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Variations of selective separability

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ABSTRACT

A space X is selectively separable if for every sequence $(D_n: n \in \omega)$ of dense subspaces of X one can select finite $F_n \subset D_n$ so that $\bigcup\{F_n: n \in \omega\}$ is dense in X. In this paper selective separability and variations of this property are considered in two special cases: C_p spaces and dense countable subspaces in 2^κ.

1. Introduction

In this paper we consider some properties that are stronger than separability in the way similar to how the covering properties of Menger, Hurevicz and Rothberger are stronger than the Lindelöf property.

Let X be a topological space, and let $\mathcal{D}$ denote the family of all dense subspaces of X. In [25] Scheepers considers the following selection principles:

$X \models S_{\mathrm{fin}}(\mathcal{D},\mathcal{D})$: for every sequence $(D_n: n \in \omega)$ of elements of $\mathcal{D}$, one can pick finite $F_n \subset D_n$ so that $\bigcup\{F_n: n \in \omega\} \in \mathcal{D}$.

$X \models S_1(\mathcal{D},\mathcal{D})$: for every sequence $(D_n: n \in \omega)$ of elements of $\mathcal{D}$, one can pick $p_n \in D_n$ so that $\{p_n: n \in \omega\} \in \mathcal{D}$.

In [3], selection principle $S_{\mathrm{fin}}(\mathcal{D},\mathcal{D})$ was called *selective separability*. We call spaces X satisfying $S_{\mathrm{fin}}(\mathcal{D},\mathcal{D})$ or $S_1(\mathcal{D},\mathcal{D})$ *M-separable* and *R-separable*, respectively. Also, we say that X is *H-separable* if for every sequence $(D_n: n \in \omega)$ of elements of $\mathcal{D}$, one can pick finite $F_n \subset D_n$ so that for every nonempty open set $O \subset X$, the intersection $O \cap F_n$ is nonempty for all

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but finitely many n. In Section 6, we consider one more variation of selective separability, called *GN-separability*. We are not giving the definition here because it is more technical.

Naturally, "M-", "R-", and "H-", are motivated by analogy with well-known Menger, Rothberger, and Hurevicz properties. (Recall that X is *Menger* if for every sequence $(\mathcal{O}_n\colon n \in \omega)$ of open covers, one can pick finite $\mathcal{F}_n \subset \mathcal{O}_n$ so that $\bigcup\{\mathcal{F}_n\colon n \in \omega\}$ covers X; X is *Rothberger* if for every sequence $(\mathcal{O}_n\colon n \in \omega)$ of open covers, one can pick $O_n \in \mathcal{O}_n$ so that $\{O_n\colon n \in \omega\}$ covers X; X is *Hurevicz* if for every sequence $(\mathcal{O}_n\colon n \in \omega)$ of open covers, one can pick finite $\mathcal{F}_n \subset \mathcal{O}_n$ so that for every $x \in X$, $x \in \bigcup \mathcal{F}_n$ for all but finitely many n.)

The following implications are obvious:

$$\begin{array}{ccccc} & \swarrow & \text{R-separable} & \nwarrow & \\ \text{separable} \leftarrow \text{M-separable} & & & & \text{countable } \pi\text{-weight} \\ & \nwarrow & \text{H-separable} & \swarrow & \end{array}$$

On the other hand, having a countable network does not imply selective separability since not all countable spaces are selectively separable. For compact spaces, M-, R-, and H-separability are equivalent to each other and to having a countable π-base. It was noted in [3], that for M-separability this follows from the equality $\pi w(X) = \delta(X)$ which holds for compact X [10] (here, $\delta(X) = \sup\{d(Y)\colon Y \text{ is dense in } X\}$ [26]; $\delta(X) = \omega$ for every M-separable space X).

M-, R-, or H-separability are not preserved by arbitrary continuous mappings (moreover, it is easy to see that every separable space can be represented as a retract of a space having a dense countable subspace consisting of isolated points; such a space of course has a countable π-base; alternatively, it is easy to see that every separable topology is contained in a stronger topology having a dense countable subspace consisting of isolated points), but these properties are preserved by continuous mappings f such that $f(U)$ has nonempty interior for every nonempty open set U, hence in particular they are preserved by continuous open mappings, and by continuous closed irreducible mappings (see [3] for M-separability). M-, R-, or H-separability are not preserved by arbitrary subspaces, but they are preserved by open subspaces, and by dense subspaces (see [3] for M-separability). Surprisingly, it remains an open question whether or not M-separability is preserved by finite unions or by finite products of spaces [3]. An example of a Hausdorff space X such that $\delta(X) = \omega$ but $\delta(X^2) > \omega$ is given in [26]; however, one can check that this example is not M-separable.

M-separability and variations of this property were considered in the literature in many aspects, for example: topological games [25,4,7], C_p-theory [25,3], irresolvability, maximal topologies [25,3], spaces of subsets [14,5]. Here we mention just several interesting results and leave aside statements in terms of game theory.

Theorem 1. *([25], Theorem 19) Every HFD space is R-separable.*

Theorem 2. *([25], Theorem 13) For a separable metrizable space X, the following conditions are equivalent*:

(1) *All finite powers of X are Rothberger.*
(2) *$C_p(X)$ is R-separable.*

Theorem 3. *([3], Theorem 2.9) A space $C_p(X)$ is M-separable iff it is separable and has countable fan tightness.*

Theorem 4. *([14], Theorem 4) For a space X, the following conditions are equivalent*:

(1) *2^X equipped with upper Fell topology is R-separable.*
(2) *X satisfies $S_1(\mathcal{K}, \mathcal{K})$.*

($\mathcal{K}$ is the family of all k-covers of X, see [15].)

Theorem 5. *([3], Theorem 2.25) If $\mathfrak{d} = \omega_1$, then there is a maximal regular countable space which is not M-separable.*

In this paper, we consider these properties in two specific situations: (1) dense countable subspaces in 2^κ, and (2) C_p-spaces. The choice of the first topic is clear: all countable spaces are obviously separable, and so when considering a stronger form of separability, it is natural to examine countable spaces; dense countable subspaces in 2^κ seem to be one of the first natural choices for such an examination. The choice of the second topic is motivated by Theorems 2 and 3, and in general by duality between covering properties of X and tightness-type properties of $C_p(X)$ [2] (selective separability is in some sense a global version of tightness-type properties). There is a relationship between the two topics; thus, if K is a zero-dimensional metrizable compact space, then $C_p(K, 2)$ is a dense countable subspace in 2^K.

2. Terminology and preliminaries

In terminology, we in general follow [8]. We assume all spaces to be Tychonoff even if some statements below are valid under weaker separation assumptions.

A family $\mathcal{P}$ of open sets in X is called a π-base for X if every nonempty open set in X contains a nonempty element of $\mathcal{P}$; $\pi w(X) = \min\{|\mathcal{P}|\colon \mathcal{P}$ is a π-base for $X\}$ is the π-weight of X.

The i-weight of $(X, \mathcal{T})$ is $iw((X, \mathcal{T}) = \min\{\kappa\colon$ there is a Tychonoff topology $\mathcal{T}' \subset \mathcal{T}$ such that $w((X, \mathcal{T}')) = \kappa\}$. Continuous bijections are called *condensations*; thus, if $\mathcal{T}' \subset \mathcal{T}$, then we say that $(X, \mathcal{T})$ *condenses* onto $(X, \mathcal{T}')$.

