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ON AN OPEN QUESTION OF RICCERI CONCERNING A NEUMANN PROBLEM

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Abstract. In this paper we solve partially an open problem raised by B. Ricceri (*Bull. London Math. Soc.* **33** (2001), 331–340). Infinitely many solutions for a Neumann problem are obtained through a direct variational approach where the nonlinearity has an oscillatory behaviour at infinity.

2000 *Mathematics Subject Classification.* 35J20, 35J25.

1. Introduction. This paper is motivated by a problem raised by B. Ricceri in [**4**] (see also [**5**]) where the existence of infinitely many weak solutions for a Neumann problem has been proved under a highly oscillatory assumption on the nonlinearity. For the sake of clarity we recall the main result from [**4**] which led the author to formulate the open question we are dealing with.

Throughout the paper, $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary, ν is the outer unit normal to $\partial\Omega$, $\lambda \in L^\infty(\Omega)$ with $\mathrm{essinf}_\Omega \lambda > 0$, $\alpha \in L^1(\Omega)$ with $\alpha \geq 0$.

THEOREM 1 [**4**, Theorem 1]. *Assume $p > N$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, $\{r_n\} \subset \mathbb{R}^+$ and $\{\xi_n\} \subset \mathbb{R}$ two sequences such that $\lim_{n\to\infty} r_n = +\infty$ and for each $n \in \mathbb{N}$, one has*

$$\frac{|\xi_n|^p}{p} \int_\Omega \lambda(x)\, dx < r_n \tag{1}$$

and

$$\int_0^{\xi_n} f(t)\, dt = \sup_{|\xi| \leq c(pr_n)^{\frac{1}{p}}} \int_0^{\xi} f(t)\, dt, \tag{2}$$

where

$$c = \sup_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\sup_{x\in\Omega} |u(x)|}{\left(\int_\Omega |\nabla u(x)|^p\, dx + \int_\Omega \lambda(x)|u(x)|^p\, dx\right)^{1/p}}.$$

Finally, assume that

$$\limsup_{|\xi|\to+\infty} \frac{\int_\Omega \alpha(x)\,dx \int_0^\xi f(t)\,dt}{|\xi|^p} > \frac{\int_\Omega \lambda(x)\,dx}{p}. \tag{3}$$

Then, problem

$$\begin{cases} -\triangle_p u + \lambda(x)|u|^{p-2}u = \alpha(x)f(u) & \text{in } \Omega \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega \end{cases} \tag{P}$$

admits an unbounded sequence of weak solutions in $W^{1,p}(\Omega)$.

In [**4**, Remark 2, p. 335] we read: "..., we observe that from condition (1) it follows that $|\xi_n| < c(pr_n)^{1/p}$. This observation leads to the following open question: assume that all the assumptions, except for (1) and (2), of Theorem 1 hold, and suppose that there is a divergent sequence $\{b_n\}$ in $\mathbb{R}^+$ such that, for each $n \in \mathbb{N}$, one has

$$\int_0^{\xi_n} f(t)\,dt = \sup_{|\xi|\le b_n} \int_0^\xi f(t)\,dt \tag{4}$$

for some ξ_n with $|\xi_n| < b_n$. Then, does the conclusion of Theorem 1 hold?"

In this paper we will give a partial answer to this question in the affirmative. Before doing this, note that in Theorem 1 Ricceri controlled the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ on both the negative *and* the positive axis, cf. (2), applying a recent variational principle proved by himself, see [**3**]. Beside the direct approach, the advantage of our method consists of assuming a suitable oscillatory behaviour of the nonlinear term on *either* the positive *or* the negative axis, together with an additional technical condition, in order to obtain the same conclusion as in Theorem 1. More precisely, we may prove the following theorem.

