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Deterministic and unambiguous two-dimensional languages over one-letter alphabet ☆

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ABSTRACT

The paper focuses on deterministic and unambiguous recognizable two-dimensional languages with particular attention to the case of a one-letter alphabet. The family DREC(1) of deterministic languages over a one-letter alphabet is characterized as both $\mathcal{L}$(DOTA)(1), the class of languages accepted by deterministic on-line tessellation acceptors, and $\mathcal{L}$(2AFA)(1), the class of languages recognized by 2-way alternating finite automata. We show that there are inherently ambiguous languages and unambiguously recognizable languages that cannot be deterministically recognized even in the case of a one-letter alphabet. In particular we show that on-line tessellation acceptors are more powerful than their deterministic counterpart, even in the case of a one-letter alphabet. Finally we show that DREC(1) is complex enough not to be characterized in terms of classical operations.

1. Introduction

Since the sixties, many studies were devoted to two-dimensional languages or picture languages. These are sets of two-dimensional arrays of symbols over a finite alphabet and they generalize the classical one-dimensional languages or string languages. This generalization has led to the definition of many different classes of picture languages and these classes are interesting as formal methods of image recognition, as well as mathematical objects in their own right. In particular, in [11], the family REC(Σ) of *recognizable picture languages* was introduced: it generalizes the class of recognizable string languages using their characterization in terms of local languages and projections (cf. [7]). The pair of a local picture language and a projection is called the *tiling system*.

The definition of REC(Σ) is implicitly non-deterministic and it seems not possible to eliminate this non-determinism without a loss in power of recognition: any deterministic finite model for two-dimensional languages defines a class that is strictly included in REC(Σ) (see e.g. [6,13,26]). This result fits the fact that REC(Σ) family is not closed under complementation, whereas any deterministic family must have this closure property.

Deterministic languages play an important role in the recognition process of pictures. Indeed the parsing problem for two-dimensional languages in REC(Σ) is an NP-complete problem [18]. In order to decide whether a given picture p belongs to the language recognized by a given tiling system we have to scan p in order to find a picture p', in the local language, whose projection is equal to p. In general, such a recognition process is non-deterministic: at each step, one can have a backtracking on all already scanned positions. So it is important to have tiling systems that lead to computations with no backtracking. After this, deterministic tiling systems were recently defined in [2], and the given definition generalizes the

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one-dimensional (string) case. Indeed a tiling system corresponds, in one dimension, to a set of undirected transitions. To consider determinism we have to fix a computation direction. Remark that determinism, even in the one-dimensional case, is a notion related to a fixed direction (usually understood): we have determinism, along the left-to-right direction, and co-determinism, along the right-to-left direction. In two dimensions this reasoning leads to define determinism along four main directions, each one starting in one of the four corners. Deterministic languages are defined as languages that can be recognized by a tiling system that is deterministic according to some corner-to-corner direction. The class of deterministic languages over an alphabet Σ is denoted DREC(Σ). Once again, the generalization from one to two dimensions, results in a more complex notion.

An intermediate notion between determinism and recognizability is *unambiguity*. A tiling system for $L \subseteq \Sigma^{**}$ is unambiguous [10] when any picture in L has a unique local counter-image and L is unambiguous if it is recognized by an unambiguous tiling system. UREC(Σ) denotes the family of all unambiguous two-dimensional languages over Σ. As one may expect, determinism implies unambiguity. Furthermore in [2] the proper inclusion of DREC(Σ) in UREC(Σ) is shown, whereas in [4] the proper inclusion of UREC(Σ) in REC(Σ) is stated. A main result of this paper states that the same proper inclusions hold even for one-letter alphabet.

In this paper we study deterministic and unambiguous languages over a *one-letter alphabet*, whose classes are denoted DREC(1) and UREC(1) respectively. Note that the investigation on a one-letter alphabet is a necessary step in studying recognizability: if a language belongs to REC(Σ) then necessarily its projection over a one-letter alphabet must belong to REC(1). Considering a one-letter alphabet is equivalent to study the shapes of pictures before their contents.

The tiling recognizability of unary languages has been considered in [9,11]. More precisely there are considered tiling recognizable languages of pictures where the number of columns is a function of the number of rows (or vice versa) and it is shown that such functions cannot grow more quickly than an exponential function or slower than a logarithmic one, apart from the constant functions. Regular expressions for languages over one-letter alphabet are studied in [21,3]. Some comparisons between the different kinds of automata accepting two-dimensional languages, in the special case of a one-letter alphabet, are contained in [13,14,20,19]. In general, the same separation results hold in the one-letter case as in the several-letters case. Very recently the authors of [5] investigated the complexity of unary tiling recognizable picture languages.

This paper mainly focuses on DREC(1) and UREC(1) families, but it concerns other families of unary languages too along with some results on the general alphabet case. First, it is shown that DREC(1) coincides with both $\mathcal{L}$(2AFA)(1), the class of languages accepted by 2-way alternating finite automata, and $\mathcal{L}$(DOTA)(1), the family of languages accepted by deterministic on-line tessellation acceptors (Proposition 7): when the cardinality of the alphabet is one, these three approaches are equivalent. Hence any result stated for one of these families immediately holds for the other two ones too. Then we prove some closure properties of DREC(Σ) and DREC(1).

A main result is a necessary condition for languages in DREC(1) that states some periodicity in the local language corresponding to a deterministic tiling system (Proposition 10). As application, we provide an example of a language L_{mult} not in DREC(1) (Proposition 15): the proof needs a careful analysis of its local pictures for any deterministic tiling system. Proposition 15 has several applications. It allows us to show that in the *one-letter case*: (1) there are unambiguously but not deterministically recognizable languages (Proposition 17); (2) there exist languages accepted by on-line tessellation acceptors (OTA), but not by deterministic on-line tessellation acceptors (DOTA) (Corollary 20); (3) DREC(1) (and hence $\mathcal{L}$(DOTA)(1), and $\mathcal{L}$(2AFA)(1) too) is not closed under row and column star operations (Proposition 16). All these results are contained in Section 4.

