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Harnack inequality and smoothness for quasilinear degenerate elliptic equations

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Abstract

We prove local and global regularity for the positive solutions of a quasilinear variational degenerate equation, assuming minimal hypothesis on the coefficients of the lower order terms. As an application we obtain Hölder continuity for the gradient of solutions to nonvariational quasilinear equations.

Keywords: Harnack inequality; Muckenhoupt weights; Degenerate elliptic equations; Morrey classes

1. Introduction

In the last decades some papers have been devoted to the study of local regularity properties for linear degenerate elliptic equations. We are interested in the case in which the eigenvalues associated to the principal part of the operators are controlled by a Muckenhoupt weight of the class A_2. In this direction previous results are contained in the papers [2,4]. In [2] the authors study the linear degenerate homogeneous equation

$$-(a_{ij}u_{x_i})_{x_j} = 0. \tag{1}$$

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The equation is degenerate in the following sense

$$\exists\lambda > 0:\ \lambda^{-1}\omega|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant \lambda\omega|\xi|^2 \quad \text{a.e. } x \in \Omega \text{ and } \forall\xi \in \mathbb{R}^n, \tag{2}$$

where ω is a function in the Muckenhoupt class A_2.

For Eq. (1) Harnack inequality and subsequence interior and boundary smoothness results are proved in [2].

The degenerate equation in which a potential appears, has been treated by Gutierrez in [4]. There he considers the equation

$$-(a_{ij}u_{x_i})_{x_j} + Vu = 0, \tag{3}$$

where the potential V is assumed to be in a Stummel–Kato type class. For solutions of Eq. (3) Harnack inequality and Hölder continuity have been proved.

Similar regularity results have been obtained in [7] for the following complete linear degenerate equation with all the coefficients in some Morrey classes

$$-(a_{ij}u_{x_i} + d_j u)_{x_j} + b_i u_{x_i} + cu = f - (h_i)_{x_i}. \tag{4}$$

The assumptions on the lower order terms are minimal in the sense that they are necessary too (see e.g. [1,6] for the linear uniformly elliptic case) at least in some cases.

The purpose of this note is to study the local regularity of weak solutions for quasilinear degenerate elliptic equations of the following kind

$$-(a_{ij}u_{x_i} + d_j u)_{x_j} + \frac{b_0}{\lambda}\omega|Du|^2 + b_i u_{x_i} + cu = f - (h_i)_{x_i}. \tag{5}$$

Here the equation is degenerate in the sense (2) and the lower order terms belong to Morrey classes. We stress that Eq. (5) has quadratic growth in the gradient.

We obtain our regularity results by showing that positive weak solutions of Eq. (5) satisfy a Harnack inequality (see Theorem 3.3). As a consequence of Harnack inequality we obtain interior and boundary Hölder continuity of the weak solutions of Eq. (5).

Inspired by the technique in [3,5], we extend the results in [7] to Eq. (5) in which quadratic growth in the gradient is allowed ($b_0 \neq 0$).

Moreover, we also show $C^{1,\alpha}$ estimates for a nondivergence type quasilinear degenerate equation of the following kind

$$Qu = a^{ij}(x, u, Du)u_{x_i x_j} + b(x, u, Du) = 0. \tag{6}$$

2. Muckenhoupt weights and related Sobolev and Morrey classes

In this section we collect some definitions and known results we will use in the sequel.

Let ω be a nonnegative function, locally integrable in $\mathbb{R}^n$ and $1 < p < +\infty$. We say that ω is an A_p weight and write $\omega \in A_p$, if

$$\sup_B\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \left[\omega(x)\right]^{\frac{-1}{p-1}}\,dx\right)^{p-1} \equiv C_0 < +\infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. The number C_0 is called the A_p constant of ω.

Functions in A_p enjoy several properties. Denoted by $B_r(x)$ the ball centered in x with radius r, we recall the doubling property, i.e. there exist positive constants $C_d > 1$ such that

$$\omega\big(B_{2r}(x_0)\big) \leqslant C_d\,\omega\big(B_r(x_0)\big),$$

for every $x_0 \in \mathbb{R}^n$, $r > 0$, where $\omega(B_r(x_0)) = \int_{B_r(x_0)} \omega\,dx$.

Any $\omega \in A_p$ defines a measure. We can define weighted Lebesgue and Sobolev classes by using the measure $\omega(x)\,dx$.

Let Ω be an open bounded set in $\mathbb{R}^n$.

Definition 2.1. Let ω be an A_p weight with $1 < p < \infty$. We say that a locally integrable function u belongs to the weighted Lebesgue space $L^p(\Omega, \omega)$ if

$$\|u\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x)\,dx\right)^{\frac{1}{p}} < +\infty. \tag{7}$$

Let k be a positive integer. We say that a locally integrable function u belongs to the weighted Sobolev class, $W^{k,p}(\Omega,\omega)$, if u and its distributional partial derivatives $u_{x_{j_1}x_{j_2}\ldots x_{j_i}}$, $1 \leqslant j_i \leqslant n$, for all $i = 1, 2, \ldots, k$ belong to the weighted Lebesgue class $L^p(\Omega,\omega)$.

The weighted Lebesgue classes $L^p(\Omega,\omega)$ are Banach spaces with respect to the norm (7) and, in particular, Hilbert space if $p = 2$. The weighted Sobolev classes $W^{k,p}(\Omega,\omega)$ are Banach spaces with respect to the following norm

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \|u\|_{L^p(\Omega,\omega)} + \sum_{i=1}^{k} \sum_{\{j_1,j_2,\ldots,j_i\}} \|u_{x_{j_1}x_{j_2}\ldots x_{j_i}}\|_{L^p(\Omega,\omega)}. \tag{8}$$

We denote by $W_0^{k,p}(\Omega,\omega)$ the closure of the smooth and compactly supported functions in $W^{k,p}(\Omega,\omega)$ with respect to the norm (8).

We can define the classes $L^p_{\mathrm{loc}}(\Omega,\omega)$ and $W^{k,p}_{\mathrm{loc}}(\Omega,\omega)$ in a similar way. We state an embedding theorem for weighted Sobolev spaces.

Theorem 2.1. *Let ω be an A_2 weight. There exist constants C_Ω and $\delta > 0$ such that for all $u \in C_0^\infty(\Omega)$ and for all τ satisfying $1 \leqslant \tau \leqslant \frac{n}{n-1} + \delta$*

$$\|u\|_{L^{2\tau}(\Omega,\omega)} \leqslant C_\Omega \|\nabla u\|_{L^2(\Omega,\omega)},$$

where C_Ω depends only on n, the A_2 constant of ω and the diameter of Ω.

Proof. See Theorem 1.3 in [2]. □

Now we define the weighted Morrey classes (see [6]).

Definition 2.2. For any locally integrable function V in Ω and $\sigma \in \mathbb{R}$ we set

$$\|V\|_{\sigma,\Omega} = \sup_{\substack{x\in\Omega \\ 0<r<2R}} \frac{1}{r^\sigma} \int\limits_{\{y\in\Omega:\ |x-y|<r\}} \big|V(y)\big| \int\limits_{|x-y|}^{4R} \frac{s}{\omega(B_s(x))}\, ds\,\omega(y)\, dy.$$

If $\|V\|_{\sigma,\Omega}$ is finite we say that the function V belongs to the class $M_\sigma(\Omega,\omega)$.

We remark that $\omega \equiv 1$ gives back the classical Morrey classes.

The following result will be quite useful in the sequel.

