt,$$

that is well defined and not increasing. It is easy to prove that (2) is equivalent to

$$\inf_{\rho>0} \inf_{\sigma<\rho} \frac{\phi(\sigma) - \phi(\rho)}{\rho^2 - \sigma^2} < \frac{1}{2},$$

which is fulfilled if there exists $\rho > 0$ such that

$$\liminf_{\tau \to 0+} \frac{\phi(\rho) - \phi(\rho + \tau)}{\tau} < \rho. \tag{3}$$

We are going to estimate the left-hand side of (3). As in [1], if $\rho > 0$, $0 < \tau < \rho$ then by using (A), we obtain

$$\begin{aligned}
\frac{\phi(\rho) - \phi(\rho+\tau)}{\tau} &= \frac{1}{\tau}\Bigg| \inf_{\|v\|\leqslant\rho} \int_0^T \Bigg[\int_0^1 \nabla_u F\big(t, sv(t)\big)\cdot v(t)\, ds + F(t,0)\Bigg] dt \\
&\qquad - \inf_{\|v\|\leqslant\rho+\tau} \int_0^T \Bigg[\int_0^1 \nabla_u F\big(t, sv(t)\big)\cdot v(t) ds + F(t,0)\Bigg] dt \Bigg| \\
&= \frac{1}{\tau}\Bigg| \inf_{\|v\|\leqslant 1} \int_0^T \int_0^\rho \nabla_u F\big(t, sv(t)\big)\cdot v(t)\, ds\, dt \\
&\qquad - \inf_{\|v\|\leqslant 1} \int_0^T \int_0^{\rho+\tau} \nabla_u F\big(t, sv(t)\big)\cdot v(t)\, ds\, dt \Bigg| \\
&\leqslant \frac{1}{\tau} \sup_{\|v\|\leqslant 1} \Bigg| \int_0^T \int_{\rho+\tau}^\rho \nabla_u F\big(t, sv(t)\big)\cdot v(t)\, ds\, dt \Bigg| \\
&\leqslant \frac{1}{\tau} \sup_{\|v\|\leqslant 1} \int_0^T \int_\rho^{\rho+\tau} \big|\nabla_u F\big(t, sv(t)\big)\big| \big|v(t)\big|\, ds\, dt \\
&\leqslant \frac{1}{\tau} \sup_{\|v\|\leqslant 1} \int_0^T \int_\rho^{\rho+\tau} \bar{a}\big(\big|sv(t)\big|\big)\bar{b}(t)\big|v(t)\big|\, ds\, dt \\
&\leqslant \Big(\max_{[0,(\rho+\tau)c]} \bar{a}\Big) \sup_{\|v\|\leqslant 1} \int_0^T \bar{b}(t)\big|v(t)\big|\, dt \leqslant \Big(\max_{[0,(\rho+\tau)c]} \bar{a}\Big) c\|\bar{b}\|_1.
\end{aligned}$$

Therefore

$$\liminf_{\tau\to 0+}\frac{\phi(\rho)-\phi(\rho+\tau)}{\tau}\leqslant\Bigl(\max_{[0,\rho c]}\bar{a}\Bigr)c\|\bar{b}\|_1,$$

since by the continuity of $\bar{a}$

$$\lim_{\tau\to 0+}\max_{[0,(\rho+\tau)c]}\bar{a}=\max_{[0,\rho c]}\bar{a}.$$

Therefore J has a local minimum $u_0\in\mathbb{H}^1_T$ such that $\|u_0\|<\rho$, provided that (A_1)(i) holds.

Step 2. Existence of a global minimum for J. Following the arguments of [9], with mild modifications, it is possible to prove that (A_2) implies the coercivity of J. Due to the weakly lower sequential semicontinuity of Ψ and Φ, the functional J has a global minimum, let us say u_1.