THEOREM 2 (Oscillation at $+\infty$). *Assume $p > N$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, $\{b_n\}$ and $\{\xi_n\}$ sequences in $\mathbb{R}^+$ with $\xi_n < b_n$ and $\lim_{n\to\infty} b_n = +\infty$ such that, for each $n \in \mathbb{N}$ one has*

$$\int_0^{\xi_n} f(t)\,dt = \sup_{0\le\xi\le b_n} \int_0^\xi f(t)\,dt. \tag{4_+}$$

Assume that

$$\limsup_{\xi\to+\infty} \frac{\int_\Omega \alpha(x)\,dx \int_0^\xi f(t)\,dt}{\xi^p} > \frac{\int_\Omega \lambda(x)\,dx}{p} \tag{5}$$

and that there exists a non-degenerate interval $I \subset \mathbb{R}^-$ such that $f|_I \ge 0$. Then, the same conclusion as in Theorem 1 holds.

THEOREM 3 (Oscillation at $-\infty$). *Assume $p > N$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, $\{b_n\}$ and $\{\xi_n\}$ sequences in $\mathbb{R}^-$ with $b_n < \xi_n$ and $\lim_{n\to\infty} b_n = -\infty$ such that, for each $n \in \mathbb{N}$ one has*

$$\int_0^{\xi_n} f(t)\,dt = \sup_{b_n\le\xi\le 0} \int_0^\xi f(t)\,dt. \tag{4_-}$$

Assume that

$$\limsup_{\xi\to-\infty} \frac{\int_\Omega \alpha(x)\,dx \int_0^\xi f(t)\,dt}{|\xi|^p} > \frac{\int_\Omega \lambda(x)\,dx}{p} \tag{6}$$

and that there exists a non-degenerate interval $I \subset \mathbb{R}^+$ *such that* $f|_I \le 0$. *Then, the same conclusion as in Theorem 1 holds.*

REMARK 1. Note that Theorems 2 and 3 are *equivalent* in the sense that they are deducible from each other. Indeed, let $b_n' := -b_n$, $\xi_n' := -\xi_n$, $g(s) = -f(-s)$, $s \in \mathbb{R}$, where b_n, ξ_n, f fulfill the hypotheses of Theorem 3. After an elementary calculation one can see that b_n', ξ_n', and g verify all the assumptions of Theorem 2. Thus, the problem

$$\begin{cases} -\triangle_p(-u) + \lambda(x)|-u|^{p-2}(-u) = \alpha(x)f(-u) & \text{in } \Omega \\ \partial(-u)/\partial\nu = 0 & \text{on } \partial\Omega \end{cases}$$

admits an unbounded sequence of weak solutions in $W^{1,p}(\Omega)$, which concludes the argument.

REMARK 2. Comparing the hypotheses of Theorems 2 and 3 with the original problem raised by Ricceri, we mention the following differences:

1. We need not have full control of the nonlinearity f on the whole real axis; compare (4) with (4_+) and (4_-), respectively.
2. Conditions (5) and (6) are stronger than (3).
3. We need extra condition on f on the other side of the real axis where the oscillatory behaviour is assumed.

REMARK 3. A similar question as we quoted earlier was formulated by Ricceri for the case when $\{b_n\}$ tends to zero. This problem has been partially solved by Anello and Cordaro in [**1**]. Note that in Anello and Cordaro's framework a suitable truncation of the nonlinearity can be employed, due to the convergence of $\{b_n\}$ to zero. Unfortunately, this technique fails in our context since $\{b_n\}$ diverges. This fact is compensated for in a certain sense by 3. of Remark 2, which cannot be avoided in our argument. It would be interesting to prove/disprove that this condition can be removed.

Our approach is variational; weak solutions of (P) will be obtained as local minima of the energy functional associated to (P). To be more precise, let $W^{1,p}(\Omega)$ be endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p\,dx + \int_\Omega \lambda(x)|u(x)|^p\,dx\right)^{1/p}$$