Theorem 2.2. *Let $V : \Omega \to \mathbb{R}^n$ be a function such that $\frac{V}{\omega} \in M_\sigma(\Omega,\omega)$. Then for any $0 < \varepsilon < 1$ there exists $\delta > 0$ such that*

$$\int\limits_\Omega \big|V(x)\big| u^2(x)\, dx \leqslant \varepsilon \int\limits_\Omega \big|\nabla u(x)\big|^2 \omega(x)\, dx + C\varepsilon^{-\delta} \int\limits_\Omega u^2(x)\omega(x)\, dx,$$

for all $u \in C_0^\infty(\Omega)$, where C is a constant depending on ν, σ, n and $\|\frac{V}{\omega}\|_{\sigma,\Omega}$.

Proof. See Theorem 2.7 in [7]. □

3. Harnack inequalities for variational quasilinear equations

In this section first we prove an invariant Harnack inequality for a quasilinear equation with quadratic growth in the gradient.

Let Ω be a bounded domain of $\mathbb{R}^n$ and ω a Muckenhoupt A_2 weight. Let $\{a_{ij}(x)\}$ be a matrix of measurable functions in Ω satisfying the following ellipticity condition

$$\exists\lambda > 0:\ \lambda^{-1}\omega(x)|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant \lambda\omega(x)|\xi|^2 \quad \text{a.e. } x \in \Omega \text{ and } \forall\xi \in \mathbb{R}^n. \tag{9}$$

Consider the equation

$$-(a_{ij}w_{x_i} + d_j w)_{x_j} + \frac{b_0}{\lambda}\omega|Dw|^2 + b_i w_{x_i} + cw = f - (h_i)_{x_i}, \tag{10}$$

where

$$b_0 \in \mathbb{R}\setminus\{0\}, \quad \left(\frac{b_i}{\omega}\right)^2, \frac{c}{\omega}, \left(\frac{d_i}{\omega}\right)^2, \frac{f}{\omega}, \left(\frac{h_i}{\omega}\right)^2 \in M_\sigma(\Omega,\omega), \quad \sigma > 0. \tag{11}$$

Definition 3.1. We say that $w \in W^{1,2}_{\mathrm{loc}}(\Omega,\omega)$ is a local weak supersolution (subsolution) of (10) if $\forall\varphi \in W_0^{1,2}(\Omega,\omega)$, with $\varphi \geqslant 0$

$$\int\limits_\Omega \left[(a_{ij}w_{x_j} + d_j w)\varphi_{x_j} + \left(\frac{b_0}{\lambda}\omega|Dw|^2 + b_i w_{x_i} + cw\right)\varphi\right] dx \geqslant (\leqslant) \int\limits_\Omega (f\varphi + h_{x_i}\varphi_{x_i})\, dx.$$

We say that $w \in W^{1,2}_{\mathrm{loc}}(\Omega, \omega)$ is a local weak solution of (10) if w is a supersolution and a subsolution.

Now we prove

Theorem 3.1. *Let w be a weak nonnegative supersolution of Eq.* (10) *in a ball $B_{3r} \Subset \Omega$. Assume* (9) *and* (11). *Let $M > 0$ be a constant such that $w \leqslant M$ in B_{3r}. Then there exists c depending on n, M, λ and the A_2 constant of ω, such that*

$$\omega^{-1}(B_{2r}) \int_{B_{2r}} w\omega\, dx \leqslant c\left\{\min_{B_r} w + r^\sigma \left\| \frac{f}{\omega}\right\|_{\sigma, B_{3r}} + \left(r^\sigma \sum_{i=1}^n \left\| \left(\frac{h_i}{\omega}\right)^2 \right\|_{\sigma, B_{3r}}\right)^{\frac{1}{2}}\right\}.$$

Proof. We may assume $r = 1$. Let $k = \|\frac{f}{w}\|_{\sigma, B_3} + (\sum_{i=1}^n \|(\frac{h_i}{\omega})^2\|_{\sigma, B_3})^{\frac{1}{2}}$ and $v = w + k$. We take $\phi(x) = \eta^2(x)v^\beta(x)e^{-|b_0|v(x)}$, $\beta < 0$ as test function, where $\eta \in C^1_0(B_3)$, $\eta \geqslant 0$. Since w is supersolution in B_3 of (10) we have

$$\begin{aligned}
&\int_{B_3} \Big[2\eta(a_{ij}w_{x_i} + d_j w - h_j)\eta_{x_j} v^\beta e^{-|b_0|v} \\
&\quad + \big(-|\beta|v^{\beta-1} - |b_0|v^\beta\big)\eta^2 e^{-|b_0|v}(a_{ij}w_{x_i} + d_j w - h_j)v_{x_j} \\
&\quad + \frac{b_0}{\lambda}\omega|Dw|^2\eta^2 v^\beta e^{-|b_0|v} + (b_i w_{x_i} + cw - f)\eta^2 v^\beta e^{-|b_0|v}\Big]\, dx \geqslant 0,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B_3} \eta^2 e^{-|b_0|v}\big(b_0 v^\beta + |\beta|v^{\beta-1}\big)|Dv|^2\omega\, dx \\
&\quad \leqslant \int_{B_3} \eta^2 e^{-|b_0|v}\big(|b_0|v^\beta + |\beta|v^{\beta-1}\big)|Dv|^2\omega\, dx \\
&\quad \leqslant \lambda \int_{B_3} \eta^2 e^{-|b_0|v}\big(|b_0|v^\beta + |\beta|v^{\beta-1}\big)a_{ij}v_{x_i}v_{x_j}\, dx \\
&\quad \leqslant \lambda \int_{B_3} \eta^2 e^{-|b_0|v}\big(|\beta|v^{\beta-1} + |b_0|v^\beta\big)(h_j - d_j w)v_{x_j}\, dx \\
&\qquad + 2\lambda \int_{B_3} \eta(a_{ij}v_{x_i} + d_j w - h_j)\eta_{x_j} v^\beta e^{-|b_0|v}\, dx + \int_{B_3} b_0\omega|Dv|^2\eta^2 v^\beta e^{-|b_0|v}\, dx \\
&\qquad + \lambda \int_{B_3} (b_i v_{x_i} + cw - f)\eta^2 v^\beta e^{-|b_0|v}\, dx. \qquad (12)
\end{aligned}$$

From (12) it follows

$$\begin{aligned}
&\int_{B_3} \eta^2 e^{-|b_0|v} |\beta| v^{\beta-1} |Dv|^2 \omega\, dx \\
&\quad \leqslant \lambda \int_{B_3} \eta^2 e^{-|b_0|v} \big(|\beta| v^{\beta-1} + |b_0| v^\beta\big)(h_j - d_j w) v_{x_j}\, dx \\
&\qquad + 2\lambda \int_{B_3} \eta (a_{ij} v_{x_i} + d_j w - h_j) \eta_{x_j} v^\beta e^{-|b_0|v}\, dx \\
&\qquad + \lambda \int_{B_3} (b_i v_{x_i} + cw - f) \eta^2 v^\beta e^{-|b_0|v}\, dx.
\end{aligned}$$