We claim that the global minimum is different to the local minimum. We have the following estimate of J on the ball centered at zero of radius ρ:

$$\frac{1}{2}\|u\|^2+\Phi(u)\geqslant\int_0^T F\bigl(t,u(t)\bigr)\,dt\geqslant-\int_0^T a\bigl(\bigl|u(t)\bigr|\bigr)b(t)\,dt\geqslant-\Bigl(\max_{[0,c\rho]}a\Bigr)\|b\|_1.$$

From assumption (A_1)(ii) if $w_0(t)=v_0$ for all $t\in[0,T]$, $w_0\in\mathbb{H}^1_T$ and

$$\frac{1}{2}\|w_0\|^2+\Phi(w_0)=\frac{1}{2}\int_0^T A(t)v_0\cdot v_0\,dt+\int_0^T F(t,v_0)\,dt$$
$$\leqslant\frac{1}{2}T\|A\||v_0|^2+\int_0^T F(t,v_0)\,dt<-\Bigl(\max_{[0,c\rho]}a\Bigr)\|b\|_1.$$

Hence, the global minimum is outside the ball of radius ρ, so it is different to u_0.

Step 3. Existence of a third critical point of J. A third solution of (S) is obtained applying a well-known result due to Pucci and Serrin (see [7]): it is easily seen that Φ' is compact and that Ψ' admits a continuous inverse on $(\mathbb{H}^1_T)^\star$. Therefore, by Proposition 38.25 of [13], we deduce that $\Psi/2+\Phi$ has the Palais–Smale property.

Our conclusion follows by Corollary 1 of [7] which proves our claim. □

In the next theorem we do not require the coercivity of the functional J.

Theorem 3. *Assume* (A) *and*

(A_1)(i) *there exists* $\rho>0$ *such that* $\max_{[0,c\rho]}\bar{a}<\frac{\rho}{c\|\bar{b}\|_1}$;
(A_3) *there exist* $q>2$ *and* $R_0>0$ *such that*

$$\nabla_u F(t,u)\cdot u\leqslant qF(t,u)<0\quad\textit{for all } u\colon\ |u|\geqslant R_0 \textit{ and a.e. in } [0,T].$$

Then, system (S) *has at least two solutions.*

Proof. A first solution u_0 with norm less than ρ is obtained as in Theorem 2.

In a standard way it is possible to prove that J has the Palais–Smale property, as it follows from (A_3).

J is unbounded from below. We have

$$u|u|^q \frac{\partial}{\partial u}\big(|u|^{-q}F(t,u)\big) = -qF(t,u) + \nabla_u F(t,u)\cdot u \leqslant 0 \quad \text{a.e. in } [0,T],\ \text{all } |u| \geqslant R_0;$$

and so, for all $|u| \geqslant R_0$,

$$F(t,u) \leqslant -R_0^{-q} \min\big\{-F(t,-R_0), -F(t,R_0)\big\}|u|^q = -k(t)|u|^q,$$

with $k(t) \equiv R_0^{-q} \min\{-F(t,-R_0), -F(t,R_0)\} > 0$.

Now, the function k belongs to the space $\mathbb{L}^1([0,T])$ and it is positive a.e. in $[0,T]$ on the strength of (A_3).

If $|u| \leqslant R_0$, $F(t,u) \leqslant (\max_{[0,R_0]} a)b(t)$ a.e. in $[0,T]$, so we obtain that

$$F(t,u) \leqslant -R_0^{-q} k(t)|u|^q + \Big(\max_{[0,R_0]} a\Big)b(t) \quad \text{for all } u \text{ and a.e. in } [0,T].$$

Therefore if $\tau > 0$, and $u \in \mathbb{H}_T^1$,

$$J(\tau u) = \frac{1}{2}\tau^2|u|^2 - R_0^{-q}\tau^q \int_0^T k(t)|u|^q\,dt + \Big(\max_{[0,R_0]} a\Big)\|b\|_1$$

that goes to $-\infty$ as τ tends to $+\infty$.