which is equivalent to the standard norm in $W^{1,p}(\Omega)$. Let $F:\mathbb{R}\to\mathbb{R}$ be defined by $F(\xi) = \int_0^\xi f(t)\,dt$. The functional $\Phi : W^{1,p}(\Omega) \to \mathbb{R}$ defined by $\Phi(u) = -\int_\Omega \alpha(x)F(u(x))\,dx$ is sequentially weakly continuous on $W^{1,p}(\Omega)$ due to the compact embedding of $W^{1,p}(\Omega)$ into $C^0(\overline{\Omega})$ $(p > N)$. (As usual $C^0(\overline{\Omega})$ is endowed with the sup-norm.) Moreover Φ is continuously Gâteaux differentiable with derivative given by $\Phi'(u)(v) = -\int_\Omega \alpha(x)f(u(x))v(x)\,dx$ for every $u, v \in W^{1,p}(\Omega)$. Thus, critical points (in particular, local minima) of the energy functional $\mathcal{E}(u) \overset{def.}{=} \frac{1}{p}\|u\|^p + \Phi(u)$ are weak solutions of problem (P).

In order to guarantee the existence of infinitely many local minima of $\mathcal{E}$ we construct a sequence of subsets in $C^0(\overline{\Omega})$ such that the *relative* minima of the energy $\mathcal{E}$ on these sets are actually *local* minima for the energy on $W^{1,p}(\Omega)$; this technique has been suggested by an idea of Saint Raymond [**6**].

2. Proof of Theorem 2. Let $\{b_n\}$ and $\{\xi_n\}$ be as in the statement of Theorem 2. Notice that the sequence $\{\xi_n\}$ is unbounded; otherwise we would obtain a contradiction of (5). Thus, without loss of generality we may assume (up to subsequences) that $b_{n-1} < \xi_n < b_n$. By (4_+), one can deduce the existence of a sequence $\{\xi_n'\}$ in $\mathbb{R}^+$ such that $\xi_n < \xi_n' < b_n$ and

$$F(\xi) \leq F(\xi_n), \quad \text{for all } \xi \in [\xi_n, \xi_n']. \tag{7}$$

In the same way, since there exists an interval $I \subset \mathbb{R}^-$ such that $f|_I \geq 0$, it is possible to find $\xi_0' < \xi_0 < 0$ such that

$$F(\xi) \leq F(\xi_0), \quad \text{for all } \xi \in [\xi_0', \xi_0]. \tag{8}$$

Define the set

$$E_n = \{u \in W^{1,p}(\Omega) : \xi_0' \leq u(x) \leq \xi_n' \quad \text{for all } x \in \Omega\}.$$

Claim 1. *$\mathcal{E}$ is bounded from below on E_n and its infimum on E_n is attained.*
It is clear that E_n is convex. Moreover, it is closed in $W^{1,p}(\Omega)$ due to the continuity of the embedding $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$; then E_n is weakly closed. Since

$$\mathcal{E}(u) = \frac{\|u\|^p}{p} - \int_\Omega \alpha(x)F(u) \geq -\|\alpha\|_1 \max_{[\xi_0', \xi_n']} F \qquad \text{for all } u \in E_n,$$

$\mathcal{E}$ is bounded from below on E_n. Let $\beta_n = \inf_{E_n} \mathcal{E}$, and $\{u_k\}$ a sequence in E_n such that $\beta_n \leq \mathcal{E}(u_k) \leq \beta_n + 1/k$ for all $k \in \mathbb{N}$. Then,

$$\|u_k\|^p/p \leq \beta_n + 1 + \|\alpha\|_1 \max_{[\xi_0', \xi_n']} F$$

for all $k \in \mathbb{N}$, i.e. $\{u_k\}$ is bounded in $W^{1,p}(\Omega)$. So, up to a subsequence, $\{u_k\}$ weakly converges in $W^{1,p}(\Omega)$ to some $\tilde{u}_n \in E_n$. By the sequentially weakly lower semicontinuity of $\mathcal{E}$ we conclude that $\mathcal{E}(\tilde{u}_n) = \beta_n = \inf_{E_n} \mathcal{E}$.