Since v is bounded we may drop the exponential to obtain

$$\begin{aligned}
&\int_{B_3} \eta^2 |\beta| v^{\beta-1} |Dv|^2 \omega\, dx \\
&\quad \leqslant c(M, b_0) \bigg[2\lambda \int_{B_3} \eta a_{ij} v_{x_i} \eta_{x_j} v^\beta\, dx + \lambda |\beta| \int_{B_3} |d_j| |v_{x_i}| v^\beta \eta^2\, dx \\
&\qquad + 2\lambda \int_{B_3} |d_j| v^{\beta+1} \eta_{x_j} \eta\, dx + 2\lambda \int_{B_3} |h_j| v^\beta \eta_{x_j} \eta\, dx + \lambda \int_{B_3} |b_i| |v_{x_i}| \eta^2 v^\beta\, dx \\
&\qquad + \lambda \int_{B_3} c \eta^2 v^{\beta+1}\, dx + \lambda \int_{B_3} |f| \eta^2 v^\beta\, dx \\
&\qquad + \lambda |\beta| \int_{B_3} h_j v_{x_j} v^{\beta-1} \eta^2\, dx + \lambda \int_{B_3} |d_j| |v_{x_j}| \eta^2 v^\beta\, dx \bigg].
\end{aligned}$$

Now, set

$$V = \sum_{i=1}^n \frac{|b_i|^2}{\omega} + |c| + \sum_{j=1}^n \frac{|d_j|^2}{\omega} + k^{-1}|f| + k^{-2} \sum_{i=1}^n \frac{|h_i|^2}{\omega}.$$

Use of Young inequality yields

$$\begin{aligned}
&\int_{B_3} \eta^2 v^{\beta-1} |Dv|^2 \omega\, dx \\
&\quad \leqslant c(M, b_0, \lambda) \bigg[\frac{|\beta|+1}{\beta^2} \int_{B_3} v^{\beta+1} |D\eta|^2 \omega\, dx + \left(\frac{|\beta|+1}{\beta}\right)^2 \int_{B_3} V \eta^2 v^{\beta+1}\, dx \bigg].
\end{aligned}$$

We get the thesis arguing as the proof of Theorem 4.1 in [7]. $\square$

We state the following weak Harnack inequality for subsolutions.

Theorem 3.2. *Let w be a weak nonnegative subsolution of* (10) *in $B_{3r} \Subset \Omega$. Assume* (9) *and* (11). *Let $M > 0$ be a constant such that $w \leqslant M$ in B_{3r}. Then there exists c depending on n, M, λ and the A_2 constant of ω, such that*

$$\max_{B_r} w \leqslant c\left\{\omega^{-1}(B_{2r})\int_{B_{2r}} w\omega\,dx + r^\sigma \left\|\frac{f}{\omega}\right\|_{\sigma,B_{3r}} + \left(r^\sigma \sum_{i=1}^n \left\|\left(\frac{h_i}{\omega}\right)^2\right\|_{\sigma,B_{3r}}\right)^{\frac{1}{2}}\right\}.$$

The proof closely follows the lines of the previous one.

Putting together our previous results we obtain

Theorem 3.3. *Let w be a weak nonnegative solution of* (10) *in $B_{3r} \Subset \Omega$. Assume* (9) *and* (11). *Let M be a constant such that $w \leqslant M$ in B_{3r}. Then there exists c depending on n, M, λ and the A_2 constant of ω such that*

$$\max_{B_r} w \leqslant c\left\{\min_{B_r} w + r^\sigma \left\|\frac{f}{\omega}\right\|_{\sigma,B_{3r}} + \left(r^\sigma \sum_{i=1}^n \left\|\left(\frac{h_i}{\omega}\right)^2\right\|_{\sigma,B_{3r}}\right)^{\frac{1}{2}}\right\}.$$

Our next step is to show a Harnack inequality near the boundary of Ω for weak supersolutions and subsolutions to Eq. (10) when $d_i = 0$ and $c = 0$, namely for the equation

$$-(a_{ij}w_{x_i})_{x_j} + \frac{b_0}{\lambda}\omega|Dw|^2 + b_i w_{x_i} = f - (h_i)_{x_i}. \tag{13}$$

Let B_r be a ball such that $B_{3r} \cap \Omega \neq \emptyset$. We define $f = 0$ and $h_i = 0$ outside Ω. If w is a weak supersolution we set

$$\tilde{w}(x) = \begin{cases} \min\{w, m\} & \text{if } x \in \Omega \cap B_{3r}, \\ m & \text{if } x \in \mathbb{R}^n \setminus (\Omega \cap B_{3r}), \end{cases}$$

where $m = \inf_{\partial\Omega \cap B_{3r}} w$.

Theorem 3.4. *Let $w \in W^{1,2}(\Omega \cap B_{3r}, \omega)$ be a weak nonnegative supersolution of* (13) *in $\Omega \cap B_{3r}$. Assume* (9) *and* (11). *Let M be a constant such that $w \leqslant M$ on $\Omega \cap B_{3r}$. Then there exists c depending on n, M, λ, the A_2 constant of ω such that*

$$\omega^{-1}(B_{2r})\int_{B_{2r}} \tilde{w}\omega\,dx \leqslant c\left\{\min_{B_r} \tilde{w} + r^\sigma \left\|\frac{f}{\omega}\right\|_{\sigma,B_{3r}} + \left(r^\sigma \sum_{i=1}^n \left\|\left(\frac{h_i}{\omega}\right)^2\right\|_{\sigma,B_{3r}}\right)^{\frac{1}{2}}\right\}. \tag{14}$$

Proof. We may assume $r = 1$. Set $k = \|\frac{f}{\omega}\|_{\sigma,B_3} + (\sum_{i=1}^n \|(\frac{h_i}{\omega})^2\|_{\sigma,B_3})^{\frac{1}{2}}$ and $v = \tilde{w} + k$. Let $\eta \in C_0^1(B_3)$ and $\eta \geqslant 0$. For $\beta < 0$ we take $\varphi(x) = \eta^2[v^\beta - (m+k)^\beta]e^{-|b_0|v(x)} \in W_0^{1,2}(B_3, \omega)$ as test function. Since w is a supersolution of (10) we have

$$
\begin{aligned}
&\int_{B_3} \eta^2 e^{-|b_0|v}\big[b_0\big(v^\beta-(m+k)^\beta\big)+|\beta|v^{\beta-1}\big]|Dv|^2\omega\,dx\\
&\quad\leqslant \lambda\int_{B_3}\eta^2 e^{-|b_0|v}\big[|b_0|\big(v^\beta-(m+k)^\beta\big)+|\beta|v^{\beta-1}\big]h_j v_{x_j}\,dx\\
&\qquad+2\lambda\int_{B_3}\eta(a_{ij}v_{x_i}-h_j)\eta_{x_j}\big(v^\beta-(m+k)^\beta\big)e^{-|b_0|v}\,dx\\
&\qquad+\int_{B_3} b_0\omega|Dv|^2\eta^2\big(v^\beta-(m+k)^\beta\big)e^{-|b_0|v}\,dx\\
&\qquad+\lambda\int_{B_3}(b_i v_{x_i}-f)\eta^2\big(v^\beta-(m+k)^\beta\big)e^{-|b_0|v}\,dx.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_{B_3} \eta^2 e^{-|b_0|v}|\beta|v^{\beta-1}|Dv|^2\omega\,dx\\
&\quad\leqslant \lambda\int_{B_3}\eta^2 e^{-|b_0|v}\big[|b_0|\big(v^\beta-(m+k)^\beta\big)+|\beta|v^{\beta-1}\big]h_j v_{x_j}\,dx\\
&\qquad+2\lambda\int_{B_3}\eta(a_{ij}v_{x_i}-h_j)\eta_{x_j}\big(v^\beta-(m+k)^\beta\big)e^{-|b_0|v}\,dx\\
&\qquad+\lambda\int_{B_3}(b_i v_{x_i}-f)\eta^2\big(v^\beta-(m+k)^\beta\big)e^{-|b_0|v}\,dx.
\end{aligned}
$$

The result follows as Theorem 3.1 since $v^\beta-(m+k)^\beta\leqslant v^\beta$. □

Now, let w be a subsolution of (13). We define the function

$$
\overline{w}(x)=\begin{cases}\max\{w,M\} & \text{if } x\in\Omega\cap B_{3r},\\ M & \text{if } x\in\mathbb{R}^n\setminus(\Omega\cap B_{3r}),\end{cases}
$$

where $M=\sup_{\partial\Omega\cap B_{3r}} w$.