In Section 5 we consider determinism, unambiguity and recognizability and show that they yield different classes, even for one-letter alphabets.

In Section 6 we compare DREC(1) with some other families of one-letter languages defined using boolean operations, row-, column-, diagonal-concatenations and stars. We show that the structure of DREC(1) cannot be captured by such operations: there are languages in DREC(1) that cannot be expressed using union, concatenations and stars and there are languages constructed by means of these operations that are not in DREC(1). Recall that this is also the case for the whole REC(Σ): the only known characterization needs also some alphabetic projection [11].

Finally in Section 7 we state some conclusions and open problems.

A preliminary and partial version of this paper can be found in [1].

2. Preliminaries

We introduce some definitions about two-dimensional languages. The notations used, some examples and results and more details can be mainly found in [11].

A *two-dimensional string* (or a *picture*) over a finite alphabet Σ is a two-dimensional rectangular array of elements of Σ. The set of all pictures over Σ is denoted by Σ^{**} and a *two-dimensional language* over Σ is a subset of Σ^{**}. Given a picture $p \in \Sigma^{**}$, let $\ell_1(p) = m$, the number of rows and $\ell_2(p) = n$ the number of columns; the pair (m, n) is the *size* of p. Note that when a one-letter alphabet is concerned, a picture p is totally defined by its size (m, n), and we will write $p = (m, n)$. Unlike the one-dimensional case, we can define an infinite number of empty pictures, namely all the pictures of size $(m, 0)$ and of size $(0, n)$, for all $m, n \geq 0$, that we denote by $\lambda_{m,0}$ and $\lambda_{0,n}$ respectively. For any picture p of size (m, n), we consider the *bordered picture* $\widehat{p}$ of size $(m + 2, n + 2)$ obtained by surrounding p with a special *boundary symbol* $\# \notin \Sigma$.

A *tile* is a picture of size $(2, 2)$ and $B_{2,2}(p)$ is the set of all sub-pictures of size $(2, 2)$ of a picture p. Given an alphabet Γ, a two-dimensional language $L \subseteq \Gamma^{**}$ is *local* if there exists a finite set Θ of tiles over $\Gamma \cup \{\#\}$ such that $L = \{p \in \Gamma^{**} | B_{2,2}(\widehat{p}) \subseteq \Theta\}$, and we will write $L = L(\Theta)$.

A *tiling system* is a quadruple $(\Sigma, \Gamma, \Theta, \pi)$ where Σ and Γ are finite alphabets, Θ is a finite set of tiles over $\Gamma \cup \{\#\}$ and $\pi : \Gamma \to \Sigma$ is a projection. A two-dimensional language $L \subseteq \Sigma^{**}$ is *tiling recognizable* if there exists a tiling system $(\Sigma, \Gamma, \Theta, \pi)$ such that $L = \pi(L(\Theta))$ (extending π in the usual way). We denote by *REC(Σ)* the family of all *tiling recognizable* picture languages over Σ. Note that when a unary alphabet is considered a tiling system can be specified by giving only the local alphabet and the set of tiles; we will write (Γ, Θ).

Furthermore, in this paper, for any family of languages $\mathcal{F}(\Sigma)$ over an alphabet Σ, we denote by $\mathcal{F}(1)$ the corresponding family over a one-letter alphabet.

Example 1. Consider the language $L_{m,m} = \{(m, m) \mid m \geq 0\}$ of square pictures over a one-letter alphabet, say $\Sigma = \{a\}$, that is pictures with same number of rows and columns. $L_{m,m}$ belongs to REC(1). Indeed it can be obtained as projection of the language of squares over the alphabet $\{0, 1\}$ in which all the symbols in the main diagonal are 1, whereas the remaining positions carry symbol 0 (cf. [11]).

Let p and q be two pictures over an alphabet Σ. The *column-concatenation* of p and q (denoted by $p \obar q$) and the *row-concatenation* of p and q (denoted by $p \ominus q$) are partial operations, defined only if $\ell_1(p) = \ell_1(q)$ and if $\ell_2(p) = \ell_2(q)$, respectively, and are given by:

$$p \obar q = \begin{array}{|c|c|} \hline \quad p \quad & \quad q \quad \\ \hline \end{array} \qquad\qquad p \ominus q = \begin{array}{|c|} \hline p \\ \hline q \\ \hline \end{array}.$$

Only in the case of one-letter alphabet, one can also define the diagonal-concatenation [3]. The *diagonal-concatenation* of $p = (m, n)$ and $q = (m', n')$ is the picture $p \obslash q = (m + m', n + n')$. It can be represented by

$$p \obslash q = \begin{array}{|c|c|} \hline \quad p \quad & \\ \hline & q \\ \hline \end{array}.$$

The definitions of picture-concatenations can be extended to languages. By iterating these operations, one can define the *row-, column- and diagonal-transitive closures (or stars) of languages*, denoted by $*\ominus$, $*\obar$, and $*\obslash$, respectively. REC(Σ) family is closed under row- and column-concatenation and their closures, under union, intersection and rotation; on the contrary it is *not* closed under complementation (see [11]) even in the case of one-letter alphabet [23]. Further REC(1) is closed under diagonal-concatenation and its closure.

Example 2. Let $L_{2m,2n}$ be the language of pictures over a one-letter alphabet with even dimensions, that is $L_{2m,2n} = \{p \,| \ell_1(p) = 2m, \ell_2(p) = 2n,\ m, n \geq 0\}$. We have that $L_{2m,2n} = (\{(2, 2)\}^{*\ominus})^{*\obar}$, and also $L_{2m,2n} = \{\lambda_{0,2}\}^{*\obslash} \obslash \{\lambda_{2,0}\}^{*\obslash}$.

Example 3. Let $L_{fc=lc}$ be the language of pictures over $\Sigma = \{a, b\}$ whose first column is equal to the last one. Language $L_{fc=lc} \in$ REC(Σ). Informally we can define a local language where information about first column symbols of a picture p is brought along horizontal direction, by means of subscripts, to match the last column of p (see [4]).