Our conclusion follows by Theorem 1 of [7]: there exists $u_1 \in \mathbb{H}_T^1$ different to u_0, that is a critical point of J, i.e. a second solution of (S). □

4. Examples and remarks

In Theorem 2, we propose a new set of conditions under which the existence of three solutions for (S) is ensured. We underline that without the main assumption (A_1) the result does not hold as the following example shows:

Example 4. Let $N = 1$, $T = 1$, $A = a_{11} = 1$, and $q \in\,]1,2[$.

Then, the problem

$$\begin{cases} u'' - u = q|u|^{q-2}u & \text{in } [0,1],\\ u(0) - u(1) = u'(0) - u'(1) = 0 \end{cases}$$

has only the trivial solution.

We notice that $F(t,u) = |u|^q$ is nonnegative and so it cannot satisfy (A_1)(ii). It is easy to see that the problem admits only the null solution. Indeed, if there exists a nontrivial solution $u(t)$, then $\max_{[0,1]} u > 0$ or $\min_{[0,1]} u < 0$. Now, if $u(t^\star) = \max_{[0,1]} u > 0$, for some $t^\star \in\,]0,1[$, then $u''(t^\star) < 0$ although $u(t^\star) + q|u(t^\star)|^{q-2}u(t^\star) > 0$. Analogously we have a contradiction if we assume that $\min_{[0,1]} u < 0$.

Remark 5. At the present stage we do not know whether Theorem 2 still holds without assuming (A_2).

Let us present an example of potential F satisfying the assumptions of Theorem 2.

Example 6. Let $\beta \in \mathbb{L}^1([0,T],\mathbb{R})$ such that

$$\int_0^T \beta(t)\,dt \neq 0, \qquad \|\beta\|_1 < \frac{m}{4}\left(\max\left\{\sqrt{T}, \frac{1}{\sqrt{T}}\right\}\right)^{-2},$$

where $m = \min\{1, \alpha\}$.

If $\int_0^T \beta(t)\,dt > 0$, choose $\bar{n}$ such that

$$\left(\frac{3}{2}\pi + 2\bar{n}\pi\right)\left(-\left(\frac{3}{2}\pi + 2\bar{n}\pi\right)^{1/2}\int_0^T \beta(t)\,dt + \frac{1}{2}T\|A\|\right) < -\|\beta\|_1. \tag{4}$$

If $\int_0^T \beta(t)\,dt < 0$, choose $\bar{n}$ such that

$$\left(\frac{\pi}{2} + 2\bar{n}\pi\right)\left(\left(\frac{\pi}{2} + 2\bar{n}\pi\right)^{1/2}\int_0^T \beta(t)\,dt + \frac{1}{2}T\|A\|\right) < -\|\beta\|_1. \tag{5}$$

Define $M^2 = \frac{\pi}{2} + 2(\bar{n}+1)\pi$ and

$$\gamma(u) = \begin{cases} |u|^3 \sin|u|^2 & \text{if } |u| \leqslant M, \\ M^3 + 3M^2(|u| - M) & \text{if } |u| > M. \end{cases}$$

Then the problem

$$\begin{cases} \ddot{u} - A(t)u = \beta(t)\nabla\gamma(u) & \text{a.e. in } [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases}$$

has at least three solutions in $\mathbb{H}^1_T$.

Let us notice that condition (A) is satisfied with $b(t) = \bar{b}(t) = |\beta(t)|$,

$$a(|u|) = \begin{cases} |u|^3 & \text{if } |u| \leqslant M, \\ M^3 + 3M^2(|u| - M) & \text{if } |u| > M \end{cases}$$

and

$$\bar{a}(|u|) = \begin{cases} 3|u|^2|\sin|u|^2| + 2|u|^4|\cos|u|^2| & \text{if } |u| \leqslant M, \\ 3M^2 & \text{if } |u| > M \end{cases}$$

since γ is continuously differentiable with

$$\nabla\gamma(u) = \begin{cases} 3u|u|\sin|u|^2 + 2u|u|^3\cos|u|^2 & \text{if } |u| \leqslant M, \\ 3M^2\frac{u}{|u|} & \text{if } |u| > M. \end{cases}$$