Theorem 3.5. *Let $w\in W^{1,2}(\Omega\cap B_{3r},\omega)$ be a weak nonnegative subsolution of* (13) *in $\Omega\cap B_{3r}$. Assume* (9) *and the conditions* (11) *for b_i, f and h_i. Let M be a constant such that $w\leqslant M$ on $\Omega\cap B_{3r}$. Then there exists c depending on n, M, λ and the A_2 constant of ω such that*

$$
\max_{B_r}\overline{w}\leqslant c\left\{\omega^{-1}(B_{2r})\int_{B_{2r}}\overline{w}\omega\,dx+r^\sigma\left\|\frac{f}{\omega}\right\|_{\sigma,B_{3r}}+\left(r^\sigma\sum_{i=1}^n\left\|\left(\frac{h_i}{\omega}\right)^2\right\|_{\sigma,B_{3r}}\right)^{\frac12}\right\}.
$$

4. Global regularity for variational quasilinear equations

In this section we derive local Hölder continuity of weak solutions of (10) as a consequence of Harnack inequality.

Theorem 4.1. *Let w be a locally bounded weak solution of* (10) *in Ω. Assume that* (9) *and* (11) *hold true. Then w is locally Hölder continuous in Ω.*

Proof. Let $\Omega' \Subset \Omega$ and $B_r(x_0) \equiv B_r \subset \Omega'$. Set $M = M(r) = \max_{B_r} w$, $m = m(r) = \min_{B_r} w$. We note that $M - w$ is a nonnegative bounded solution of

$$-(a_{ij} w_{x_i} + d_j w)_{x_j} - \frac{b_0}{\lambda}\omega|Dw|^2 + b_i w_{x_i} + cw = cM - f - (Md_i - h_i)_{x_i}$$

in B_r.

We also note that

$$\frac{Mc - f}{\omega}, \left(\frac{Md_i - f_i}{\omega}\right)^2 \in M_\sigma(\Omega, \omega), \quad \sigma > 0.$$

Moreover

$$\left\|\frac{Mc - f}{\omega}\right\|_{\sigma, B_\rho} \leqslant L\left\|\frac{c}{\omega}\right\|_{\sigma, B_\rho} + \left\|\frac{f}{\omega}\right\|_{\sigma, B_\rho}$$

and

$$\sum_{i=1}^n \left\|\left(\frac{Md_i - f_i}{\omega}\right)^2\right\|_{\sigma, B_\rho} \leqslant 2L^2 \sum_{i=1}^n \left\|\left(\frac{d_i}{\omega}\right)^2\right\|_{\sigma, B_\rho} + 2\sum_{i=1}^n \left\|\left(\frac{f_i}{\omega}\right)^2\right\|_{\sigma, B_\rho}$$

for every $B_\rho \subseteq \Omega'$.

By Harnack inequality

$$\sup_{B_{\frac{r}{3}}}(M - w) \leqslant C\Biggl\{\inf_{B_{\frac{r}{3}}}(M - w) + \left(\frac{r}{3}\right)^\sigma \left(L\left\|\frac{c}{\omega}\right\|_{\sigma,\Omega} + \left\|\frac{f}{\omega}\right\|_{\sigma,\Omega}\right)$$
$$+ \left[\left(\frac{r}{3}\right)^\sigma \left(2L^2 \sum_{i=1}^n \left\|\left(\frac{d_i}{\omega}\right)^2\right\|_{\sigma,\Omega} + 2\sum_{i=1}^n \left\|\left(\frac{f_i}{\omega}\right)^2\right\|_{\sigma,\Omega}\right)\right]^{\frac{1}{2}}\Biggr\}$$

and then

$$M(r) - m\left(\frac{r}{3}\right) \leqslant C\left\{M(r) - M\left(\frac{r}{3}\right) + Hr^{\frac{\sigma}{2}}\right\}, \tag{15}$$

where we put

$$H = \left(\frac{1}{3}\right)^{\sigma} R^{\frac{\sigma}{2}} \left(L \left\| \frac{c}{\omega} \right\|_{1,\sigma,\Omega} + \left\| \frac{f}{\omega} \right\|_{1,\sigma,\Omega} \right)$$
$$+ \left(\frac{1}{3}\right)^{\sigma} \left(2L^2 \sum_{i=1}^{n} \left\| \left(\frac{d_i}{\omega}\right)^2 \right\|_{\sigma,\Omega} + 2 \sum_{i=1}^{n} \left\| \left(\frac{f_i}{\omega}\right)^2 \right\|_{\sigma,\Omega} \right)^{\frac{1}{2}}.$$

Arguing in the same way we obtain

$$M\left(\frac{r}{3}\right) - m(r) \leqslant C\left\{ m\left(\frac{r}{3}\right) - m(r) + Hr^{\frac{\sigma}{2}} \right\}, \tag{16}$$

where C and H are as in (15).

Adding (15) and (16) we get

$$M\left(\frac{r}{3}\right) - m\left(\frac{r}{3}\right) \leqslant \frac{C-1}{C+1}\left[M(r) - m(r)\right] + \frac{2C}{C+1} Hr^{\frac{\sigma}{2}}.$$

Set, for $\rho > 0$

$$\phi(\rho) = M(\rho) - m(\rho),$$
$$\theta = \frac{C-1}{C+1}$$

and

$$K = \frac{2C}{C+1} H,$$

we have

$$\phi\left(\frac{r}{4}\right) \leqslant \phi\left(\frac{r}{3}\right) \leqslant \theta\phi(r) + Kr^{\frac{\sigma}{2}}, \quad 0 < r < R,$$

and the conclusion follows by Lemma 5.1 in [7]. □

In order to get regularity up to the boundary of the domain we need some geometric assumptions.

Definition 4.1. Let Ω be a domain in $\mathbb{R}^n$ and $x_0 \in \partial\Omega$. We say that Ω satisfies the condition A_ω at x_0 if there exist positive constants R_0 and A such that

$$\frac{\omega(B_r(x_0) \setminus \Omega)}{\omega(B_r(x_0))} \geqslant A, \quad 0 < r < R_0.$$

We say that Ω satisfies the condition A_ω if it satisfies the condition at any point of the boundary.

Condition A_ω has been already considered in [2]. In the case $\omega = 1$ the A_ω condition gives back the outer sphere condition.

Using the geometric assumption A_ω we give an estimate for the oscillation of solutions near the boundary.