Consider also the language $L_{fc=c'}$ of pictures where the first column is equal to some ith column, $i \neq 1$. Note that $L_{fc=c'} = L_{fc=lc} \obar \Sigma^{**}$ and thus $L_{fc=c'} \in$ REC(Σ). Similarly $L_{c'=lc} = \Sigma^{**} \obar L_{fc=lc}$ and $L_{fc=c'} \cap L_{c'=lc}$ are all in REC(Σ).

Two-dimensional on-line tessellation acceptors (OTA) were introduced in [13]. A run of an OTA on a picture consists in associating a state to each position of the picture. Such state for some position (i, j) is given by the transition function and depends on the symbol in that position and on the states already associated to positions $(i, j-1)$, $(i-1, j-1)$ and $(i-1, j)$ (note that an equivalent definition is possible with the state not depending on the state in the top-left corner, $(i-1, j-1)$ [11]). A deterministic version of this model is obtained, as usual, requiring that the state associated by the transition function to any position is unique; a deterministic OTA is referred to as *DOTA*. We have that the family *$\mathcal{L}$(DOTA)(Σ)* of two-dimensional languages recognized by a DOTA is strictly included in the family *$\mathcal{L}$(OTA)(Σ)* of two-dimensional languages recognized by an OTA. Moreover, despite this kind of automaton is quite difficult to manage, this is actually the machine counterpart of a tiling system: REC(Σ) = $\mathcal{L}$(OTA)(Σ) [16].

Another model of automaton recognizing two-dimensional languages is the *4-way automaton* (*4NFA* or *4DFA* for the deterministic version); a four-way automaton is defined as an extension of the two-way automaton that recognizes strings (cf. [6]) by allowing it to move in four directions: *Left, Right, Up, Down*. A *2NFA* is a 4NFA that can move right and down only. The family of languages recognized by a 4NFA (resp. 4DFA, 2NFA) is denoted *$\mathcal{L}$(4NFA)(Σ)* (resp. *$\mathcal{L}$(4DFA)(Σ), $\mathcal{L}$(2NFA)(Σ)*).

An *alternating finite automaton (AFA)* [17] is a generalization of a finite automaton where a state can be either existential or universal. A computation that meets a universal (existential, resp.) state accepts if every (at least one, resp.) path from that state is accepting. A *two-way two-dimensional alternating automaton* (here denoted *2AFA*) is an AFA that can move right and down only. The family of languages recognized by a 2AFA is denoted *$\mathcal{L}$(2AFA)(Σ)*.

3. Deterministic recognizable languages: Some properties

Very recently the definition of deterministic tiling systems and deterministic languages was introduced and discussed [2]. Deterministic recognizable languages are defined according to one of the four corner-to-corner directions: from top-left corner towards the bottom-right one (*tl2br* for short), and all the other *corner-to-corner directions* in the set $C2C = \{tl2br, tr2bl, bl2tr, br2tl\}$. A tiling system $(\Sigma, \Gamma, \Theta, \pi)$ is *tl2br-deterministic* if for any $\gamma_1, \gamma_2, \gamma_3 \in \Gamma \cup \{\#\}$ and $\sigma \in \Sigma$ there is at most one tile $\begin{array}{|c|c|}\hline \gamma_1 & \gamma_2 \\ \hline \gamma_3 & \gamma_4 \\ \hline \end{array} \in \Theta$, with $\pi(\gamma_4) = \sigma$. Similarly d-deterministic tiling systems are defined, for any $d \in C2C$. A recognizable two-dimensional language L is *deterministic*, if it is recognized by a d-deterministic tiling system for some corner-to-corner direction d. Moreover, *DREC(Σ)* denotes the class of deterministic recognizable two-dimensional languages over the alphabet Σ. According to our notation, *DREC(1)* will denote the class of deterministic languages over a one-letter alphabet.

Example 4. The tiling system sketched in Example 1 for the language $L_{m,m}$ of square pictures is d-deterministic for $d = tl2br, br2tl$, but not for the other directions. Anyway $L_{m,m} \in$DREC(1).

Example 5. Let $L_{fc=lc}$ be the language of pictures over $\Sigma = \{a, b\}$ whose first column is equal to the last one, as defined in Example 3. The tiling system there described is d-deterministic for any $d \in C2C$ and hence $L_{fc=lc} \in$ DREC(Σ).

In [2], it is shown that $L \in \mathcal{L}$(DOTA)(Σ) if and only if L is recognized by a tl2br-deterministic tiling system and, moreover, DREC(Σ) is characterized as the closure by rotation of $\mathcal{L}$(DOTA)(Σ). Indeed, using the result that a language is in $\mathcal{L}$(DOTA)(Σ) if and only if its 180° rotation is in $\mathcal{L}$(2AFA)(Σ) (see [15]), we have that DREC(Σ) also coincides with the closure by rotation of $\mathcal{L}$(2AFA)(Σ). Furthermore, when $|\Sigma| = 1$, these characterizations can be strengthened in view of the following remark.

Remark 6. Consider languages on one-letter alphabet. As observed in Example 4, a tiling system may be d-deterministic along a given $d \in C2C$, but not along the other directions. Even though, if $L \in$ DREC(1) then, for any $d \in C2C$, there exists a d-deterministic tiling system recognizing L. Indeed from a d-deterministic tiling system recognizing L, one can construct a d'-deterministic tiling system, by replacing its tiles by some proper rotations. For example, consider the tiling system for language $L_{m,m}$ of square pictures shown in Example 1. It is tl2br-deterministic, but not tr2bl-deterministic. We can obtain a tr2bl-deterministic tiling system recognizing $L_{m,m}$, by replacing its tiles by their 90° rotations. This way, the local image of a square is a square with symbol 1 on the counter-diagonal and 0 elsewhere. The same holds for the other corner-to-corner directions.