A space X has *countable fan tightness*, see [2], if whenever $x \in \overline{A_n}$ for all $n \in \omega$, one can choose finite $F_n \subset A_n$ so that $x \in \overline{\bigcup\{F_n\colon n \in \omega\}}$. It is natural to say that X has *countable fan tightness with respect to dense subspaces* if this statement is true for A_n dense in X, that is for every $x \in X$ and every sequence $(A_n\colon n \in \omega)$ of dense subspaces of X one can choose finite $F_n \subset A_n$ so that $x \in \overline{\bigcup\{F_n\colon n \in \omega\}}$.

A space X has *Reznichenko property* (see, for example, [12]) if whenever $x \in \overline{A} \setminus A$, there are pairwise disjoint finite $F_n \subset A$ $(n \in \omega)$ such that every neighborhood of x intersects all but finitely many F_n. Sakai calls Reznichenko property *weakly Fréchet* [23]. It is natural to say that X is *weakly Fréchet with respect to dense subspaces* if for every dense $D \subset X$ and every $x \in X \setminus D$, there are pairwise disjoint nonempty finite $F_n \subset D$ $(n \in \omega)$ such that every neighborhood of x intersects all but finitely many F_n.

Also Sakai calls X *weakly Fréchet in the strict sense* if whenever $x \in \overline{A_n}$ for all $n \in \omega$, there are finite $F_n \subset A_n$ such that every neighborhood of x intersects all but finitely many F_n [23]. This property is the conjunction of Reznichenko property and the countability of fan tightness. It is natural to say that X is *weakly Fréchet in the strict sense with respect to dense subspaces* if for every sequence $(D_n\colon n \in \omega)$ of dense subspaces of X and for every $x \in X$ there are finite $F_n \subset D_n$ such that every neighborhood of x intersects all but finitely many F_n.

A space X has *countable strong fan tightness* [22] if whenever $x \in \overline{A_n}$ for $n \in \omega$, there are $x_n \in A_n$ such that $x \in \overline{\{x_n\colon n \in \omega\}}$. It is natural to say that X has *countable strong fan tightness with respect to dense subspaces* if for every $x \in X$ and every sequence $(D_n\colon n \in \omega)$ of dense subspaces of X one can pick $x_n \in D_n$ so that $x \in \overline{\{x_n\colon n \in \omega\}}$.

Terminology related with Gerlitz–Nagy property $(*)$ and GN-separability will be introduced in Section 6.

For $f, g \in \omega^\omega$, $f \leqslant^* g$ means that $f(n) \leqslant g(n)$ for all but finitely many $n \in \omega$ (see e.g. [6]). A set of functions $X \subset \omega^\omega$ is *bounded* if there is $g \in \omega^\omega$ such that $f \leqslant^* g$ for all $f \in X$; X is *dominating* if for every $h \in \omega^\omega$, there is $f \in X$ such that $h \leqslant^* f$; $\mathfrak{b}$ and $\mathfrak{d}$ denote the minimum of cardinality of an unbounded set, and of a dominating set in ω^ω, respectively [6].

A family of functions $X \subset \omega^\omega$ can be *guessed* by a function $g \in \omega^\omega$ if for every $f \in X$ the set $\{n \in \omega\colon f(n) = g(n)\}$ is infinite.

Let $\mathcal{M}$ denote the family of all meager subsets of $\mathbb{R}$. Minimum of cardinalities of subfamilies of $\mathcal{M}$ covering $\mathbb{R}$ is denoted by $cov(\mathcal{M})$. Minimum of cardinalities of subfamilies $\mathcal{A} \subset \mathcal{M}$ such that $\bigcup \mathcal{A} \notin \mathcal{M}$ is denoted by $add(\mathcal{M})$. We will need the following results:

Theorem 6. *([16]) $add(\mathcal{M}) = \min\{\mathfrak{b}, cov(\mathcal{M})\}$.*

Theorem 7. *([16,17]) $cov(\mathcal{M}) = \min\{|X|\colon X \subset \omega^\omega$ and X cannot be guessed$\}$.*

Theorem 8. *([19,24]) $add(\mathcal{M}) = \min\{|X|\colon X \subset \omega^\omega$ and there do not exist $f, g \in \omega^\omega$ such that f is strictly increasing and for every $x \in X$, for all but finitely many n, there is $j \in [f(n), f(n+1))$ such that $x(j) = g(j)\}$.*

(Theorem 8 is a combination of Proposition 21 in [19], Theorem 21 in [19], and a theorem from [24] mentioned in the end of [19]).

Let κ be a cardinal. In [24], Scheepers introduces the following statement:

$A(\kappa)$: For every sequence $(\mathcal{U}_n\colon n \in \omega)$ of partitions of a set K of cardinality κ into countably many pieces, there are a strictly increasing $f \in \omega^\omega$ and $U_n \in \mathcal{U}_n$ $(n \in \omega)$ such that each element of K is included into all but finitely many sets $\bigcup\{U_n\colon f(i) \leqslant n < f(i+1)\}$.

We will need the following:

Theorem 9. *([24]) $add(\mathcal{M}) = \min\{\kappa\colon A(\kappa)$ fails$\}$.*

3. M-separability

In this section we recall some results from [3] and extend some of them.

Definition 10. A space X is M-separable if for every sequence $(D_n\colon n \in \omega)$ of dense subspaces of X one can select finite $F_n \subset D_n$ so that $\bigcup\{F_n\colon n \in \omega\}$ is dense in X.

Theorem 11. *([3]) If $\delta(X) = \omega$ and $\pi w(X) < \mathfrak{d}$, then X is M-separable.*

Corollary 12. *(See also [3].) If $\kappa < \mathfrak{d}$, then every countable subspace of 2^κ is M-separable.*

Theorem 13. *([3]) The space $2^{\mathfrak{d}}$ contains a dense countable subspace which is not M-separable.*

Corollary 14. *The smallest π-weight of a countable non-M-separable space is $\mathfrak{d}$.*

Proposition 15. *A separable space X is M-separable iff X has countable fan tightness with respect to dense subspaces.*

Proof. Suppose X has countable fan tightness with respect to dense subspaces, and let $D = \{d_n\colon n \in \omega\}$ be a dense subspace of X. Given a sequence of dense subspaces of X, enumerate it as $(D_{n,m}\colon n, m \in \omega)$. For each $n \in \omega$, pick finite $F_{n,m} \subset D_{n,m}$ so that $d_n \in \overline{\bigcup\{F_{n,m}\colon m \in \omega\}}$. Then $\bigcup\{F_{n,m}\colon n, m \in \omega\}$ is dense in X.

The reverse implication is obvious. □

The previous proposition is not true if we drop *with respect to dense subspaces*: consider, for example, the countable Fréchet–Urysohn fan (see also [3]) or $\beta\omega$.

Proposition 16. *The following conditions are equivalent*:

(1) *X is hereditarily M-separable.*
(2) *X is hereditarily separable and all countable subspaces of X are M-separable.*

Proof. Assume (2). Let M be a subspace of X and $(Y_n\colon n \in \omega)$ be a sequence of dense subspaces of M. Since X is hereditarily separable, there exist countable $Z_n \subset Y_n$, $n \in \omega$, such that Z_n is dense in Y_n for each $n \in \omega$. Put $Z = \bigcup\{Z_n\colon n \in \omega\}$. Since Z is countable, it is M-separable; then, since $\{Z_n\colon n \in \omega\}$ is a sequence of dense subsets of Z, there exist finite $F_n \subset Z_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} F_n$ is dense in Z. Since Z is dense in M, $\bigcup_{n\in\omega} F_n$ is dense in M. Then X is hereditarily M-separable. □

Theorem 17. *(Arhangelskii, [1], Theorem 2.2.2 in [2]) The following conditions are equivalent*:

(1) *$C_p(X)$ has countable fan tightness.*
(2) *All finite powers of X are Menger.*

Corollary 18. *If all finite powers of X are Menger, then every separable subspace of $C_p(X)$ is M-separable.*

In the proof of (1) ⇒ (2) in Theorem 17, Arhangelskii indicates that the sets A_n he uses (see the definition of countable fan tightness in Section 2) are dense in $C_p(X)$. This implies the following:

Proposition 19. *the following condition is equivalent to conditions* (1) *and* (2) *in Theorem* 17:

(1′) *$C_p(X)$ has countable fan tightness with respect to dense subspaces.*

Recently, Okunev and Tkachuk have noticed the similar fact about the “usual” tightness (see [20], Theorem 2.8).