Theorem 4.2. *Let Ω be a domain satisfying the A_ω condition at $x_0 \in \partial\Omega$. Let $w \in W^{1,2}(\Omega,\omega)$ be a locally bounded weak solution of* (13)*. Then, there exists $R_0 > 0$ such that for any ball $B_r(x_0)$, with $0 < r \leqslant R_0$ we have*

$$\operatorname*{osc}_{B_r\cap\Omega} w \leqslant c\Big\{r^\alpha\Big(R_0^{-\alpha}\sup_{B_{R_0}\cap\Omega}|w|\Big) + \operatorname*{osc}_{B_{\sqrt{rR_0}}\cap\partial\Omega} w + (rR_0)^{\sigma/4}H\Big\},$$

where R_0 is the number in Definition 4.1 *and $H = \|\frac{f}{\omega}\|_{\sigma,\Omega} + (\sum_{i=1}^n \|(\frac{h_i}{\omega})^2\|_{\sigma,\Omega})^{\frac{1}{2}}$.*

Proof. Let $M(\rho) = \sup_{B_\rho\cap\Omega} w$ and $m(\rho) = \inf_{B_\rho\cap\Omega} w$. Let $r \leqslant R_0/3$, the functions $M(3r) - w$ and $w - m(3r)$ are supersolutions of (13), then by (14) and A_ω condition, we have

$$M(4r) - M \leqslant c\big[M(4r) - M(r) + r^{\sigma/2}H\big]$$

and

$$m - m(4r) \leqslant c\big[m(r) - m(4r) + r^{\sigma/2}H\big],$$

where $M = \sup_{B_{4r}\cap\partial\Omega} w$ and $m = \inf_{B_{4r}\cap\partial\Omega} w$.

By addition we obtain

$$M(r) - m(r) \leqslant \theta\big[M(4r) - m(4r)\big] + M - m + cr^{\sigma/2}H,$$

from which applying Lemma 8.23 in [5] we obtain the thesis. □

By the interior regularity Theorems 4.1 and 4.2 we get

Corollary 4.1. *Let Ω be a domain satisfying A_ω condition. Let w be a bounded solution of* (13) *in Ω. Assume that $\operatorname{osc}_{B_r(x_0)\cap\partial\Omega} w \to 0$, as $r \to 0$ for all $x_0 \in \partial\Omega$. Then w is uniformly continuous in Ω.*

5. Interior regularity for nonvariational quasilinear equations

In this section we prove Hölder continuity estimates for the first derivatives of $W^{2,2}(\Omega,\omega)$ solutions of the following equation

$$Qu = a^{ij}(x,u,Du)u_{x_ix_j} + b(x,u,Du) = 0 \quad \text{in } \Omega. \tag{17}$$

We assume that the functions $a^{ij}(x,u,p)$, $b(x,u,p)$ are differentiable in $\Omega\times\mathbb{R}\times\mathbb{R}^n$ and the following degenerate ellipticity condition to hold true:

$$\exists\lambda > 0\text{: for a.e. } x \in \Omega,\ \forall u \in \mathbb{R},\ \forall p \in \mathbb{R}^n \text{ and } \forall \xi \in \mathbb{R}^n$$
$$\lambda^{-1}\omega(x)|\xi|^2 \leqslant \sum_{i,j=1}^{n} a^{ij}(x,u,p)\xi_i\xi_j \leqslant \lambda\omega(x)|\xi|^2. \tag{18}$$

Theorem 5.1. *Let $u \in W^{2,2}_{\mathrm{loc}}(\Omega,\omega)$ be a solution of Eq.* (17). *We set*

$$f(x) = \sup\bigl\{\bigl|a^{ij}_u\bigl(x,u(x),Du(x)\bigr)\bigr|, \bigl|a^{ij}_x\bigl(x,u(x),Du(x)\bigr)\bigr|, \bigl|b\bigl(x,u(x),Du(x)\bigr)\bigr|\bigr\}$$

and assume that $(\frac{f}{\omega})^2 \in M_\sigma(\Omega,\omega)$ for some $\sigma > 0$.

Let B_r be a ball in Ω. Assume that there exist M and K such that $|Du| \leqslant M$ and $\frac{|a^{ij}_p(x,u(x),Du(x))|}{\omega(x)} \leqslant K$ in B_r. Then there exists $0 < \alpha < 1$ such that $|Du|$ is α-Hölder continuous in B_r and, for any $0 < \rho < r$ we have

$$\operatorname*{osc}_{B_\rho} u_{x_l} \leqslant c\left(\frac{\rho}{r}\right)^{\alpha}, \quad l = 1,2,\dots,n,$$

where $\alpha > 0$ depends on λ, n, M, K and the A_2 constant of ω.

Proof. Let u be a $W^{2,2}_{\mathrm{loc}}(\Omega,\omega)$ solution of (17) and $k = 1,2,\dots,n$, we have

$$\int_{B_r} \bigl(a^{ij}(x,u,Du)u_{x_ix_j} + b(x,u,Du)\bigr)\varphi_{x_k} = 0 \quad \forall\varphi \in W^{1,2}_0(B_r,\omega). \tag{19}$$

By density argument we may assume $u \in C^3(B_r)$. Then

$$\int_{B_r} a^{ij}(x,u,Du)u_{x_ix_j}\varphi_{x_k}\,dx = \int_{B_r} \bigl(-a^{ij}_{x_k}u_{x_ix_j}\varphi - a^{ij}_u u_{x_k}u_{x_ix_j}\varphi - a^{im}_{p_j}u_{x_jx_k}u_{x_ix_m}\varphi - a^{ij}u_{x_ix_jx_k}\varphi\bigr)\,dx \tag{20}$$

and

$$-\int_{B_r} a^{ij}u_{x_ix_jx_k}\varphi\,dx = \int_{B_r} \bigl(a^{ij}u_{x_jx_k}\varphi_{x_i} + a^{ij}_{x_i}u_{x_jx_k}\varphi + a^{ij}_u u_{x_i}u_{x_jx_k}\varphi + a^{ij}_{p_m}u_{x_mx_i}u_{x_jx_k}\varphi\bigr)\,dx, \tag{21}$$

from (20) and (21) it follows

$$\int_{B_r} \bigl\{a^{ij}u_{x_kx_j}\varphi_{x_i} + \bigl(a^{ij}_m u_{x_mx_i}u_{x_jx_k} + a^j u_{x_kx_j} + b^{ij}_k u_{x_ix_j}\bigr)\varphi + b\varphi_{x_k}\bigr\}\,dx = 0, \tag{22}$$

where

$$a_m^{ij} = a_{p_m}^{ij}(x, u, Du) - a_{p_j}^{im}(x, u, Du),$$
$$a^j = a_u^{ij}(x, u, Du)u_{x_i} + a_{x_i}^{ij}(x, u, Du),$$
$$b_k^{ij} = -a_u^{ij}(x, u, Du)u_{x_k} - a_{x_k}^{ij}(x, u, Du).$$

Let $\eta \geqslant 0$, $\eta \in C_0^1(B_r)$, we choose $\varphi = u_{x_k}\eta(x)$, as test function in (22) and we get

$$\int_{B_r} \left\{ a^{ij} u_{x_k x_i} u_{x_k x_j} \eta + \frac{1}{2} a^{ij} v_{x_j} \eta_{x_i} + \frac{1}{2} a_m^{ij} u_{x_m x_i} v_{x_j} \eta + \frac{1}{2} a^j v_{x_j} \eta + b_k^{ij} u_{x_i x_j} u_{x_k} \eta + b \Delta u \eta + b u_{x_k} \eta_{x_k} \right\} dx = 0, \tag{23}$$

where $v = |Du|^2$.