Proposition 7. *DREC(1)* $= \mathcal{L}$*(DOTA)(1)* $= \mathcal{L}$*(2AFA)(1).*

Proof. $L \in \mathcal{L}$(DOTA)(1) iff L is recognized by a tl2br-deterministic tiling system; thus $\mathcal{L}$(DOTA)(1) $\subseteq$ DREC(1). Moreover, if L is recognized by a d-deterministic tiling system for some $d \in C2C$ then it is also recognized by a tl2br-deterministic tiling system (see Remark 6); thus DREC(1) $= \mathcal{L}$(DOTA)(1). In [15] the authors show that a language is in $\mathcal{L}$(DOTA)(Σ) iff its 180° rotation is in $\mathcal{L}$(2AFA)(Σ). But, when $|\Sigma| = 1$, then any language coincides with its 180° rotation. □

Now we study closure properties of DREC(Σ) under the boolean operations, and compare the general alphabet case with the one-letter case. Different results hold for the unary alphabet since in the general case, mixing tiling systems that are deterministic from different corner-to-corner directions does not yield any determinism.

Proposition 8. *Let Σ be a finite alphabet, $|\Sigma| \geq 2$. Then DREC(Σ) is closed under complementation, but it is not closed under union and intersection.*

Proof. The closure under complementation is in [2]. Let $L_{fc=c'}$ and $L_{c'=lc}$ as in Example 3. These languages are both in DREC(Σ), but their intersection is not [2]. Hence DREC(Σ) is not closed under union (otherwise the closure under union and complementation would yield the one under intersection). □

Corollary 9. *DREC(1) is closed under union, intersection and complementation.*

Proof. The result follows from Proposition 7 and the analogous closure properties of $\mathcal{L}$(DOTA)(Σ) [13]. □

4. A necessary condition for DREC(1)

In this section we state a necessary condition for languages in DREC(1); it will provide an example of a recognizable language that cannot be deterministically recognized. Such example will yield several important consequences.

Roughly speaking the necessary condition says that the (local) pictures with same number m of rows satisfy some "periodicity" condition. Note that, in the general case of a language $L \in$ REC(Σ), the Horizontal Iteration Lemma holds (cf. [11]): any sufficient long (local) picture with m rows has a factor that can be arbitrarily repeated still remaining in L. In the case of $L \in$ DREC(1) the result is much stronger: because of determinism, all (local) pictures with m rows in L can be obtained (column) concatenating a fixed picture (x_m) with some repetitions of another picture (y_m) (any local picture is the prefix of the longer ones). Furthermore, Corollary 14 states that the "period" (the number of columns of minimal y_m) can be divided only by prime numbers less than or equal to the number of tiles in the referred tiling system.

Remark that the result is strongly based on the cardinality one of the alphabet and it does not hold in general (for example the local language for $L_{fc=lc}$ on a two-letter alphabet as in Example 3 does not satisfy this condition, even if it is tl2br-deterministic).

Let us write $y' \prec y$ if $y = y' \varobar y''$ for some $y'' \in \Sigma^{**}$, say that y' is a *prefix* of y, and denote by $Pref(L) = \{y' \mid y' \prec y$ for some $y \in L\}$. Further we introduce for any picture p of size (m, n), the *half-bordered picture* $\tilde{p}$ of size $(m+1, n+1)$ obtained by surrounding p with the boundary symbol only on its top and left borders. We will denote by $\tilde{L}(\Theta) = \{p \in \Gamma^{**} | B_{2,2}(\tilde{p}) \subseteq \Theta\}$ and $\tilde{L}_m(\Theta)$ the set of pictures in $\tilde{L}(\Theta)$ with m rows.

Proposition 10. *Let $L \in DREC(1)$ and (Γ, Θ) be a tl2br-deterministic tiling system for L with $|\Gamma| = \gamma$.*

For any $m > 0$, there exist $x_m, y_m \in \Gamma^{**}$, with $\ell_2(x_m), \ell_2(y_m) \le \gamma^m$, such that for any $p \in \tilde{L}_m(\Theta)$, with $\ell_2(p) > \gamma^m$, we have $p \in Pref(x_m \varobar (y_m^{*\varobar}))$.

Moreover if, for any $m > 0$, $\bar{y}_m$ denotes some y_m as above with minimal number of columns, then $\ell_2(\bar{y}_{m+1}) = c \cdot \ell_2(\bar{y}_m)$ for some $c \in \{1, 2, \ldots, \gamma\}$.

Proof. Let $m > 0$: every picture in $\tilde{L}_m(\Theta)$ can have at most γ^m distinct columns. If $\tilde{L}_m(\Theta)$ is finite, the statement is vacuously true. Otherwise, consider the picture $p_0 \in \tilde{L}_m(\Theta)$ with $\ell_2(p_0) = \gamma^m + 1$: in p_0 there exist two columns, say the ith and the jth ones, with $i < j$, that are equal. Clearly $1 \le i \le \gamma^m$ (such considerations are similar to the ones in the proof of the Horizontal Iteration Lemma [11]). Set x_m the picture of size $(m, i-1)$ such that $x_m \prec p_0$, and y_m the picture of size $(m, j-i)$ such that $x_m \varobar y_m \prec p_0$. Since (Γ, Θ) is tl2br-deterministic, then for any picture $p \in \tilde{L}_m(\Theta)$ with $\ell_2(p) > \gamma^m$, $p_0 \prec p$. So the ith column of p is equal to its jth one. Furthermore determinism implies that also the $(i+1)$th column of p is equal to the $(j+1)$th one, and, in general, the nth column of p is equal to the $(n + \ell_2(y_m))$th one, for any $n > i$. Therefore we have $p \in Pref(x_m \varobar (y_m^{*\varobar}))$.

Moreover, if we choose in p_0 the indexes i and j such that $(j-i)$ is minimal, then in any $p \in \tilde{L}_m(\Theta)$ with $\ell_2(p) > \gamma^m$, there cannot exist two equal columns at a distance less than $j - i = \ell_2(y_m)$ (apply again the determinism).