Claim 2. *$\xi_0 \leq \tilde{u}_n(x) \leq \xi_n$ for all $x \in \Omega$.*
Let $X = \{x \in \Omega : \tilde{u}_n(x) \notin [\xi_0, \xi_n]\}$ and suppose that $X \neq \emptyset$. Thus, $m(X) > 0$ (where $m(X)$ denotes the Lebesgue measure of X), due to the continuity of $\tilde{u}_n$. Define

$$h(\xi) = \begin{cases} \xi_0, & \text{if } \xi < \xi_0; \\ \xi, & \text{if } \xi \in [\xi_0, \xi_n]; \\ \xi_n, & \text{if } \xi > \xi_n. \end{cases}$$

Set $\tilde{v}_n = h \circ \tilde{u}_n$. Due to Marcus and Mizel [**2**], $\tilde{v}_n$ belongs to $W^{1,p}(\Omega)$ (since h is uniformly Lipschitz). Moreover $\tilde{v}_n \in E_n$. Denoting by

$$X_1 = \{x \in X : \tilde{u}_n(x) < \xi_0\} \qquad \text{and} \qquad X_2 = \{x \in X : \tilde{u}_n(x) > \xi_n\},$$

we have that $\tilde{v}_n(x) = \tilde{u}_n(x)$ for all $x \in \Omega\backslash X$, $\tilde{v}_n(x) = \xi_0$ for all $x \in X_1$ and $\tilde{v}_n(x) = \xi_n$ for all $x \in X_2$. Then,

$$\begin{aligned}\mathcal{E}(\tilde{v}_n) - \mathcal{E}(\tilde{u}_n) &= -\frac{1}{p}\int_X |\nabla \tilde{u}_n|^p + \frac{1}{p}\int_X \lambda(x)[|\tilde{v}_n|^p - |\tilde{u}_n|^p] - \int_X \alpha(x)[F(\tilde{v}_n) - F(\tilde{u}_n)]\\ &= -\frac{1}{p}\int_X |\nabla \tilde{u}_n|^p + \frac{1}{p}\int_{X_1} \lambda(x)[|\xi_0|^p - |\tilde{u}_n|^p] + \frac{1}{p}\int_{X_2} \lambda(x)\big[\xi_n^p - \tilde{u}_n^p\big]\\ &\quad + \int_{X_1} -\alpha(x)[F(\xi_0) - F(\tilde{u}_n)] + \int_{X_2} -\alpha(x)[F(\xi_n) - F(\tilde{u}_n)].\end{aligned}$$

From (7) and (8) we obtain that every term of the above expression is not positive. On the other hand, since $\mathcal{E}(\tilde{v}_n) \geq \mathcal{E}(\tilde{u}_n) = \inf_{E_n} \mathcal{E}$, then in particular,

$$\int_X |\nabla \tilde{u}_n|^p = 0,$$

$$\int_{X_1} \lambda(x)[|\xi_0|^p - |\tilde{u}_n|^p] = \int_{X_2} \lambda(x)\big[\xi_n^p - \tilde{u}_n^p\big] = 0.$$

From the first equality we deduce the existence of a positive measured subset Y of X and a constant C such that $\tilde{u}_n = C$ on Y. Then, either $Y \subset X_1$ or $Y \subset X_2$. Assume that the first case occurs (analogously if $Y \subset X_2$). So,

$$\begin{aligned}0 = \int_{X_1} \lambda(x)[|\xi_0|^p - |\tilde{u}_n|^p] &\leq \int_Y \lambda(x)[|\xi_0|^p - |C|^p]\\ &\leq \operatorname{essinf}_\Omega \lambda[|\xi_0|^p - |C|^p]m(Y) < 0,\end{aligned}$$

a contradiction. This shows that X has zero measure, therefore, $X = \emptyset$.

Claim 3. *$\tilde{u}_n$ is a local minimum of $\mathcal{E}$ in $W^{1,p}(\Omega)$.*
Suppose the contrary. Then there exists a sequence $\{u_k\} \subset W^{1,p}(\Omega)$ such that it converges to $\tilde{u}_n$ and $\mathcal{E}(u_k) < \mathcal{E}(\tilde{u}_n)$ for all $k \in \mathbb{N}$. From the latter inequality, it follows that $u_k \notin E_n$ for any $k \in \mathbb{N}$. Since $u_k \to \tilde{u}_n$ in $W^{1,p}(\Omega)$, then $u_k \to \tilde{u}_n$ in $C^0(\overline{\Omega})$. In particular, for every $0 < \varepsilon < \min\{\xi_n' - \xi_n, \xi_0 - \xi_0'\}/2$, there exists $k_\varepsilon \in \mathbb{N}$ such that $\sup_{x\in\Omega} |u_k(x) - \tilde{u}_n(x)| < \varepsilon$ for every $k \geq k_\varepsilon$. Taking into account the choice of the number ε, and using Claim 2 we conclude that