Set $w_l^+ = \gamma u_{x_l} + v$ and $w_l^- = -\gamma u_{x_l} + v$ for $l = 1, \dots, n$ and $\gamma = 10nM$. Now we denote by w the function w_l^+. After substituting $k = l$ and $\varphi = \eta$ from (23) and (22) we obtain

$$\int_{B_r} \left\{ a^{ij} u_{x_i x_k} u_{x_k x_j} \eta + \left(\frac{1}{2} a^{ij} w_{x_j} + b u_{x_i} + \frac{1}{2} \gamma b \delta_i^l \right) \eta_{x_i} \right\} dx$$
$$= - \int_{B_r} \left\{ \frac{1}{2} a_m^{ij} u_{x_m x_i} w_{x_j} + \frac{1}{2} a^j w_{x_j} + \left(\frac{1}{2} \gamma b_l^{ij} + b_k^{ij} u_{x_k} + b \delta_i^j \right) u_{x_i x_j} \right\} \eta \, dx.$$

Then

$$\int_{B_r} \left(a^{ij} w_{x_j} + 2b u_{x_i} + \gamma b \delta_i^l \right) \eta_{x_i} \, dx$$
$$\leqslant \int_{B_r} \left\{ -2\lambda^{-1} \omega \left| D^2 u \right|^2 - a_m^{ij} u_{x_m x_i} w_{x_j} - a^j w_{x_j} - \left(\gamma b_{ij}^l + 2 b_{ij}^k u_{x_k} + 2b \delta_i^j \right) u_{x_i x_j} \right\} \eta \, dx$$
$$\leqslant \int_{B_r} \left\{ -2\lambda^{-1} \omega \left| D^2 u \right|^2 + \left(\sum_{m,i} \left(a_m^{ij} w_{x_j} \right)^2 \right)^{\frac{1}{2}} \left| D^2 u \right| + |Dw| f + \left| D^2 u \right| f \right\} \eta \, dx$$
$$\leqslant \int_{B_r} \left\{ -2\lambda^{-1} \omega \left| D^2 u \right|^2 + \lambda^{-1} \omega \left| D^2 u \right|^2 + \frac{\lambda}{\omega} \sum_{m,i} \left(a_m^{ij} w_{x_j} \right)^2 + \lambda^{-1} \omega |Dw|^2 + \frac{2\lambda f^2}{\omega} + \lambda^{-1} \omega \left| D^2 u \right|^2 \right\} \eta \, dx$$
$$\leqslant c(K, \lambda) \int_{B_r} \left\{ \omega |Dw|^2 + \frac{f^2}{\omega} \right\} \eta \, dx. \tag{24}$$

From (24) the function $w(x)$ is a subsolution in B_r to the following equation

$$-(\tilde{a}_{ij} w_{x_i})_{x_j} - c(K, \lambda)\omega |Dw|^2 = \frac{f^2}{\omega} - \bigl(F_i(x)\bigr)_{x_i}, \tag{25}$$

where

$$\tilde{a}_{ij}(x) = a^{ij}\bigl(x, u(x), Du(x)\bigr)$$

and

$$F_i(x) = -2b\bigl(x, u(x), Du(x)\bigr) u_{x_i}(x) - \gamma b\bigl(x, u(x), Du(x)\bigr)\delta_i^l.$$

Note that $F_i \leqslant c(M) f$, $i = 1, 2, \ldots, n$, that implies $(\frac{F_i}{\omega})^2 \in M_\sigma(B_r)$ and

$$\left\| \left(\frac{F_i}{\omega}\right)^2 \right\|_{M_\sigma(B_r)} \leqslant \left\| \left(\frac{f}{\omega}\right)^2 \right\|_{M_\sigma(\Omega)}.$$

The function w_l^- satisfies an inequality similar to (24).

Now, fix $0 < \rho < \min\{1, \frac{1}{3} r\}$ and choose $r \leqslant n$ such that

$$\operatorname*{osc}_{B_{3\rho}} u_{x_r} \geqslant \operatorname*{osc}_{B_{3\rho}} u_{x_l} \quad \forall l = 1, 2, \ldots, n.$$

We have ($w^+ = w_r^+$, $w^- = w_r^-$)

$$\begin{aligned} \operatorname*{osc}_{B_{3\rho}} w^+ &\leqslant \operatorname*{osc}_{B_{3\rho}} (10nMu_{x_r}) + \operatorname{osc}_{B_{3\rho}} |Du|^2 \\ &\leqslant 10nM \operatorname*{osc}_{B_{3\rho}} u_{x_r} + \operatorname*{osc}_{B_{3\rho}} \left(\sum_{i=1}^n u_{x_i}^2\right) \\ &\leqslant 10nM \operatorname*{osc}_{B_{3\rho}} u_{x_r} + 2M \operatorname*{osc}_{B_{3\rho}} \left(\sum_{i=1}^n u_{x_i}\right) \leqslant 12nM \operatorname*{osc}_{B_{3\rho}} u_{x_r} \end{aligned} \tag{26}$$

and

$$\operatorname*{osc}_{B_{3\rho}} w^+ \geqslant 10nM \operatorname*{osc}_{B_{3\rho}} u_{x_r} - \operatorname{osc}_{B_{3\rho}} \left(\sum_{i=1}^n u_{x_i}^2\right) \geqslant 8nM \operatorname*{osc}_{B_{3\rho}} u_{x_r}.$$

In the same way

$$8nM \operatorname*{osc}_{B_{3\rho}} u_{x_r} \leqslant \operatorname*{osc}_{B_{3\rho}} w^- \leqslant 12nM \operatorname{osc}_{B_{3\rho}} u_{x_r}.$$

In order to estimate the oscillation of w^+ and w^-, set $W^+ = \sup_{B_{3\rho}} w_r^+$ and $W^- = \sup_{B_{3\rho}} w_r^-$. The functions $W^+ - w^+$ and $W^- - w^-$ are supersolutions of (25) and then, by Theorem 3.1 (writing $W - w$ instead of $W^+ - w^+$ or $W^- - w^-$) we easily get

$$\omega^{-1}(B_{2\rho})\int_{B_{2\rho}}(W-w)\omega\,dx \leqslant c\left(W-\sup_{B_\rho}w+\rho^\sigma\left\|\left(\frac{f}{\omega}\right)^2\right\|_{\sigma,B_r}+\left(\rho^\sigma\sum_{i=1}^n\left\|\left(\frac{F_i}{\omega}\right)^2\right\|_{\sigma,B_r}\right)^{\frac{1}{2}}\right)$$
$$\leqslant c\left(W-\sup_{B_\rho}w+\rho^{\sigma/2}L\right), \tag{27}$$

where L is a constant depending on $\|(\frac{f}{\omega})^2\|_{\sigma,\Omega}$.