Now, for any $m > 0$, let us choose y_m and y_{m+1} with minimal number of columns and denote them by $\bar{y}_m$ and $\bar{y}_{m+1}$: we show that $\ell_2(\bar{y}_{m+1}) = c \cdot \ell_2(\bar{y}_m)$ for some $c \in \{1, 2, \ldots, \gamma\}$. Indeed, any $q \in \tilde{L}_{m+1}(\Theta)$, with $\ell_2(q) > \gamma^{m+1}$, is in $Pref(x_{m+1} \varobar (\bar{y}_{m+1}^{*\varobar}))$. By erasing the last row of q we obtain a picture $p \in \tilde{L}_m(\Theta)$, that is in $Pref(x_v \varobar (y_v^{*\varobar}))$, with $\ell_2(\bar{y}_{m+1}) = \ell_2(y_v)$. On the other hand, we have $p \in Pref(x_m \varobar (\bar{y}_m^{*\varobar}))$. In such situation we have that necessarily $\ell_2(\bar{y}_{m+1}) = \ell_2(y_v)$ is a multiple of $\ell_2(\bar{y}_m)$ by some factor $c \in \{1, 2, \ldots, \gamma\}$. Indeed, we cannot have $\ell_2(\bar{y}_{m+1}) < \ell_2(\bar{y}_m)$. Moreover it cannot be $\ell_2(\bar{y}_{m+1}) \equiv h \bmod \ell_2(\bar{y}_m)$, $h \ne 0$. Otherwise, $y_v = y_0 \varobar \cdots \varobar y_0 \varobar y'_0$ with $y_0 = y'_0 \varobar y''_0$ and $\ell_2(y'_0) = h$. This would imply that in y_0 the first column and the $(h+1)$th one are equal, against the minimality of $\ell_2(\bar{y}_m)$. At last, it cannot be $\ell_2(\bar{y}_{m+1}) = k \cdot \ell_2(\bar{y}_m)$ with $k > \gamma$, otherwise $y_v = x_0 \varobar \cdots \varobar x_0$, k times, for some x_0 with $\ell_2(x_0) = \ell_2(\bar{y}_m)$. But, since below the first column of x_0, in p, at most γ different symbols can occur, this is against the minimality of $\ell_2(\bar{y}_{m+1}) = k \cdot \ell_2(x_0)$. □

The necessary condition as just stated for the local language associated to a language in DREC(1) has a weaker consequence on the language itself.

Corollary 11. *Let $L \in DREC(1)$. Then there exists a constant $\gamma > 0$ such that, for any $m > 0$, there exist integers c_m, p_m, with $0 \le c_m, p_m \le \gamma^m$ and, if $p_m > 0$, then, for any $n > c_m$, we have $(m, n) \in L$ iff $(m, n + p_m) \in L$.*

Moreover if, for any $m > 0$, $\bar{p}_m$ denotes the minimal p_m as above, then $\bar{p}_{m+1} = c\bar{p}_m$ for some $c \in \{1, 2, \ldots, \gamma\}$.

Example 12. Let $L = \{(m, m+2k) \mid m, k \ge 0\}$. A tl2br-deterministic tiling system recognizing L is the one associating, for any $m, k \ge 0$, to the mth row of length $m + 2k$, the local row $0^{m-1}1(ab)^k$, and to the mth row of length $m + 2k + 1$, the local row $0^{m-1}1(ab)^k a$. In this case the minimal x_m is a picture of size (m, m) (with 1 on the diagonal, 0 under the diagonal and a proper prefix of $(ab)^*$ in each row above the diagonal) and the minimal y_m is a picture of size $(m, 2)$ (where rows ab and ba properly alternate).

Example 13. Let $L = \{(m, 2^m) \mid m \ge 0\}$. The tiling system for L that can be constructed from the DOTA given in [13] (following the canonical construction, cf. [11]) provides a more involved example of a tl2br-deterministic tiling system, where the number of columns of x_m grows exponentially.

Proposition 10 states some periodicity on any local language for a deterministic language in DREC(1). The next corollary states some relation between the "period" and the number of tiles. More precisely, given a tl2br-deterministic tiling system (Γ, Θ), let us denote, for any $m > 0$, by $k_m^{(\Gamma,\Theta)}$ (or simply k_m when the tiling system is clearly stated) the number of columns of $\bar{y}_m$ constructed as in Proposition 10. Then the prime divisors of k_m are only numbers less than or equal to the number of tiles.

Corollary 14. *Let $L \in DREC(1)$, let (Γ, Θ) be a tl2br-deterministic tiling system for L and let $|\Gamma| = \gamma$.*
Then, for any $m > 0$, we have $k_m = 1^{h_1} 2^{h_2} 3^{h_3} \ldots \gamma^{h_\gamma}$ for some $h_i \ge 0$, $i = 1, \ldots, \gamma$.

Proof. The proof is by induction on m. Since $L \in$ DREC(1), k_1 is at most γ. For the inductive step note that, from Proposition 10, we have $k_{m+1} = ck_m$ for some $c \in \{1, 2, \ldots, \gamma\}$ and that for k_m the inductive hypothesis holds. □

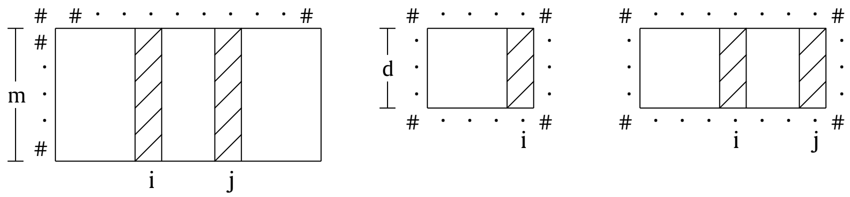
We now apply the necessary condition for DREC(1) stated in Proposition 10 in order to show that the language $L_{mult} = \{(m, km) \mid m \geq 0, k \geq 0\}$ does not belong to DREC(1). Note that we cannot use Corollary 11 for this goal. For this we need to deeply analyze the local pictures in L_{mult}, from a computational point of view, looking at what local columns must "represent" and from a more analytical point of view, looking at the periodicity of the divisors (less than or equal to a given threshold) in a sequence of consecutive integers. First, for any $m, n > 0$, let us denote by $D_m(n)$ the set of all the divisors of n that are less than or equal to m.