$$\xi_0' < u_k(x) < \xi_n' \quad \text{for all } x \in \Omega,\ k \geq k_\varepsilon,$$

which clearly contradicts the fact $u_k \notin E_n$.

Claim 4. $\lim_{n\to\infty} \beta_n = -\infty$. (Recall that $\beta_n = \inf_{E_n} \mathcal{E}$.)
From (5) there exist a sequence $\{\tilde{\xi}_k\} \subset \mathbb{R}^+$ tending to $+\infty$ and a constant $M > 0$ such that

$$\frac{F(\tilde{\xi}_k)\int_\Omega \alpha(x)\,dx}{\tilde{\xi}_k^p} > M > \frac{\int_\Omega \lambda(x)\,dx}{p}.$$

Since ξ_n tends to $+\infty$, there exist a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ and $\bar{k} \in \mathbb{N}$ such that $\tilde{\xi}_k < \xi_{n_k}$, for $k \geq \bar{k}$. Then, the constant function $w_k = \tilde{\xi}_k$ belongs to E_{n_k} and

$$\beta_{n_k} = \inf_{E_{n_k}} \mathcal{E} \leq \mathcal{E}(w_k) = \frac{\|w_k\|^p}{p} - F(\tilde{\xi}_k) \int_\Omega \alpha(x)\, dx \leq \frac{1}{p} \tilde{\xi}_k^p \int_\Omega \lambda(x)\, dx - M\tilde{\xi}_k^p$$
$$= \tilde{\xi}_k^p \left(\frac{1}{p} \int_\Omega \lambda(x)\, dx - M \right) \to -\infty.$$

Since $\{\beta_n\}$ is non-increasing, our claim is achieved.

Proof of Theorem 2 *concluded.* Since $\tilde{u}_n$ are local minima of $\mathcal{E}$ (cf. Claim 3), they are critical points of $\mathcal{E}$, thus weak solutions of (P). Due to Claim 4 there are infinitely many pairwise distinct $\tilde{u}_n$. Moreover, one has $\|\tilde{u}_n\| \to \infty$ as $n \to \infty$. Indeed, arguing by contradiction, there is a subsequence $\{\tilde{u}_{n_k}\}$ of $\{\tilde{u}_n\}$ which is bounded in $W^{1,p}(\Omega)$. Thus, it is bounded in $C^0(\overline{\Omega})$ as well. In particular we can find $n_0 \in \mathbb{N}$ such that $\tilde{u}_{n_k} \in E_{n_0}$ for every $k \in \mathbb{N}$. For every $n_k \geq n_0$ one has

$$\beta_{n_0} \geq \beta_{n_k} = \inf_{E_{n_k}} \mathcal{E} = \mathcal{E}(\tilde{u}_{n_k}) \geq \inf_{E_{n_0}} \mathcal{E} = \beta_{n_0},$$

which proves that $\beta_{n_k} = \beta_{n_0}$ for all $n_k \geq n_0$, contradicting Claim 4.

3. Consequences, examples. In the sequel, we assume $p > N$, and α, λ are as in Section 1. The next result gives a simple criterion for applying Theorem 2.

Corollary 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which fulfills* (5) *and let $I \subset \mathbb{R}^-$ a non-degenerate interval such that $f|_I \geq 0$. Assume that there are sequences $\{a_n\}$ and $\{b_n\}$ in $\mathbb{R}^+$ with $a_n < b_n$ and $\lim_{n\to\infty} b_n = +\infty$ such that, for every $n \in \mathbb{N}$ one has*

$$f(t) \leq 0 \quad \textit{for all } t \in [a_n, b_n]. \tag{9}$$

Then, problem (P) admits an unbounded sequence of weak solutions in $W^{1,p}(\Omega)$. In particular, if $f \geq 0$ on $\mathbb{R}^-$, the solutions are non-negative.