As a consequence of (26) we have

$$\begin{aligned}\sum_{+,-}\left(W^\pm-w^\pm\right)&=\sup_{B_{3\rho}}w^++\sup_{B_{3\rho}}w^--2v\\&\geqslant \sup_{B_{3\rho}}w^++\sup_{B_{3\rho}}w^--2\sup_{B_{3\rho}}v\\&\geqslant \sup_{B_{3\rho}}(10nMu_{x_r})-\inf_{B_{3\rho}}(10nMu_{x_r})+2\inf_{B_{3\rho}}v-2\sup_{B_{3\rho}}v\\&\geqslant 10nM\operatorname*{osc}_{B_{3\rho}}u_{x_r}-4nM\operatorname*{osc}_{B_{3\rho}}u_{x_r}\geqslant\frac{1}{2}\operatorname*{osc}_{B_{3\rho}}w^\pm\quad\forall x\in B_{3\rho}.\end{aligned}\tag{28}$$

The following inequality holds true for w^+ or w^-. Let us write if it is true for w^+

$$\frac{1}{4}\operatorname*{osc}_{B_{3\rho}}w^+\leqslant\inf_{B_{2\rho}}\left(W^+-w^+\right)\leqslant\omega^{-1}(B_{2\rho})\int_{B_{2\rho}}\left(W^+-w^+\right)\omega\,dx$$

by (27) we give an estimate for the oscillation of w^+, i.e.

$$\operatorname*{osc}_{B_{3\rho}}w^+\leqslant c\left(W^+-\sup_{B_\rho}w^++\rho^{\sigma/2}L\right)\leqslant c\left(\operatorname*{osc}_{B_{3\rho}}w^+-\operatorname*{osc}_{B_\rho}w^++\rho^{\sigma/2}L\right),$$

from which

$$\operatorname*{osc}_{B_\rho}w^+\leqslant(1-1/c)\operatorname*{osc}_{B_{3\rho}}w^++\rho^{\sigma/2}L.$$

Now we can apply Lemma 5.1 in [7]. There exist two positive constants $\alpha<1$ and k such that

$$\operatorname*{osc}_{B_\rho}w^+\leqslant k\rho^\alpha,$$

from which

$$\operatorname*{osc}_{B_\rho}u_{x_i}\leqslant c\rho^\alpha\quad\forall i=1,\dots,n.\qquad\square$$

6. Boundary estimates for nonvariational quasilinear equations

Let $\Omega \subset \mathbb{R}^n$ be of class $C^{1,1}$ and consider the operator Q satisfying the condition (18) in Ω. Let $u \in W^{2,2}(\Omega, \omega)$ be a solution of $Qu = 0$ such that $u = 0$ on $\partial\Omega$.

Theorem 6.1. *Let $\partial\Omega$ be of class $C^{1,1}$, $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \omega)$ be a solution of* (17) *such that $u = 0$ in $\partial\Omega$. Suppose that, set*

$$f(x) = \sup\{\left|a_u^{ij}\big(x, u(x), Du(x)\big)\right|, \left|a_x^{ij}\big(x, u(x), Du(x)\big)\right|, \left|b\big(x, u(x), Du(x)\big)\right|\},$$

$(\frac{f}{\omega})^2 \in M_\sigma(\Omega, \omega)$.

Moreover, if $\forall x \in \partial\Omega$ there exists a ball $B = B_R(x)$ such that $|Du| \leqslant M$ and $\frac{|a_p^{ij}(x,u(x),Du(x))|}{\omega(x)} \leqslant K$ in $B \cap \Omega$, then Du is Hölder continuous in $B \cap \Omega$.

Proof. By the hypothesis on $\partial\Omega$, it suffices to consider Eq. (17) in the neighborhood B of a flat boundary portion of Ω such that the hypothesis of the theorem holds true. Let $B^+ = B \cap \Omega \subset \mathbb{R}^n_+$ and $B \cap \partial\Omega \subset \partial\mathbb{R}^n_+$. In B^+ define $v' = \sum_{i=1}^{n-1} |u_{x_i}|^2$, $w = \gamma u_{x_l} + v'$, with $l = 1, 2, \ldots, n-1$ and $\gamma \in \mathbb{R}$. It results $w = 0$ on $B \cap \partial\Omega$ and, as in Section 5, w satisfies an inequality as (24), that implies that w is a subsolution in B^+ of the equation

$$-(\tilde{a}_{ij} w_{x_i})_{x_j} - \frac{b_0}{\lambda}\omega|Dw|^2 = \frac{f^2}{\omega} - \big(F_i(x)\big)_{x_i}. \tag{29}$$

Take $B_r(y) \subset B$ with $y \in B \cap \partial\Omega$, $\rho \leqslant \frac{1}{3}r$ and choose r such that for $l = 1, 2, \ldots, n-1$, $\operatorname{osc}_{B_{3\rho}(y)\cap B^+} u_{x_r} \geqslant \operatorname{osc}_{B_{3\rho}(y)\cap B^+} u_{x_l}$. Set $W = \sup_{B_{3\rho}(y)\cap B^+} w$, as (28) we obtain the estimate (for definition of $\widetilde{W - w}$ see Section 3)

$$\begin{aligned}
\omega^{-1}\big(B_{2\rho}(y)\big) \int\limits_{B_{2\rho}(y)} (\widetilde{W - w})\omega\,dx &\geqslant \omega^{-1}\big(B_{2\rho}(y)\big) \int\limits_{B_{2\rho}(y)\cap B^+} (\widetilde{W - w})\omega\,dx \\
&\geqslant \inf_{B_{2\rho}(y)\cap B^+} (\widetilde{W - w}) \frac{\omega(B_{2\rho}(y)\cap B^+)}{\omega(B_{2\rho}(y))} \\
&\geqslant c \inf_{B_{2\rho}(y)\cap B^+} (\widetilde{W - w}) \geqslant c \inf_{B_{3\rho}(y)\cap B^+} (\widetilde{W - w}) \\
&\geqslant \operatorname*{osc}_{B^+\cap B_{3\rho}(y)} w,
\end{aligned}$$

since, if $\bar{y} = (y_1, y_2, \ldots, \rho/2)$ it results

$$\omega\big(B_{2\rho}(y) \cap B^+\big) \geqslant \omega\big(B_\rho(\bar{y})\big) \geqslant c\omega\big(B_{4\rho}(\bar{y})\big) \geqslant c\omega\big(B_{2\rho}(y)\big).$$

Then, using Theorem 3.4 we obtain that for any ball $B_r(y) \subset B$ with $y \in B \cap \partial\Omega$ and $\rho \leqslant r$

$$\operatorname*{osc}_{B^+\cap B_\rho(y)} u_{x_i} \leqslant c\rho^\alpha, \quad i = 1, 2, \ldots, n-1.$$

Now, let $B' \Subset B$, $d = \operatorname{dist}(B' \cap B^+, \partial B)$, $x_0 \in B' \cap B^+$, $\rho \leqslant d/3$ and $\eta \in C_0^1(B_{2\rho}(x_0))$ such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in $B_\rho(x_0)$, and $|D\eta| \leqslant 2/\rho$. Let $C = \inf_{B_{2\rho}(x_0)} w$ if $B_{2\rho}(x_0) \subset B^+$ and $C = 0$ if $B_{2\rho}(x_0) \cap \partial\Omega$. Consider the function $\varphi = \eta^2 \sup(w - C, 0) e^{|b_0| \sup(w-C,0)} = \eta^2 v e^{|b_0| v}$, it results $\varphi \geqslant 0$ in B^+, $\varphi \in W_0^{1,2}(B^+)$.