Proposition 15. *The language L_{mult} does not belong to DREC(1).*

Proof. The proof is by contradiction and it consists in showing that if $L_{mult} \in$ DREC(1) then Corollary 14 does not hold. Suppose that $L_{mult} \in$ DREC(1) and let (Γ, Θ) be a tl2br-deterministic tiling system recognizing it with $|\Gamma| = \gamma$. From Proposition 10, for any $m > 0$, there exist $x_m, \bar{y}_m \in \Gamma^{**}$, such that for any $p \in \tilde{L}_m(\Theta)$, with $\ell_2(p) > \gamma^m$, we have $p \in Pref(x_m \oplus (\bar{y}_m^{*\oplus}))$ with $\ell_2(\bar{y}_m) = k_m$. We are going to show that, under such hypothesis, for any $m > 0$, k_m is a multiple of m. This contradicts Corollary 14, since for any prime integer z such that $z > \gamma$, k_z would be a multiple of z, against the fact that the prime divisors of k_z must be less than or equal to γ.

Let us show that for any $m > 0$, k_m is a multiple of m.

First we show that, if $p \in \tilde{L}_m(\Theta)$ is such that its ith column is equal to its jth one and $j > i > \ell_2(x_m) + \ell_2(\bar{y}_m)$, then $D_m(i) = D_m(j)$ (see also the figure below). Let us denote for any $d \leq m$, $h \leq \ell_2(p)$ by $p_{d,h}$ the subpicture of p consisting of its first d rows and its first h columns. Note that the periodicity of p (i.e. $p \in Pref(x_m \oplus (\bar{y}_m^{*\oplus}))$) implies a similar periodicity for any subpicture $p_{d,h}$. Furthermore, since (Γ, Θ) is a tl2br-deterministic tiling system, if $(d, h) \in L_{mult}$ then its (unique) counter-image in $L(\Theta)$ is $p_{d,h}$. Now suppose that $d \in D_m(j)$; then $(d, j) \in L_{mult}$ (recall that d is a divisor of n iff the picture $(d, n) \in L_{mult}$). The first d symbols in the jth column of p match the # symbols (by means of allowed tiles in Θ). Then so is for the first d symbols in the ith column of p, $p_{d,i} \in L(\Theta)$ and, hence, $(d, i) \in L_{mult}$ i.e. $d \in D_m(i)$. Conversely, if $d \in D_m(i)$ then $(d, i) \in L_{mult}$, that is $p_{d,i}$ is in $L(\Theta)$. We show that also $p_{d,j}$ is in $L(\Theta)$. Indeed the top and the left borders match # symbols since $p \in \tilde{L}_m(\Theta)$; the right border matches # symbols since it is equal to the right border of $p_{d,i}$; and finally the bottom border of $p_{d,j}$ matches # symbols because the bottom border of $p_{d,i}$ does and the remaining bottom tiles are a repetition of some previous ones (recall that $j > i > \ell_2(x_m) + \ell_2(\bar{y}_m)$). This concludes the proof that, if $p \in \tilde{L}_m(\Theta)$ is such that its ith column is equal to its jth one and $j > i > \ell_2(x_m) + \ell_2(\bar{y}_m)$, then $D_m(i) = D_m(j)$.



In particular we have that for any $n > 2\gamma^m$, since the nth column of p is equal to the $(n + k_m)$th one, then $D_m(n) = D_m(n + k_m)$.

Now, using this fact, we are able to show that, for any $m > 0$, k_m is a multiple of m. By contradiction, suppose that there exists l, $1 \leq l < m$, such that $k_m \equiv l \bmod m$. Take n such that $n > 2\gamma^m$ and $n \equiv (m - l) \bmod m$. Then $n + k_m \equiv 0 \bmod m$ and this is against $D_m(n) = D_m(n + k_m)$.

As a remark, note that the same technique can be used to show that k_m is a multiple of the lowest common multiple of $1, 2, \ldots, m$. $\square$

This result has some immediate, but very meaningful consequences. As a first application we obtain some non-closure properties of DREC(1). Also note that, because of Proposition 7, the same non-closure properties hold for $\mathcal{L}$(DOTA)(1), and $\mathcal{L}$(2AFA)(1) (as far as we know, the properties are not stated even in those frameworks).

Proposition 16. *DREC(1), $\mathcal{L}$(DOTA)(1), and $\mathcal{L}$(2AFA)(1) are not closed neither under $*\oplus$ nor under $*\ominus$.*

Proof. Consider the language L_{mult} introduced in Example 4. We have shown that $L_{m,m} \in$ DREC(1). On the other hand Proposition 15 shows that $L_{mult} = L_{m,m}^{*\oplus}$ does not belong to DREC(1). In a similar way, the 90° rotation of L_{mult} is an example of non-closure under $*\ominus$. $\square$

5. Determinism and unambiguity

In this section we consider determinism and unambiguity in the frame of tiling recognizability of picture languages, as introduced in [4] and in [10], respectively. As one may expect, determinism implies unambiguity. Moreover the proper inclusion of DREC(Σ) in UREC(Σ) and of UREC(Σ) in REC(Σ) have been shown (see [2] and [4], resp.). We are now able to show that such strict inclusions hold even for a one-letter alphabet.

Proposition 17. *DREC(1) $\subset$ UREC(1), where the inclusion is strict.*

Proof. Consider the language L_{mult}. Proposition 15 shows that it does not belong to DREC(1). On the other hand we can construct an unambiguous tiling system recognizing L_{mult}. Starting from a tiling system $\mathcal{T}$ recognizing the language of square pictures, as sketched in Example 1, we can yield a tiling system for L_{mult}, following the construction of a tiling system for the column star of a language in [11]. We make two disjoint copies of $\mathcal{T}$ and we force it to alternate starting with the first copy. The resulting tiling system is unambiguous, since for any picture (m, km) the value k is unique. □

The next result uses a necessary condition on UREC(Σ) family shown in [4] that involves some matrices associated to a two-dimensional language. This approach has recently been applied to encompass some other known necessary conditions for REC(Σ) family and its closure by complementation (see [12]).