Theorem 20. *(Noble, [18], see Theorem 1.1.5 in [2]) For every Tychonoff space X, $iw(X) = d(C_p(X))$.*

Putting together the results above, we get the following:

Theorem 21. *For a Tychonoff space, the following conditions are equivalent*:

(1) *$C_p(X)$ is a separable space and has countable fan tightness.*
(1′) *$C_p(X)$ is a separable space and has countable fan tightness with respect to dense subspaces.*
(2) *$C_p(X)$ is M-separable.*
(3) *$iw(X) = \omega$ and X^n is Menger for each $n \in \mathbb{N}$.*

Corollary 22. *([3]) If $C_p(X)$ is M-separable, then for every finite n, $C_p(X^n)$ is M-separable, and $(C_p(X))^\omega$ is M-separable.*

Corollary 23. *([3]) If X is second countable, then $C_p(X)$ is M-separable iff $C_p(X)$ is hereditarily M-separable.*

Corollary 24. *([3]) If X is a second countable space of cardinality less than $\mathfrak{d}$, then $C_p(X)$ is hereditarily M-separable.*

Proposition 25. *For a zero-dimensional X, the following conditions are equivalent to the conditions of Theorem* 21:

(2_ω) *$C_p(X,\omega)$ is M-separable.*
(2_2) *$C_p(X,2)$ is M-separable.*

Proof. For convenience of proof, we extend the list of conditions adding one more:

($2_\mathbb{Q}$) $C_p(X,\mathbb{Q})$ is M-separable.

(2) ⇒ ($2_\mathbb{Q}$) Because for a zero-dimensional X, $C_p(X,\mathbb{Q})$ is dense in $C_p(X,\mathbb{R})$.

($2_\mathbb{Q}$) ⇒ (2_ω) Take $\mathbb{Z}$ instead of ω. For every $n\in\mathbb{Z}$, pick irrational point $\alpha_n\in(n,n+1)$. Define a mapping $\pi:\mathbb{Q}\to\mathbb{Z}$ by setting $\pi(q)=n$ if $q\in(\alpha_{n-1},\alpha_n)$. Then $f\mapsto\pi\circ f$ is a continuous open mapping from $C_p(X,\mathbb{Q})$ onto $C_p(X,\mathbb{Z})$ (and we know that M-separability is preserved by continuous open mappings).

(2_ω) ⇒ (2_2) For $n\in\mathbb{Z}$, put $\rho(n)=0$ if $n<0$ and $\rho(n)=1$ if $n\geqslant 0$. Then $f\mapsto\rho\circ f$ is a continuous open mapping from $C_p(X,\mathbb{Z})$ onto $C_p(X,2)$.

(2_2) ⇒ (3) Since X is zero-dimensional, and $C_p(X,2)$ is separable, $iw(X)=\omega$ (for a dense countable $C\subset C_p(X,2)$, $\{f^{-1}(0),f^{-1}(1)\colon f\in C\}$ is a base for a zero-dimensional second countable topology on X which is included into the original topology).

Now we show that X^n is Menger for each $n\in\mathbb{N}$, following the argument from ([2], the proof of (a) ⇒ (b) in Theorem 2.2.2). Let $n\in N$, and let $(\mathcal{U}_k\colon k\in\omega)$ be a sequence of open covers of X^n. A family μ of open sets of X is called $\mathcal{U}_k$-small if for every $V_1,\dots,V_n\in\mu$, there is $G\in\mathcal{U}_k$ such that $V_1\times\cdots\times V_n\subset G$. Denote by $\mathcal{E}_k$ the family of all finite $\mathcal{U}_k$-small families of clopen sets in X. For $\mu\in\mathcal{E}_k$, put $F_\mu=\{f\in C_p(X,2)\colon f(X\setminus\bigcup\mu)=\{0\}\}$. Denote $A_k=\bigcup\{F_\mu\colon \mu\in\mathcal{E}_k\}$. It is easy to see that A_k is dense in $C_p(X,2)$.

By M-separability of $C_p(X,2)$, there are finite $H_k\subset A_k$ such that $\bigcup\{H_k\colon k\in\omega\}$ is dense in $C_p(X,2)$. For every $f\in H_k$, fix $\mu_f\in\mathcal{E}_k$ such that $f\in F_{\mu_f}$. For every $V_1,\dots,V_n\in\mu_f$, pick $G(V_1,\dots,V_n)\in\mathcal{U}_k$ so that $V_1\times\cdots\times V_n\subset G(V_1,\dots,V_n)$. Then $\mathcal{V}_k=\{G(V_1,\dots,V_n)\colon V_1,\dots,V_n\in\mu_f$ and $f\in H_k\}$ is a finite subfamily of $\mathcal{U}_k$.

It follows that $\bigcup\{\mathcal{V}_k\colon k\in\omega\}$ is a cover of X^n. Indeed, let $x=\langle x_1,\dots,x_n\rangle\in X^n$. Then $U=\{\varphi\in C_p(X,2)\colon \varphi(x_1)=\cdots=\varphi(x_n)=1\}$ is a nonempty open set in $C_p(X,2)$, so there are $k\in\omega$ and $f\in H_k\cap U$. Pick $V_1,\dots,V_n\in\mu_f$ so that $\langle x_1,\dots,x_n\rangle\in V_1\times\cdots\times V_n$. Then $x=\langle x_1,\dots,x_n\rangle\in G(V_1,\dots,V_n)\in\mathcal{V}_k$. □

Corollary 26. *If $(X,\mathcal{T})$ can be condensed onto a separable metrizable space $(X,\mathcal{T}_\mathcal{M})$ such that all finite powers of $(X,\mathcal{T}_\mathcal{M})$ are Menger, then $C_p((X,\mathcal{T}))$ contains a dense M-separable subspace.*

Proof. $C_p((X,\mathcal{T}_\mathcal{M}))$ can be viewed as a subspace of $C_p((X,\mathcal{T}))$; then obviously it is dense. □

Remark. In general, it is not true that if $C_p(X)$ contains a dense M-separable subspace, then $C_p(X)$ is M-separable. Let $D(\mathfrak{c})$ be the discrete space of cardinality $\mathfrak{c}$. The proof of Theorem 46 below can be easily modified to show that $C_p(D(\mathfrak{c}))=\mathbb{R}^\mathfrak{c}$ contains a dense H-separable subspace. On the other hand, it follows from Theorem 21 that $C_p(D(\mathfrak{c}))=\mathbb{R}^\mathfrak{c}$ is not M-separable (another way to verify this is to notice that $\delta(\mathbb{R}^\mathfrak{c})=\mathfrak{c}$).

Question 27. Suppose $C_p(X)$ contains a dense M-separable subspace. Does it follow that X can be condensed onto a second countable space all finite powers of which are Menger?