We prove the inequality

$$\int_{\{x \in B^+ :\, w \geqslant C\}} \eta^2 |Dw|^2 \omega \, dx \leqslant c \int_{B^+} \left[\eta^2 + |D\eta|^2 v^2\right] \omega \, dx.$$

Since w is a subsolution in B^+ of (29) we have

$$\int_{B^+} \left[(a_{ij} w_{x_i} - F_j)\varphi_{x_j} - \frac{b_0}{\lambda} \omega |Dw|^2 \varphi\right] dx \leqslant \int_{B^+} f \varphi \, dx,$$

that is

$$\int_{B^+} (a_{ij} w_{x_i} - F_j)\left[2\eta \eta_{x_j} v e^{|b_0| v} + \eta^2 e^{|b_0| v}\left(1 + |b_0| v\right) v_{x_j}\right] dx$$
$$\leqslant \int_{B^+} \left(\frac{b_0}{\lambda} \omega |Dw|^2 + f\right) \eta^2 v e^{|b_0| v} \, dx.$$

Then

$$\begin{aligned}
&\int_{B^+} \eta^2 e^{|b_0| v} (b_0 v + 1) |Dv|^2 \omega \, dx \\
&\quad \leqslant \int_{B^+} \eta^2 e^{|b_0| v} \left(|b_0| v + 1\right) |Dv|^2 \omega \, dx \\
&\quad \leqslant \int_{B^+} \eta^2 e^{|b_0| v} \left(|b_0| v + 1\right) a_{ij} w_{x_i} w_{x_j} \, dx \\
&\quad \leqslant \lambda \int_{B^+} \left[(F_j - a_{ij} w_{x_i}) 2\eta \eta_{x_j} v e^{|b_0| v} + \eta^2 F_j \left(1 + |b_0| v\right) v_{x_j} e^{|b_0| v}\right] dx \\
&\qquad + \int_{B^+} b_0 \omega |Dw|^2 \eta^2 v e^{|b_0| v} \, dx + \lambda \int_{B^+} f \eta^2 v e^{|b_0| v} \, dx,
\end{aligned}$$

from which and since v is bounded

$$\int_{B^+} \eta^2 |Dv|^2 \omega \, dx \leqslant c \int_{B^+} \left[(F_j - a_{ij} w_{x_i}) \eta \eta_{x_j} v + \eta^2 F_j \left(1 + |b_0| v\right) v_{x_j}\right] dx + c \int_{B^+} f \eta^2 v \, dx$$

$$\leqslant c(M,b_0)\int_{B^+}\left[\eta|Dw||D\eta|v\omega+F_j\eta\eta_{x_j}v+\eta^2F_jv_{x_j}+\eta^2f\right]dx$$

$$\leqslant \epsilon\int_{B^+}\eta^2|Dv|^2\omega\,dx+c(M,b_0)\int_{B^+}|D\eta|^2v^2\omega+\int_{B^+}\eta^2\left(\frac{|F|^2}{\omega}+f\right)dx.$$

Using Theorem 2.2 we obtain

$$\int_{\{x\in B^+:\ w\geqslant C\}}\eta^2|Dw|^2\omega\,dx\leqslant c(M,b_0,k)\int_{B^+}\left[\eta^2+|D\eta|^2v^2\right]\omega\,dx.$$

Now, if $B_{2\rho}\subset B^+$ we have

$$\int_{\{x\in B_\rho\cap B^+:\ w\geqslant C\}}|Dw|^2\omega\,dx\leqslant c\omega\big(B^+\cap B_{2\rho}\big)\left(1+\frac{1}{\rho^2}\sup_{B^+\cap B_{2\rho}}v^2\right)$$

$$\leqslant c\omega\big(B^+\cap B_{2\rho}\big)\left(1+\frac{1}{\rho^2}\Big(\operatorname*{osc}_{B^+\cap B_{2\rho}}w\Big)^2\right)\leqslant c\omega(B_{2\rho})\rho^{2\alpha-2}.$$

Consider $\gamma=1$ and $\gamma=0$ to obtain for $r=1,\dots,n-1$

$$\int_{B_\rho}|Du_{x_r}|^2\omega\,dx\leqslant 2\int_{B_\rho}\big(|Dv'|^2+|Dw|^2\big)\omega\,dx\leqslant c\omega(B_{2\rho})\rho^{2\alpha-2}.$$

If $B_{2\rho}\cap\partial\Omega\neq\emptyset$ take $z\in B_{2\rho}\cap\partial\Omega$ to obtain

$$\int_{\{x\in B_\rho\cap B^+:\ w\geqslant 0\}}|Dw|^2\omega\,dx\leqslant c\int_{B^+}\left[\eta^2+|D\eta|^2\big(w(x)-w(z)\big)^2\right]\omega\,dx$$

$$\leqslant c\omega\big(B^+\cap B_{2\rho}\big)\left(1+\frac{1}{\rho^2}\Big(\operatorname*{osc}_{B^+\cap B_{2\rho}}w\Big)^2\right)$$

$$\leqslant c\omega(B_{2\rho})\rho^{2\alpha-2},$$

from which for $r=1,\dots,n-1$

$$\int_{B_\rho\cap B^+}|Du_{x_r}|^2\omega\,dx\leqslant 2\int_{B_\rho}\big(|Dv'|^2+|Dw|^2\big)\omega\,dx\leqslant c\omega(B_{2\rho})\rho^{2\alpha-2}.$$

Then we have proved $\forall y\in B'\cap B^+$, $\rho<d/3$ and $r=1,\dots,n-1$

$$\left(\int_{B_\rho\cap B^+}|Du_{x_r}|^2\omega\,dx\right)^{1/2}\leqslant c\big(\omega(B_{2\rho})\big)^{1/2}\rho^{\alpha-1}. \tag{30}$$

Moreover, since $u_{x_n x_n} = -\frac{1}{a_{nn}}(\sum_{(i,j)\neq(n,n)} a_{ij} u_{x_i x_j} + b)$ the estimate (30) holds also for $r = n$. Finally by Hölder inequality, inequality (30), definition of A_2 weights and doubling property of ω

$$\int\limits_{B_\rho \cap B^+} |D^2 u|\,dx \leqslant \left(\int\limits_{B_\rho \cap B^+} |D^2 u|^2 \omega\,dx\right)^{\frac{1}{2}} \left(\int\limits_{B_\rho \cap B^+} \omega^{-1}\,dx\right)^{\frac{1}{2}}$$

$$\leqslant c\rho^{\alpha-1}\omega(B_{2\rho})^{\frac{1}{2}} \left(\int\limits_{B_\rho} \omega^{-1}\,dx\right)^{\frac{1}{2}} \leqslant c(d)\rho^{n+\alpha-1}.$$

Apply Lemma 7.19 in [3] to obtain that Du is Hölder continuous in $B' \cap B^+$. $\square$

Finally we obtain the global estimate

Theorem 6.2. *Let $\partial\Omega$ be of class $C^{1,1}$ and $u \in W^{2,2}(\Omega, \omega)$ be a solution of* (17) *such that $u = 0$ in $\partial\Omega$. If $(\frac{f}{\omega})^2 \in M_\sigma(\Omega, \omega)$ and $\frac{|a_p^{ij}(x, u(x), Du(x))|}{\omega(x)} \leqslant K$ and $|Du| \leqslant M$ in Ω, then Du is Hölder continuous in Ω.*

References

[1] G. Di Fazio, Dirichlet problem. Characterization of regularity, Manuscripta Math. 84 (1994) 47–56.

[2] E. Fabes, C. Kenig, R. Serapioni, The local regularity of solutions of degenerate elliptic equations, Comm. Partial Differential Equations 7 (1) (1982) 77–116.

[3] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 1983.

[4] C.E. Gutierrez, Harnack's inequality for degenerate Schrödinger operators, Trans. Amer. Math. Soc. 312 (1989) 403–419.

[5] N.S. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic equations, Comm. Pure Appl. Math. XX (1967) 721–747.

[6] C. Vitanza, P. Zamboni, Necessary and sufficient conditions for Hölder continuity of solutions of degenerate Schrödinger operators, Matematiche 52 (2) (1997) 393–409.

[7] P. Zamboni, Hölder continuity for solutions of linear degenerate elliptic equations under minimal assumptions, J. Differential Equations 182 (1) (2002) 121–140.