Let us recall the terminology and the necessary condition for UREC(Σ) family. Let $L \subseteq \Sigma^{**}$ be a picture language. For any $m \geq 1$, we can consider the subset $L(m) \subseteq L$ containing all pictures in L with exactly m rows. Note that the language $L(m)$ can be viewed as a string language over the alphabet $\Sigma^{m,1}$ of the columns, i.e. words in $L(m)$ have a "fixed height m". Moreover, for any string language L, one can define the infinite boolean matrix $M_L = \|a_{\alpha\beta}\|_{\alpha\in\Sigma^*,\beta\in\Sigma^*}$ where $a_{\alpha\beta} = 1$ if and only if $\alpha\beta \in L$. Observe that, since every regular language has a finite index (Myhill–Nerode Theorem), the number of different rows of M_L is finite. Moreover, given a matrix M, we denote by $Rank_Q(M)$, the rank of M over the field of rational numbers Q.

Now let us recall the necessary condition for UREC(Σ) family.

Proposition 18 *([4]). Let $L \subseteq \Sigma^{**}$. If $L \in$ UREC, then there is a $k \in N$ such that, for all $m \geq 1$, $Rank_Q(M_{L(m)}) \leq k^m$.*

We will use this necessary condition to show that UREC(1) is strictly contained in REC(1). First, let us fix some notations: we denote by λ the empty string and, for $\Sigma = \{a\}$ and $n \in N$, by a^n the string over Σ^* of length n. Moreover, for $m \in N$, we denote by $lcm(1, 2, \ldots, m)$ the lowest common multiple of $1, 2, \ldots, m$.

Proposition 19. *UREC(1) ⊂ REC(1), where the inclusion is strict.*

Proof. The inclusion UREC(1) ⊆ REC(1) is trivial. Now let us define, for any $m \geq 0$, the function $f(m) = lcm(2^m + 1, \ldots, 2^{m+1})$ and the language $L = \{(m, n) \mid n$ is not multiple of $f(m)\}$ over the unary alphabet $\Sigma = \{a\}$. It was shown that $L \in$ REC(1) (see [22,23]). Now, we will show that L does not satisfy the necessary condition of Proposition 18 so that $L \in$ REC(1) \ UREC(1). Indeed, for any $m > 1$, consider language $L(m)$ as defined above and the corresponding boolean matrix $M_{L(m)}$. Let us denote by c the picture over the alphabet Σ with m rows and one column, i.e. $c = (m, 1)$, and consider, in $M_{L(m)}$, the sub-matrix M_c composed by the $f(m) + 1$ rows and the $f(m) + 1$ columns indexed by $\lambda, c, c^2, \ldots, c^{f(m)}$, in this order. Then, for $i = 1, \ldots, f(m)+1$, the ith row of M_c will have symbol 1 in all its entries except for the $f(m)+2-i$ position that will carry symbol 0. Therefore $Rank_Q(M_{L(m)}) \geq f(m) + 1$. Moreover, it was shown (see [22,23]) that $f(m) = 2^{\theta(2^m)}$ and then $Rank_Q(M_{L(m)})$ cannot be bounded by k^m where k is a constant. □

In view of known characterizations involving tiling recognizability and OTA, we obtain the following result.

Corollary 20. *$\mathcal{L}$(DOTA)(1) ⊂ $\mathcal{L}$(UOTA)(1) ⊂ $\mathcal{L}$(OTA)(1), with all strict inclusions.*

Proof. Propositions 17 and 19 can be restated using the following characterizations: DREC(1) = $\mathcal{L}$(DOTA)(1) (Proposition 7), UREC(Σ) = $\mathcal{L}$(UOTA)(Σ) (see [4,25]) and REC(Σ) = $\mathcal{L}$(OTA)(Σ) (see [16]). □

Remark 21. It is well known that OTA are more powerful than DOTA [13], but the examples given in the literature are all on alphabets with cardinality greater than one. The language L_{mult} gives an example of a language that is in $\mathcal{L}$(OTA)(1) but not in $\mathcal{L}$(DOTA)(1). Also note that a different proof of the strict inclusion of $\mathcal{L}$(DOTA)(1) in $\mathcal{L}$(OTA)(1) could be obtained observing that $\mathcal{L}$(OTA)(1) = REC(1) and REC(1) is not closed under complementation [23], while $\mathcal{L}$(DOTA)(1) is closed under complementation [13].

6. DREC(1) and some regular families

In this section we look for a characterization of DREC(1) in terms of (regular) operations already introduced for two-dimensional languages over a one-letter alphabet. Hence we will compare DREC(1) with some families REG(1), $\mathcal{L}$(D), $\mathcal{L}$(CRD), of languages over a one-letter alphabet, that can be constructed using union, row-, column- and diagonal-concatenations and their closures. REG(Σ) is defined in [21] and has been recently considered in [24] as a possible counterpart of regular one-dimensional languages, while $\mathcal{L}$(D), $\mathcal{L}$(CRD) are defined and investigated in [3]. Unfortunately we will find that DREC(1) family is not captured by such considered classes.

Let us recall some definitions.

- REG(1) is the smallest family containing the singleton and closed under union, row- and column-concatenations and stars.
- $\mathcal{L}(D)$ is the smallest family containing the empty set, $\lambda_{0,0}$, $\lambda_{0,1}$, $\lambda_{1,0}$ and closed under union, diagonal-concatenation and star.
- $\mathcal{L}(CRD)$ is the smallest family containing the empty set, $\lambda_{0,0}$, $\lambda_{0,1}$, $\lambda_{1,0}$ and closed under union, row-, column-, and diagonal-concatenations and stars.

Now, let us show some properties and characterizations of these classes that will be useful later, to yield some comparison results.