4. H-separability

Definition 28. A space X is *H-separable* if for every sequence $(D_n\colon n\in\omega)$ of dense subspaces of X, one can pick finite $F_n\subset D_n$ so that for every nonempty open set $O\subset X$, the intersection $O\cap F_n$ is nonempty for all but finitely many n.

Theorem 29. *If $\delta(X)=\omega$ and $\pi w(X)<\mathfrak{b}$, then X is H-separable.*

Proof. Let $\mathcal{B}=\{B_\alpha\colon \alpha<\kappa\}$ be a π-base for X. We assume that $\kappa<\mathfrak{b}$, and all sets B_α are nonempty. Let $(Y_n\colon n\in\omega)$ be a sequence of dense subspaces of X. Select for each $n\in\omega$ a dense countable $D_n=\{d_{n,m}\colon m\in\omega\}\subset Y_n$. For $\alpha<\kappa$, define functions $f_\alpha\in\omega^\omega$ by letting $f_\alpha(n)=\min\{m\colon d_{n,m}\in B_\alpha\}$. Since $\kappa<\mathfrak{b}$, there is a function $f^*\in\omega^\omega$ such that for every $\alpha<\kappa$, $f^*(n)>f_\alpha(n)$ for all but finitely many n. For $n\in\omega$, put $F_n=\{d_{n,m}\colon m\leqslant f^*(n)\}$. Then F_n is a finite subset of D_n, and each B_α meets all but finitely many F_n. □

Corollary 30. *If $\kappa<\mathfrak{b}$, then every countable subspace of 2^κ is H-separable.*

Theorem 31. *The space $2^\mathfrak{b}$ contains a dense countable subspace which is not H-separable.*

Proof. Let $X = \{x_m\colon m \in \omega\}$ be a countable dense subspace in $2^{\mathfrak{b}}$, and let $\{f_\alpha\colon \alpha < \mathfrak{b}\} \subset \omega^\omega$ be an unbounded family. We define points $y_{n,m} \in 2^{\mathfrak{b}}$ $(n, m \in \omega)$ as follows:

$$y_{n,m}(\alpha) = \begin{cases} 1 & \text{if } m < f_\alpha(n), \\ x_m(\alpha) & \text{otherwise.} \end{cases}$$

Put $Y_n = \{y_{n,m}\colon m \in \omega\}$ and $Y = \bigcup\{Y_n\colon n \in \omega\}$; Y is the subspace of $2^{\mathfrak{b}}$ declared in the statement of the theorem.

First, we claim that Y_n is dense in $2^{\mathfrak{b}}$. Fix a canonical open set $U(s) \subset 2^{\mathfrak{b}}$ where $s : \mathrm{dom}(s) \to 2$ is a finite function, that is $\mathrm{dom}(s) \subset \mathfrak{b}$, $|\mathrm{dom}(s)| < \omega$, and $U(s) = \{x \in 2^{\mathfrak{b}}\colon x \restriction_{\mathrm{dom}(s)} = s\}$. Let $g = \max\{f_\alpha\colon \alpha \in \mathrm{dom}(s)\}$. Since X is dense in $2^{\mathfrak{b}}$, the set $U(s)$ contains infinitely many points of X, and so we can find some $m > g(n)$ such that $x_m \restriction_{\mathrm{dom}(s)} = s$. If $\alpha \in \mathrm{dom}(s)$, then we have $m > g(n) \geqslant f_\alpha(n)$, and therefore $y_{n,m}(\alpha) = x_m(\alpha)$. This means that $y_{n,m} \restriction_{\mathrm{dom}(s)} = s$, and so $Y_n \cap U(s) \neq \emptyset$. This suffices to assert that Y_n is dense in $2^{\mathfrak{b}}$.

To conclude the proof we claim that for any choice of finite subsets $F_n \subset Y_n$, there is a basic open set in $2^{\mathfrak{b}}$ that misses infinitely many F_n. Define a function $f \in \omega^\omega$ so that $F_n \subset \{y_{n,m}\colon m < f(n)\}$ and choose β so that $f \not>^* f_\beta$, i.e. the set $N = \{n \in \omega\colon f(n) < f_\beta(n)\}$ is infinite. Now, if $n \in N$, and $y_{n,m} \in F_n$, we have $m < f(n) \leqslant f_\beta(n)$, and so $y_{n,m}(\beta) = 1$. In other words, we have shown that $y(\beta) = 1$ for every $y \in F$. This means that the set F_n does not intersect the basic open set $U = \{x \in 2^{\mathfrak{b}}\colon x(\beta) = 0\}$. □

Corollary 32. *The smallest π-weight of a countable non-H-separable space is $\mathfrak{b}$.*

Corollary 33. *The existence of a countable M-separable space which is not H-separable is consistent with ZFC.*

Question 34. Does there exist a ZFC example of a M-separable space which is not H-separable?

Proposition 35. *A separable space is H-separable iff it is weakly Fréchet in the strict sense with respect to dense subspaces.*

Proof. The "only if" part is obvious.

Now suppose X is weakly Fréchet in the strict sense with respect to dense subspaces, let $D = \{d_k\colon k \in \omega\}$ be a dense subspace of X, and let $(D_n\colon n \in \omega)$ be an arbitrary sequence of dense subspaces of X. For every $k, n \in \omega$, pick finite $F_{k,n} \subset D_n$ so that every neighborhood of d_k intersects infinitely many $F_{k,n}$. For every n, put $F_n = \bigcup\{F_{k,n}\colon 0 \leqslant k \leqslant n\}$. Then F_n is a finite subset of D_n, and every nonempty open set in X intersects all but finitely many F_n. □

Proposition 36. *The following conditions are equivalent*:

(1) *X is hereditarily H-separable.*
(2) *X is hereditarily separable, and all countable subsets of X are H-separable.*

Theorem 37. *([12], stated as in [23]) The following conditions are equivalent*:

(1) *$C_p(X)$ is weakly Fréchet in the strict sense.*
(2) *All finite powers of X are Hurewicz.*

Proposition 38. *Conditions of Theorem* 37 *are equivalent to the following condition*:

(1′) *$C_p(X)$ is weakly Fréchet in the strict sense with respect to dense subspaces.*

Proof. We only have to prove $(1') \Rightarrow (2)$. Again the proof is a minor modification of the argument from [2], the proof of (a) ⇒ (b) in Theorem 2.2.2. Let $n \in \mathbb{N}$, and let $(\mathcal{U}_k\colon k \in \omega)$ be a sequence of open covers of X^n. Let $\mathcal{E}_k$ F_μ and A_k be like in the proof of Proposition 25 $(2_2) \Rightarrow (3)$. Consider the function $e \equiv 1$ on X. By (1′), there are finite $H_k \subset A_k$ such that every neighborhood of e intersects all but finitely many H_k. For every $f \in H_k$, fix $\mu_f \in \mathcal{E}_k$ such that $f \in F_{\mu_f}$. For every $V_1, \ldots, V_n \in \mu_f$, pick $G(V_1, \ldots, V_n) \in \mathcal{U}_k$ such that $V_1 \times \cdots \times V_n \subset G(V_1, \ldots, V_n)$. Then $\mathcal{V}_k = \{G(V_1, \ldots, V_n)\colon V_1, \ldots, V_n \in \mu_f$ and $f \in H_k\}$ is a finite subfamily of $\mathcal{U}_k$.