Condition (A_1)(i) holds with $\rho < \min\{\frac{1}{5\|b\|_1 c^3}, \frac{1}{c}\}$. Let us verify (A_1)(ii): let $v_0 \in \mathbb{R}^N$ such that

$$|v_0|^2 = \begin{cases} \frac{3}{2}\pi + 2\bar{n}\pi & \text{if } \int_0^T \beta(t)\,dt > 0, \\ \frac{\pi}{2} + 2\bar{n}\pi & \text{if } \int_0^T \beta(t)\,dt < 0, \end{cases}$$

where $\bar{n}$ is the integer chosen in (4), (5) according to the sign of $\int_0^T \beta(t)\,dt$. We have that $|v_0|^2 \leqslant M^2$ by the definition of M.

If $\int_0^T \beta(t)\,dt > 0$,

$$\int_0^T F(t, v_0)\,dt = -|v_0|^3 \int_0^T \beta(t)\,dt,$$

and by the assumptions,

$$|v_0|^2 \left(-|v_0| \int_0^T \beta(t)\,dt + \frac{1}{2} T \|A\| \right) < -\|b\|_1.$$

If $\int_0^T \beta(t)\,dt < 0$,

$$\int_0^T F(t, v_0)\,dt = |v_0|^3 \int_0^T \beta(t)\,dt,$$

and by the assumptions,

$$|v_0|^2 \left(|v_0| \int_0^T \beta(t)\,dt + \frac{1}{2} T \|A\| \right) < -\|b\|_1.$$

Let us verify now condition (A_2). If $|u| \geqslant M$,

$$\frac{F(t,u)}{|u|^2} = \beta(t) \left[\frac{M^3 + 3M^2(|u| - M)}{|u|^2} \right]$$

and

$$\lim_{|u| \to \infty} \frac{M^3 + 3M^2(|u| - M)}{|u|^2} = 0.$$

Hence, there exists a $\delta > 0$ such that $|\frac{M^3+3M^2(|u|-M)}{|u|^2}| \leqslant 1$, for $|u| \geqslant \delta$. Therefore,

$$\frac{F(t,u)}{|u|^2} \geqslant -\bigl|\beta(t)\bigr| \quad \text{a.e. in } [0, T] \text{ and for all } |u| \geqslant \max\{\delta, M\}.$$

(A_2) follows by the estimate

$$c \leqslant \sqrt{\frac{2}{m}} \max\left\{ \sqrt{T}, \frac{1}{\sqrt{T}} \right\}$$

(see [5]). All the assumptions of Theorem 2 are satisfied and our claim follows.

Remark 7. Example 6 shows that the sign condition (Sh) in [9], does not apply to our case. Indeed if $|u| \leqslant M$, $F(t,u) = \beta(t)|u|^3 \sin|u|^2$ changes sign for every t.

Remark 8. When the nonlinearity is $G(t,u) = \frac{1}{2}A(t)u \cdot u + F(t,u)$, the two problems $(S^\star)$ and (S) are equal and condition (BN) reads as follows:

$$-\frac{1}{2}(k+1)^2 w^2 |u|^2 \leqslant \frac{1}{2}A(t)u \cdot u + F(t,u) - F(t,0) \leqslant -\frac{1}{2}k^2 w^2 |u|^2.$$

We notice that the function F in Example 6 does not satisfy condition (BN):

$$\frac{1}{2}A(t)u \cdot u + F(t,u) - F(t,0) = \frac{1}{2}A(t)u \cdot u + \beta(t)\big[|u|^3 \sin|u|^2\big]$$

and so

$$\frac{1}{2|u|^2}A(t)u \cdot u + \frac{F(t,u) - F(t,0)}{|u|^2} \geqslant \frac{1}{2}\alpha + \beta(t)\big[|u| \sin|u|^2\big].$$

The right-hand side goes to $\frac{1}{2}\alpha > 0$ as $|u|$ tends to zero, hence it cannot be bounded from above by a negative constant.