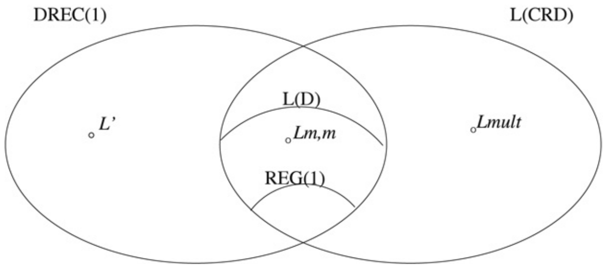


Fig. 1. Relations among DREC(1), REG(1), $\mathcal{L}$(D) and $\mathcal{L}$(CRD).

In [21] the author showed that REG(1) is closed under the boolean operations (as well as DREC(1)) and that the following characterization holds. Roughly it says that languages in REG(1) contain pictures where the number of rows and the number columns respect some periodicity, and this independently from each other.

Proposition 22 (*[21]*). *L is in REG(1) if and only if it is a finite union of Cartesian product of ultimately periodic sets.*

Example 23. Let $L_{m,m}$ be the language of square pictures (see Example 1). We have $L_{m,m} = \{(1, 1)\}^{*\oslash} = \{\lambda_{0,1} \oslash \lambda_{1,0}\}^{*\oslash} \in \mathcal{L}(\mathrm{D})$, while $L_{m,m} \notin \mathrm{REG}(1)$. In fact in $L_{m,m}$ there are an infinite number of pairs of pictures (n, n) and (n', n') with $n \neq n'$ while $(n, n') \notin L_{m,m}$ (use Proposition 22).

On the other hand in [3] it is shown that $L \in \mathcal{L}(\mathrm{D})$ if and only if the set of sizes of pictures in L is a rational relation and if and only if L is accepted by a 2NFA. We give here another characterization, more similar to the one in Proposition 22 for REG(1).

Proposition 24. *$L \in \mathcal{L}(D)$ if and only if it is a finite union of languages of the form $c^{*\oslash} \oslash P^{*\oslash}$, where c is a single picture and P is a finite set of pictures. Furthermore $\mathcal{L}(D)$ is closed under union, intersection and complementation.*

Proof. We use the characterization of rational relations of $\mathbb{N}^2$ in terms of semilinear sets of $\mathbb{N}^2$ given in [8]. Recall that a semilinear set of $\mathbb{N}^2$ is a finite union of linear sets, i.e. sets of the form $\{c\} + P$ where c is an element of $\mathbb{N}^2$ and P is a finite subset of $\mathbb{N}^2$. $\square$

We are now able to show the following relations among DREC(1), REG(1), $\mathcal{L}$(D), and $\mathcal{L}$(CRD), as summarized in Fig. 1.

Proposition 25. *DREC(1) is incomparable with $\mathcal{L}$(CRD).*

Proof. The language L_{mult} is in $\mathcal{L}$(CRD) since $L_{mult} = L^{*\obar}_{m,m}$ and $L_{m,m} \in \mathcal{L}(\mathrm{D})$ (Example 23). On the other hand $L_{mult} \notin \mathrm{DREC}(1)$ (see Proposition 15).

Consider now $L' = \{(m, 2^m) \mid m \geq 0\}$. We have $L' \in \mathcal{L}(\mathrm{DOTA})(1)$ (see [13]) and hence $L' \in \mathrm{DREC}(1)$ (see Proposition 7). On the other hand $L' \notin \mathcal{L}(\mathrm{CRD})$ (cf. [3]). $\square$

Proposition 26. *REG(1) $\subsetneq$ $\mathcal{L}$(D) $\subseteq$ DREC(1) $\cap$ $\mathcal{L}$(CRD).*

Proof. We have $\mathrm{REG}(1) \subset \mathcal{L}(\mathrm{D})$. Indeed using Proposition 22 any language in REG(1) can be accepted by a 2NFA that first verifies the number of columns moving right on the first row (simulating a classical finite automaton on strings), and then verifies the number of rows moving down on the last column. The inclusion is strict: for example $L_{m,m} \in \mathcal{L}(\mathrm{D}) \backslash \mathrm{REG}(1)$ (see Example 23).

$\mathcal{L}(\mathrm{D}) \subseteq \mathrm{DREC}(1)$ since $\mathcal{L}(\mathrm{D}) = \mathcal{L}(\mathrm{2NFA}) \subseteq \mathcal{L}(\mathrm{2AFA}) = \mathrm{DREC}(1)$.

$\mathcal{L}(\mathrm{D}) \subseteq \mathcal{L}(\mathrm{CRD})$ follows from definition. $\square$

Previous propositions say that the structure of DREC(1) cannot be encompassed by classical operations: DREC(1) coincides neither with REG(1), nor with $\mathcal{L}$(D), nor even with $\mathcal{L}$(CRD). And the same result holds if we would consider other operations, such as intersection and/or complementation. Indeed $\mathcal{L}$(D) is closed under intersection and complementation, while considering also intersection and/or complementation of languages in $\mathcal{L}$(CRD) would result in a class equal or bigger than $\mathcal{L}$(CRD), but never equal to DREC(1).

7. Conclusions and open questions

We have investigated determinism and unambiguity for tiling recognized two-dimensional languages over a one-letter alphabet. We found that these notions yield different classes of languages even for a one-letter alphabet, whereas some equivalences with other formalisms hold. These results along with the difficulty to capture the DREC(1) family by means of classical operations, point out the richness and complexity of deterministic two-dimensional languages, even for a one-letter alphabet.

The investigation is surely not yet accomplished. For example the paper leaves incomplete the study of some closure properties of the considered classes. Furthermore, it would be interesting to examine the relationships of the UREC(1) family with the other families considered in Section 6. It is also an open question as to whether $\mathcal{L}(\mathrm{D}) = \mathrm{DREC}(1) \cap \mathcal{L}(\mathrm{CRD})$ or not.

Following the characterizations given in this paper, such a question has an equivalent statement in a different formalism: it says that $\mathcal{L}(2NFA)(1) = \mathcal{L}(2AFA)(1) \cap \mathcal{L}(CRD)$, that is a language is accepted by a 2AFA and it is in $\mathcal{L}(CRD)$ iff it can be accepted with only existential states.

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