It follows that every point $x = \langle x_1, \ldots, x_n\rangle \in X^n$ is in $\bigcup \mathcal{V}_k$ for all but finitely many k. Indeed, put $U = \{\varphi \in C_p(X)\colon \varphi(x_i) > 0$ for $0 \leqslant i \leqslant n\}$. Then U is open in $C_p(X)$ and $e \in U$. So $H_k \cap U \neq \emptyset$ for all but finitely many k. Let $H_k \cap U \neq \emptyset$, $f \in H_k \cap U$. Pick $V_1, \ldots, V_n \in \mu_f$ so that $\langle x_1, \ldots, x_n\rangle \in V_1 \times \cdots \times V_n$. Then $x = \langle x_1, \ldots, x_n\rangle \in G(V_1, \ldots, V_n) \in \mathcal{V}_k$. □

Corollary 39. *If all finite powers of X are Hurewicz, then every separable subspace of $C_p(X)$ is H-separable.*

Putting together previous results, we get the following:

Theorem 40. *For a Tychonoff space, the following conditions are equivalent*:

(1) *$C_p(X)$ is separable and weakly Fréchet in the strict sense.*
(1′) *$C_p(X)$ is separable and weakly Fréchet in the strict sense with respect to dense subspaces.*
(2) *$C_p(X)$ is H-separable.*
(3) *$iw(X)=\omega$, and X^n is Hurewicz for each $n\in\mathbb{N}$.*

Corollary 41. *If $C_p(X)$ is H-separable, then for every finite n, $C_p(X^n)$ is H-separable, and $(C_p(X))^\omega$ is H-separable.*

Corollary 42. *If X is second countable, then $C_p(X)$ is H-separable iff $C_p(X)$ is hereditarily H-separable.*

Corollary 43. *If X is a second countable space of cardinality less than $\mathfrak{b}$, then $C_p(X)$ is hereditarily H-separable.*

Proposition 44. *For a zero-dimensional X, the following conditions are equivalent to the conditions of Theorem* 40:

(2_ω) *$C_p(X,\omega)$ is H-separable.*
(2_2) *$C_p(X,2)$ is H-separable.*

The proof is similar to the proof of Proposition 25.

Corollary 45. *If $(X,\mathcal{T})$ can be condensed onto a separable metrizable space $(X,\mathcal{T}_\mathcal{M})$ such that all finite powers of $(X,\mathcal{T}_\mathcal{M})$ are Hurewicz, then $C_p((X,\mathcal{T}))$ contains a dense H-separable subspace.*

Theorem 46. *The space $2^\mathfrak{c}$ contains a dense countable H-separable subspace.*

Proof. Interpret $2^\mathfrak{c}$ as $2^{(2^\omega)}$. Put $X=C_p(2^\omega,2)$ where 2^ω bears the usual product topology that makes it homeomorphic to the Cantor set. Then X is dense in $2^{(2^\omega)}$. On the other hand, X is countable. Indeed, for every $f\in X$, the sets $f^{-1}(0)$ and $f^{-1}(1)$ are clopen subsets of the compact space 2^ω; they can therefore be partitioned into finitely many elements of the standard countable base $\mathcal{B}$ of 2^ω. So we have an injection from $C_p(X)$ to the countable set consisting of all ordered pairs of finite subfamilies of $\mathcal{B}$. Then X is H-separable by Proposition 44. □

5. R-separability

Definition 47. A space X is R-separable if for every sequence $(D_n\colon n\in\omega)$ of dense subspaces of X one can pick $p_n\in D_n$ so that $\{p_n\colon n\in\omega\}$ is dense in X.

Theorem 48. *If $\delta(X)=\omega$ and $\pi w(X)<cov(\mathcal{M})$, then X is R-separable.*

Proof. Similar to the proof of Theorem 29, but we choose f^* that guesses all f_α, and pick $p_n=d_{n,f^*(n)}\in D_n$; then $\{p_n\colon n\in\omega\}$ is dense in X. □

Corollary 49. *If $\kappa<cov(\mathcal{M})$, then every countable subspace of 2^κ is R-separable.*

Theorem 50. *The space $2^{cov(\mathcal{M})}$ contains a dense countable subspace which is not R-separable.*

Proof. Similar to the proof of Theorem 31, only we start with a family $\{f_\alpha\colon \alpha<cov(\mathcal{M})\}\subset\omega^\omega$ that cannot be guessed, and modify the last claim accordingly. □

Corollary 51. *The smallest π-weight of a countable non-R-separable space is $cov(\mathcal{M})$.*

It follows that the existence of a countable M-separable space which is not R-separable is consistent with ZFC. However such a space can be found without cardinality assumptions: the countable space $C_p(2^\omega,2)$ is H-separable (hence M-separable) but not R-separable since 2^ω does not have Rothberger property (see below).

Proposition 52. *A separable space is R-separable iff it has countable strong fan tightness with respect to dense subspaces.*

Proposition 53. *The following conditions are equivalent*:

(1) *X is hereditarily R-separable.*
(2) *X is hereditarily separable and all countable subspaces of X are R-separable.*

Theorem 54. *([22]) The following conditions are equivalent*:

(1) $C_p(X)$ *has countable strong fan tightness.*
(2) *All finite powers of X are Rothberger.*

In the proof of (1) ⇒ (2) in Theorem 54, Sakai mentions that the sets A_n he uses are dense in $C_p(X)$. The following is an immediate corollary:

Proposition 55. *Conditions* (1) *and* (2) *of Theorem* 54 *are equivalent to the following condition*:

(1′) $C_p(X)$ *has countable strong fan tightness with respect to dense subspaces.*

Corollary 56. *If all finite powers of X are Rothberger, then every separable subspace of* $C_p(X)$ *is R-separable.*

Putting together previous results, we get the following:

Theorem 57. *For a Tychonoff space, the following conditions are equivalent*:

(1) $C_p(X)$ *is separable and has countable strong fan tightness.*
(1′) $C_p(X)$ *is separable and has countable strong fan tightness with respect to dense subspaces.*
(2) $C_p(X)$ *is R-separable.*
(3) $iw(X)=\omega$, *and* X^n *is Rothberger for each* $n \in \mathbb{N}$.

Corollary 58. *If* $C_p(X)$ *is R-separable, then for every finite n,* $C_p(X^n)$ *is R-separable, and* $(C_p(X))^\omega$ *is R-separable.*

Corollary 59. *If X is second countable, then* $C_p(X)$ *is R-separable iff* $C_p(X)$ *is hereditarily R-separable*

Corollary 60. *If X is a second countable space of cardinality less than* $cov(\mathcal{M})$, *then* $C_p(X)$ *is hereditarily R-separable.*

Proposition 61. *Consistently,* $C_p(X)$ *is R-separable iff X is at most countable.*

Proof. Consistently, a second countable space X is Rothberger iff it is countable [13]. But if $C_p(X)$ is R-separable, then by Theorem 57 X condenses onto a second countable Rothberger space.

Proposition 62. *If X is zero-dimensional, then the following conditions are equivalent to the conditions of Theorem* 57:

(2_ω) $C_p(X,\omega)$ *is R-separable.*
(2_2) $C_p(X,2)$ *is R-separable.*

The proof is similar to the case of M-separability.

Corollary 63. *If* $(X,\mathcal{T})$ *can be condensed onto a separable metrizable space* $(X,\mathcal{T}_\mathcal{M})$ *such that all finite powers of* $(X,\mathcal{T}_\mathcal{M})$ *are Rothberger, then* $C_p((X,\mathcal{T}))$ *contains a dense R-separable subspace.*

Question 64. Does there exist an X such that $C_p(X)$ is not R-separable but contains a dense R-separable subspace?

Question 65. Suppose $\kappa > \omega$ and 2^κ contains a dense R-separable subspace. Does it follow that there is an X with $|X| = \kappa$ and $C_p(X)$ being R-separable?