Proof. By condition (9), one has $\int_0^\xi f(t)\, dt \leq \int_0^{a_n} f(t)\, dt$ for all $\xi \in [a_n, b_n]$. Hence, there exists a point $\xi_n \in]0, a_n]$ such that condition (4_+) is verified. Now, we can apply Theorem 2.

When $f \geq 0$ on $\mathbb{R}^-$, the solutions are non-negative. Indeed, suppose that $u \in W^{1,p}(\Omega)$ is a weak solution of (P) and the set $S = \{x \in \Omega : u(x) < 0\}$ is not empty. It is clear that S is open. Let $u_S \in W^{1,p}(\Omega)$ be defined by $u_S = \min\{u, 0\}$. Then we obtain

$$\int_\Omega (|\nabla u|^{p-2} \nabla u \nabla u_S + \lambda(x)|u|^{p-2} u u_S) = \int_\Omega \alpha(x) f(u) u_S.$$

Using the above relation and the fact that $f \geq 0$ in $\mathbb{R}^-$, we conclude that $\|u\|^p_{W^{1,p}(S)} = \int_S (|\nabla u|^p + \lambda(x)|u|^p) = \int_S \alpha(x) f(u) u \leq 0$, which contradicts the choice of the set S. This completes the proof. □

Remark 4. A similar result to Corollary 1 can be stated in view of Theorem 3. These kinds of results solve partially Problem 8 in [**5**]. Indeed, we can avoid in [**4**, Theorem 3] the condition $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ which was essential in Ricceri's approach (see e.g. examples below). Note that Theorem 3 in [**4**] is a direct consequence of Theorem 1.

EXAMPLE 1. Let $p > N$ and $\lambda \in L^\infty(\Omega)$ with $\text{essinf}_\Omega \lambda > 0$. Let $\sigma > 8$ and $A_n = [2n, 2n+1]$ for every $n \in \mathbb{N}$. Then, the problem

$$\begin{cases} -\Delta_p u = \lambda(x)|u|^{p-2}u[\sigma \operatorname{dist}(u, \mathbb{R}\setminus \cup_{n\in\mathbb{N}} A_n) - 1] & \text{in } \Omega \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega \end{cases}$$

admits an unbounded sequence of non-negative weak solutions in $W^{1,p}(\Omega)$.

Proof. We take $f(t) = \sigma|t|^{p-2}t \operatorname{dist}(t, \mathbb{R}\setminus \cup_{n\in\mathbb{N}} A_n)$ and $\alpha(x) = \lambda(x)$, $a_n = 2n+1$, $b_n = 2n+2$. Since $f \equiv 0$ in $\mathbb{R}^-$ and $\limsup_{\xi\to+\infty} \frac{\int_0^\xi f(t)\,dt}{\xi^p} \geq \frac{\sigma}{8p}$, we can apply Corollary 1. □

In the next example we denote by $[p]$ the integer part of $p \in \mathbb{R}$.

EXAMPLE 2. Let $p > N$, $\lambda \in L^\infty(\Omega)$ with $\text{essinf}_\Omega \lambda > 0$ and $\alpha \in L^1(\Omega)\setminus\{0\}$ with $\alpha \geq 0$. Then, the problem

$$\begin{cases} -\Delta_p u + \lambda(x)|u|^{p-2}u = \alpha(x)|u|^{[p]+1} \sin u & \text{in } \Omega \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega \end{cases}$$

admits an unbounded sequence of weak solutions in $W^{1,p}(\Omega)$.

Proof. We apply Corollary 1, taking $f(t) = |t|^{[p]+1} \sin t$ and $a_n = (2n+1)\pi$, $b_n = 2(n+1)\pi$. Notice that $\limsup_{\xi\to+\infty} \frac{\int_0^\xi f(t)\,dt}{\xi^p} = +\infty$, thus relation (5) is verified. □

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