Let us present now an example of potential satisfying the assumptions of Theorem 3.

Example 9. Let $\beta \in \mathbb{L}^1([0,T], \mathbb{R}^+)$, $x_0 \in \mathbb{R}^N$.

The problem

$$\begin{cases} \ddot{u} - A(t)u = \beta(t)\nabla[-|u|^4 + e^{-|u+x_0|^2|u|^3}] & \text{a.e. in } [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases}$$

has at least two solutions in $\mathbb{H}^1_T$.

Put

$$F(t,u) = \beta(t)\big[-|u|^4 + e^{-|u+x_0|^2|u|^3}\big].$$

Let us prove that the function F satisfies all the assumptions of Theorem 3.

Differentiating $F(t,u)$ with respect to u, we obtain

$$\nabla_u F(t,u) = \beta(t)\big[-4u|u|^2 - \big(2(u+x_0)|u|^3 + 3|u+x_0|^2 u|u|\big)e^{-|u+x_0|^2|u|^2}\big]$$

and condition (A) is easily obtained taking $b(t) = \bar{b}(t) = \beta(t)$ for all $t \in [0,T]$,

$$a\big(|u|\big) = \big|-|u|^4 + 1\big| \quad \text{for all } u \in \mathbb{R}^N$$

and

$$\bar{a}\big(|u|\big) = 5|u|^4 + 2\big(2 + 4|x_0|\big)|u|^3 + 3|x_0|^2|u|^2 \quad \text{for all } u \in \mathbb{R}^N.$$

It is easily seen that the maximum of $\bar{a}$ in $[0, c\rho]$ is attained in $c\rho$. Hence, for ρ sufficiently small condition (A_1)(i) is satisfied.

It is immediately seen that $F(t,u)$ is negative for $|u|$ big enough, i.e. there exists a positive constant R_1 such that $F(t,u) < 0$ for a.e. $[0,T]$ and all $|u| \geqslant R_1$.

Moreover,

$$\nabla_u F(t,u)\cdot u = b(t)\big[-4|u|^4 - \big(2(u+x_0)\cdot u|u|^3 + 3|u+x_0|^2|u|^2\big)e^{-|u+x_0|^2|u|^2}\big].$$

So,

$$\begin{aligned}&\nabla_u F(t,u)\cdot u \leqslant 4F(t,u)\\ &\quad\text{iff}\quad -\big(2(u+x_0)\cdot u|u|^3 + 3|u+x_0|^2|u|^2\big)e^{-|u+x_0|^2|u|^2} \leqslant 4e^{-|u+x_0|^2|u|^3},\end{aligned}$$

that is equivalent to

$$4 + |u|^3\big(5|u|^2 + 3|x_0|^2 - 8x_0\cdot u\big) \geqslant 0,$$

that holds for $|u| \geqslant R_2$, for some $R_2 > 0$. Taking $R_0 = \max\{R_1, R_2\}$, we have (A_3).

Remark 10. Example 9 shows that condition (BN) does not apply to our case. Indeed in the previous example we have

$$\frac{1}{2}A(t)u\cdot u + F(t,u) - F(t,0) = \frac{1}{2}A(t)u\cdot u + \beta(t)\big[-|u|^4 + e^{-|u+x_0||u|^3} - 1\big],$$

and

$$\begin{aligned}&\frac{1}{|u|^2}\left[\frac{1}{2}A(t)u\cdot u + F(t,u) - F(t,0)\right]\\ &\quad= \frac{1}{2|u|^2}A(t)u\cdot u + \beta(t)\left[-|u|^2 + \frac{e^{-|u+x_0||u|^3} - 1}{|u|^2}\right]\\ &\quad\geqslant \frac{1}{2}\alpha + \beta(t)\left[-|u|^2 + \frac{e^{-|u+x_0||u|^3} - 1}{|u|^2}\right]\end{aligned}$$

that tends to $\frac{1}{2}\alpha > 0$ as $|u|$ tends to zero.

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