Theorem 66.

(CH) *The space* $2^\mathfrak{c}$ *contains a dense countable R-separable subspace.*

Proof. First of all, note that example from the proof of Theorem 46 does not work since 2^ω is not Rothberger.

Recall that a subspace $S \subset 2^{\omega_1}$ is *finally dense* if there is $\alpha < \omega_1$ such that $\pi_{\omega_1\setminus\alpha}(S)$ is dense in $2^{\omega_1\setminus\alpha}$; $X \subset 2^{\omega_1}$ is called *Hereditarily Finally Dense (HFD)* if every infinite $S \subset X$ is finally dense. HFD spaces exist assuming (CH), and they are hereditarily separable, see [21], so there exist countable HFD spaces. Let Y be a countable HFD space, and let $\alpha < \omega_1$ be such that $X = \pi_{\omega_1\setminus\alpha}(Y)$ is dense in $2^{\omega_1\setminus\alpha}$. Taking into account (CH), view X as a subspace in $2^\mathfrak{c}$. It is countable, dense, and, again, an HFD. By Theorem 1 it is R-separable. □

Alternative proof. Let $X \subset \mathbb{R}$ be an uncountable set whose all finite powers are Rothberger. (Under (CH) such spaces exist, see the proof of Theorem 2.11 in [11].) Then $C_p(X,2)$ is R-separable. Let Y be a dense countable subspace in $C_p(X,2)$. Then Y is R-separable and can be viewed as a dense countable subspace in $2^{\mathfrak{c}}$. □

Question 67. Is it consistent that for $k > \omega$, 2^k does not contain dense countable R-separable subspaces?

6. GN-separability

In this section we discuss only spaces without isolated points. We need some preliminaries before giving the definition of GN-separability.

Recall that X has *property* $(*)$ *of Gerlitz–Nagy* [9] if for each sequence $(\mathcal{U}_n\colon n \in \omega)$ of open covers of X there is a partition $X = \bigcup\{X_n\colon n \in \omega\}$ such that for each n and m there are k and l such that $m < k < l$ and there are $U_i \in \mathcal{U}_i$, $k \leqslant i \leqslant l$ such that $X_n \subset \bigcup\{U_i\colon k \leqslant i \leqslant l\}$.

Theorem 68. *([19]) The following conditions are equivalent*:

(1) *X has property $(*)$.*
(2) *For each sequence $(\mathcal{U}_n\colon n \in \omega)$ of open covers of X, there are $U_n \in \mathcal{U}_n$ and a strictly increasing function $f : \omega \to \omega$ such that for every $x \in X$, $x \in \bigcup\{U_i\colon f(n) \leqslant i < f(n+1)\}$ for all but finitely many n.*
(3) *X is both Hurewicz and Rothberger.*

A dual form of condition (2) in Theorem 68 was considered in [14]. A countable dense subset D of X is called *groupable* if it can be partitioned as $D = \bigcup\{A_n\colon n \in \omega\}$ (where the sets A_n are nonempty and finite) so that every nonempty open set in X intersects all but finitely many A_n. Notice that if D is groupable, then D is ω-resolvable, that is it can be partitioned into ω many pairwise disjoint dense subsets.

Selection principle $S_1(\mathcal{D}, \mathcal{D}^{gp})$, introduced in [14], states that for each sequence $(D_n\colon n \in \omega)$ of dense subsets of X there are $d_n \in D_n$ such that $\{d_n\colon n \in \omega\}$ is groupable.

Definition 69. A space X is called GN-separable if it satisfies $S_1(\mathcal{D}, \mathcal{D}^{gp})$.

A symmetry between GN-separability and the condition (2) in Theorem 68 can be easier seen with the help of the following lemma:

Lemma 70. *For every family $\{A_n\colon n \in \omega\}$ of pairwise disjoint nonempty finite subsets of a countably infinite set N there exists a partition of N into pairwise disjoint finite sets B_m ($m \in \omega$) such that each B_m contains at least one A_n.*

Theorem 68 motivates the following question:

Question 71. Under what conditions is GN-separability equivalent to the conjunction of H-separability and R-separability?

Below we show that this is the case for C_p spaces. Here we present some positive results in the general case. The first one is obvious.

Proposition 72. *Every GN-separable space is R-separable.*

A partial result in the "$\mathrm{R} + \mathrm{H} \Rightarrow \mathrm{GN}$" direction will be obtained via a sequence of simple steps.

Proposition 73. *If D and D' are two dense countable sets in X, $D \supset D'$, and D' is groupable, then D is groupable.*

Proof. Assume $D' = \bigcup\{A_n\colon n \in \omega\}$, where the sets A_n are pairwise disjoint, nonempty, and finite and every nonempty open set in X intersects all but finitely many A_n. If $D \setminus D' = \emptyset$, of course D is groupable. If $D \setminus D'$ is finite, then $\{A_n, D \setminus D'\colon n \in \omega\}$ is the partition of D witnessing that D is groupable. If $D \setminus D'$ is countable, enumerate $D \setminus D' = \{d_n\colon n \in \omega\}$ and put $B_n = A_n \cup \{d_n\}$, $n \in \omega$; then $\{B_n\colon n \in \omega\}$ is the partition of D witnessing that D is groupable. □

Proposition 74. *The following conditions are equivalent*:

(1) *X is GN-separable.*
(2) *X is R-separable, and every dense countable subset of X contains a groupable subset.*

Proof. (2) ⇒ (1) Let $(D_n\colon n \in \omega)$ be a sequence of dense subsets of X. Then there exist $d_n \in D_n$, $n \in \omega$, such that $\{d_n\colon n \in \omega\}$ is dense in X, and a groupable subset $D' \subset \{d_n\colon n \in \omega\}$. By Proposition 73, we have that $\{d_n\colon n \in \omega\}$ is groupable.

(1) ⇒ (2) We already noticed that X is R-separable. So let D be a dense countable subset of X; considering the sequence $(D_n\colon n \in \omega)$, where $D_n = D$ for every $n \in \omega$, we obtain a subset $\{d_n\colon n \in \omega\}$ of D which is groupable. □

Proposition 75. *If X is H-separable, then every ω-resolvable dense countable subspace of X is groupable.*

Proof. Let D be an ω-resolvable dense countable subset of X. Let $\{A_n\colon n \in \omega\}$ be a partition of D into pairwise disjoint dense subsets. By H-separability, there exist finite $F_n \subset A_n$, $n \in \omega$, such that every nonempty open sets intersects all but finitely many F_n. Then $F = \bigcup_{n\in\omega} F_n$ is a countable dense subset of X which is groupable. Since $D \supset F$, by Proposition 73, D is groupable. □

Two previous propositions together imply

Proposition 76. *If X is both R-separable and H-separable, and every dense countable subspace of X is ω-resolvable, then X is GN-separable.*

Lemma 77. *If X is a countable space without isolated points, and $\pi w(X) < add(\mathcal{M})$, then X is ω-resolvable.*

Proof. Let $\mathcal{K}$ be a π-base for X, $|\mathcal{K}| = \kappa < add(\mathcal{M})$. Enumerate $X = \{x_n\colon n \in \omega\}$. For $n \in \omega$, put $\mathcal{U}_n = \{V_{n,m}\colon n \leqslant m < \omega\}$ where $V_{n,m} = \{P \in \mathcal{K}\colon x_m \in P \text{ and } x_j \notin P \text{ for } n \leqslant j < m\}$ (i.e. the elements of $\mathcal{K}$ are classified by the first point in the enumeration, starting from n; some $V_{n,m}$ may be empty). Then $\mathcal{U}_n$ is a partition of K. By Theorem 9, there are $f \in \omega^\omega$ and $U_n \in \mathcal{U}_n$ satisfying principle $A(\kappa)$. The sets U_n are of the form $U_n = V_{n,m(n)}$ for some $m(n)$.

For $i \in \omega$, denote $M(i) = \{m(n)\colon f(i) \leqslant n < f(i+1)\}$. Then $M(i) \subset \omega$, the sets $M(i)$ are finite, and any $m \in \omega$ may belong only to finitely many sets $M(i)$ (in fact, only to some of those for which $f(i) \leqslant m$). It follows that there is a strictly increasing sequence $(i_j\colon j \in \omega)$ such that the sets $M(i_j)$ are pairwise disjoint. Partition the set $J = \{i_j\colon j \in \omega\}$ into infinitely many pairwise disjoint infinite subsets J_l $(l \in \omega)$. Put $T_l = \bigcup\{M(i_j)\colon i_j \in J_l\}$, and $X_l = \{x_n\colon n \in T_l\}$. Then the sets X_l are pairwise disjoint (since so are $M(i_j)$) and dense in X (because each T_l contains x_j corresponding to infinitely many blocks $\bigcup\{U_n\colon f(i) \leqslant n < f(i+1)\}$, and thus T_l must meet all elements of the π-base $\mathcal{K}$). □

The following is an immediate corollary.

Lemma 78. *If X does not have isolated points, $\delta(X) = \omega$, and $\pi w(X) < add(\mathcal{M})$, then X is ω-resolvable.*

Theorem 79. *If X does not have isolated points, $\delta(X) = \omega$, and $\pi w(X) < add(\mathcal{M})$, then X is GN-separable.*

Proof. By Theorem 6, $add(\mathcal{M}) = \min\{\mathfrak{b}, cov(\mathcal{M})\}$. So by Theorems 29 and 48, X is both H-separable, and R-separable. By Lemma 78, X is ω-resolvable. Then by Proposition 76, X is GN-separable. □

Corollary 80. *If $\kappa < add(\mathcal{M})$, then every countable subspace of 2^κ is GN-separable.*

Theorem 81. *The space $2^{add(\mathcal{M})}$ contains a dense countable subspace which is not GN-separable.*

Proof. Similar to Theorem 31. Let $X = \{x_m\colon m \in \omega\}$ be a dense countable subspace in $2^{add(\mathcal{M})}$. Let $\mathcal{F} = \{f_\alpha\colon \alpha < add(\mathcal{M})\} \subset \omega^\omega$ be a family of functions such that for every $\varphi, g \in \omega^\omega$ (where φ is strictly increasing), there is $\alpha < add(\mathcal{M})$ such that for infinitely many n, for all $j \in (\varphi(n), \varphi(n+1))$, $f_\alpha(j) \neq g(j)$ (see Theorem 8). Then the last claim in the proof of Theorem 31 is modified accordingly. □

Corollary 82. *The smallest π-weight of a countable space without isolated points which is not GN-separable is $add(\mathcal{M})$.*

Proposition 83. *Let X be a space without isolated points.*

(1) *If X is weakly Fréchet with respect to dense subspaces, then every dense countable subspace of X is groupable.*
(2) *If every dense countable subspace of X is groupable and $\delta(X) = \omega$, then X is weakly Fréchet with respect to dense subspaces.*

Proof. (1) Let $D = \{d_m\colon m \in \omega\}$ be a dense countable subspace of X. Let $d_m \in D$. Put $D_{d_m} = D \setminus \{d_m\}$; it is a dense subspace of X. Fix pairwise disjoint finite $F_{m,n} \subset D_{d_m}$ so that every neighborhood of d_m intersects all but finitely many $F_{m,n}$.

Put $H_0 = F_{0,0}$.

Now suppose $k > 0$ and pairwise disjoint nonempty finite sets $H_i \subset D$ have been defined for $0 \leqslant i < k$ so that if $0 \leqslant m \leqslant i$, then H_i contains one of the sets $F_{m,n}$. Put $K_k = \bigcup\{H_i\colon 0 \leqslant i < k\}$. Let $0 \leqslant m \leqslant k$. Since the set K_k is finite, and the sets $F_{m,n}$ are pairwise disjoint, there is $n(k,m)$ such that $F_{m,n(k,m)} \cap K_k = \emptyset$. Put $H_k = \bigcup\{F_{m,n(k,m)}\colon 0 \leqslant m \leqslant k\}$. Then H_k is a finite subset of D disjoint from each of $H_{k'}$ for $k' < k$.

Thus the sets H_k are defined for all $k \in \omega$. It is easy to see that every open set in X intersects all but finitely many H_k. If there are points in D not included into any H_k, add them to H_k, not more than one to each. This makes D groupable.

(2) Obvious. □

Proposition 84. *A separable space without isolated points is GN-separable iff it has countable strong fan tightness with respect to dense subspaces and is weakly Fréchet with respect to dense subspaces.*

Proof. The necessity is obvious.

Sufficiency: having a countable family of dense subspaces of X, double enumerate it as $(D_{n,m}\colon n, m \in \omega)$. Fix one more dense countable subspace $D = \{d_n\colon n \in \omega\}$. We may assume that $d_n \notin D_{n,m}$ for any n and m. Since X has countable strong fan tightness, one can pick $d_{n,m} \in D_{n,m}$ so that $d_n \in \overline{\{d_{n,m}\colon m \in \omega\}}$. Then $\{d_{n,m}\colon n, m \in \omega\}$ is dense in X. By Proposition 83 it is groupable. □

Remark. We cannot state analog of Propositions 16, 36, or 53 for GN-separability because when considering GN-separability we assume the space not to have isolated points; it turns out that the notion of hereditary GN-separability does not make sense.

Theorem 85. *(Kočinac and Scheepers, [12]) The following conditions are equivalent*:

(1) *$C_p(X)$ has countable strong fan tightness, and is weakly Fréchet.*
(2) *All finite powers of X have property $(*)$.*

Theorem 86. *The following conditions are equivalent*:

(1) *$C_p(X)$ is separable, has countable strong fan tightness, and is weakly Fréchet.*
(1′) *$C_p(X)$ is separable, has countable strong fan tightness with respect to dense subspaces, and is weakly Fréchet with respect to dense subspaces.*
(2) *$C_p(X)$ is GN-separable.*
(2′) *$C_p(X)$ is H-separable and R-separable.*
(3) *$iw(X) = \omega$, and all finite powers of X have property $(*)$.*
(3′) *$iw(X) = \omega$, and all finite powers of X are Hurewicz and Rothberger.*

Proof. $(1) \Leftrightarrow (3') \Leftrightarrow (2')$ follows from Theorems 40, 57.

$(3) \Leftrightarrow (3')$ follows from Theorem 68.

$(1) \Rightarrow (1')$ is trivial.

$(1') \Leftrightarrow (2)$ follows from Proposition 84.

It suffices to prove $(2) \Rightarrow (3')$.

$(2) \Rightarrow (iw(X) = \omega)$ is trivial.

$(2) \Rightarrow$ (all finite powers are Rothberger) follows from Theorem 57 and Proposition 72.

It remains only to show that $(2) \Rightarrow$ (all finite powers of X are Hurewicz). This is a modification of the proof of Proposition 38.

Let $n \in \mathbb{N}$, and let $(\mathcal{U}_k\colon k \in \omega)$ be a sequence of open covers of X^n. We assume that $\mathcal{U}_l$ refines $\mathcal{U}_k$ if $l > k$. Define $\mathcal{E}_k$, A_k and F_μ like in the proof of Proposition 25 $(2_2) \Rightarrow (3)$. Then $\mathcal{E}_k \supset \mathcal{E}_l$ and $A_k \supset A_l$ if $l > k$.

Consider the function $e \equiv 1$ on X. By GN-separability, there are $f_k \in A_k$ such that the set $D = \{f_k\colon k \in \omega\}$ is groupable, that is can be partitioned as $D = \bigcup\{B_m\colon m \in \omega\}$ (where B_m are nonempty and finite) so that every nonempty open set in $C_p(X)$ (in particular, every neighborhood of e) intersects all but finitely many B_m. For $m \in \omega$, let $k(m) = \min\{k\colon f_k \in B_m\}$. Let $f_k \in B_m$. Fix $\mu_k \in \mathcal{E}_k$ such that $f_k \in F_{\mu_k}$. For every $V_1, \ldots, V_n \in \mu_k$, pick $G(V_1, \ldots, V_n) \in \mathcal{U}_{k(m)}$ such that $V_1 \times \cdots \times V_n \subset G(V_1, \ldots, V_n)$. Then $\mathcal{V}_m = \{G(V_1, \ldots, V_n)\colon V_1, \ldots, V_n \in \mu_k \text{ and } f_k \in B_m\}$ is a finite subfamily of $\mathcal{U}_{k(m)}$.

Like in the proof of Proposition 38, it follows that every point of X^n is in $\bigcup \mathcal{V}_m$ for all but finitely many m.

Now we define $\mathcal{W}_k \subset \mathcal{U}_k$ for all $k \in \omega$. If k is of the form $k(m)$ for some $m \in \omega$, we put $\mathcal{W}_k = \mathcal{V}_m$. If k is not of this form, there is the first $k' > k$ which is of this form. For every $W \in \mathcal{W}_{k'}$, pick $O(W) \in \mathcal{U}_k$ so that $O(W) \supset W$ and put $\mathcal{W}_k = \{O(W)\colon W \in \mathcal{W}_{k'}\}$. Then each $\mathcal{W}_k$ is a finite subfamily of $\mathcal{U}_k$, and each point of X^n is contained in $\bigcup \mathcal{W}_k$ for all but finitely many k. □

Corollary 87. *If all finite powers of X have property $(*)$, then every separable subspace of $C_p(X)$ is GN separable.*

Corollary 88. *If $C_p(X)$ is GN-separable, then for every finite n, $C_p(X^n)$ is GN-separable, and $(C_p(X))^\omega$ is GN-separable.*

Corollary 89. *If X is a second countable space of cardinality less than add($\mathcal{M}$), then $C_p(X)$ is GN-separable.*

Proposition 90. *Let X be zero-dimensional. The following conditions are equivalent to the conditions of Theorem* 86:

(2_ω) *$C_p(X,\omega)$ is GN-separable.*
(2_2) *$C_p(X,2)$ is GN-separable.*

The proof is similar to the M-, H-, and R-cases.

Corollary 91. *If $(X,\mathcal{T})$ can be condensed onto a separable metrizable space $(X,\mathcal{T}_\mathcal{M})$ such that all finite powers of $(X,\mathcal{T}_\mathcal{M})$ have property $(*)$, then $C_p((X,\mathcal{T}))$ contains a dense GN-separable subspace.*

Question 92. Suppose some dense subspace of $C_p(X)$ is GN-separable. Must $C_p(X)$ be GN-separable?

Question 93. Suppose $\kappa > \omega$ and 2^κ contains a dense GN-separable subspace. Does it follow that there is an X with $|X| = \kappa$ and $C_p(X)$ GN-separable?

Question 94. Can one prove, assuming (CH), that $2^\mathfrak{c}$ contains a dense countable GN-separable subspace?

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References

[1] A.V. Arhangelskii, Hurewicz spaces, analytic sets and fan tightness of function spaces, Soviet Math. Dokl. 33 (1986) 396–399.
[2] A.V. Arhangelskii, Topological Function Spaces, Kluwer Academic Publishers, 1992.
[3] A. Bella, M. Bonanzinga, M.V. Matveev, V.V. Tkachuk, Selective separability: General facts and behavior in countable spaces, Topology Proc. 32 (2008) 15–32.
[4] A. Berner, I. Juhász, Point-picking games and HFD's, in: Models and Sets, Proceedings of Logic Colloquimum'83, in: Lecture Notes in Statist., vol. 1103, Springer, Berlin, 1983, pp. 53–56.
[5] A. Caserta, G. DiMaio, Lj. Kočinac, E. Meccariello, Applications of k-covers, II, Topology Appl. 153 (2006) 3277–3293.
[6] E.K. van Douwen, Integers in topology, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier Sci. Pub. B.V., 1984, pp. 111–168.
[7] A. Dow, G. Gruenhage, A point-picking game and semi-selective filters, Topology Proc. 14 (1989) 221–238.
[8] R. Engelking, General Topology, Helderman-Verlag, Berlin, 1989.
[9] J. Gerlitz, Zs. Nagy, Some properties of $C(X)$, I, Topology Appl. 14 (1982) 151–161.
[10] I. Juhàsz, S. Shelah, $\pi(X) = \delta(X)$ for compact X, Topology Appl. 32 (1989) 289–294.
[11] W. Just, A. Miller, M. Scheepers, P. Szeptycki, Combinatorics of open covers, II, Topology Appl. 73 (1996) 241–266.
[12] Lj. Kočinac, M. Scheepers, Function spaces and a property of Reznichenko, Topology Appl. 123 (2002) 135–143.
[13] R. Laver, On the consistency of Borel's conjecture, Acta Math. 137 (1976) 151–169.
[14] G. DiMaio, Lj. Kočinac, E. Meccariello, Selection principles and hyperspace topologies, Topology Appl. 153 (2005) 912–923.
[15] R.A. McCoy, Function spaces which are k-spaces, Topology Proc. 5 (1980) 134–146.
[16] A. Miller, Some properties of measure and category, Trans. Amer. Math. Soc. 266 (1981) 93–114.
[17] A. Miller, A characterization of the least cardinal for which the Baire category theorem fails, Proc. Amer. Math. Soc. 86 (1982) 498–502.
[18] N. Noble, The density character in function spaces, Proc. Amer. Math. Soc. 42 (1) (1974) 515–531.
[19] A. Nowik, M. Scheepers, T. Weiss, The algebraic sum of sets of real numbers with strong measure zero sets, J. Symbolic Logic 63 (1998) 301–324.
[20] O. Okunev, V. Tkachuk, Density properties and points of uncountable order for families of open sets in function spaces, Topology Appl. 122 (2002) 397–406.
[21] J. Roitman, Basic S and L, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier Sci. Pub. B.V., 1984, pp. 295–326.
[22] M. Sakai, Property C'' and function spaces, Proc. Amer. Math. Soc. 104 (3) (1988) 917–919.
[23] M. Sakai, Special subsets of reals characterizing local properties of function spaces, in: Lj.D.R. Kočinac (Ed.), Selection Principles and Covering Properties in Topology, in: Quad. Mat., vol. 18, 2007, pp. 195–225.
[24] M. Scheepers, Meager sets and infinite games, in: Contemp. Math., vol. 192, 1996, pp. 77–90.
[25] M. Scheepers, Combinatorics of open covers, VI. Selectors for sequences of dense sets, Quaest. Math. 22 (1) (1999) 109–130.
[26] J.H. Weston, J. Shilleto, Cardinalities of dense sets, Gen. Topology Appl. 6 (1976) 227–240.