Configuration types and cubic surfaces

Elena Guardo [a], Brian Harbourne [b,*]

[a] *Dipartimento di Matematica e Informatica, Viale A. Doria 6, 95100 Catania, Italy*
[b] *Department of Mathematics, University of Nebraska–Lincoln, Lincoln, NE 68588-0130, USA*

Abstract

This paper is a sequel to the paper [E. Guardo, B. Harbourne, Resolutions of ideals of six fat points in $\mathbf{P}^2$, J. Algebra 318 (2) (2007) 619–640]. We relate the matroid notion of a combinatorial geometry to a generalization which we call a configuration type. Configuration types arise when one classifies the Hilbert functions and graded Betti numbers for fat point subschemes supported at $n \leqslant 8$ essentially distinct points of the projective plane. Each type gives rise to a surface X obtained by blowing up the points. We classify those types such that $n = 6$ and $-K_X$ is nef. The surfaces obtained are precisely the desingularizations of the normal cubic surfaces. By classifying configuration types we recover in all characteristics the classification of normal cubic surfaces, which is well known in characteristic 0 [J.W. Bruce, C.T.C. Wall, On the classification of cubic surfaces, J. London Math. Soc. (2) 19 (2) (1979) 245–256]. As an application of our classification of configuration types, we obtain a numerical procedure for determining the Hilbert function and graded Betti numbers for the ideal of any fat point subscheme $Z = m_1 p_1 + \cdots + m_6 p_6$ such that the points p_i are essentially distinct and $-K_X$ is nef, given only the configuration type of the points $p_1, \ldots, p_6$ and the coefficients m_i.

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1. Introduction

Matroids and combinatorial geometries. The combinatorial classification of points in projective space leads one to the concept of combinatorial geometries. Intuitively, a combinatorial geometry of rank $N + 1$ or less and size n is an abstract specification of linear dependencies among a set of n points spanning a space of dimension at most N. Formally, a *combinatorial geometry* is a matroid without loops or parallel elements [HDCM]. We are interested in the case of n points in the projective plane, in which case one can regard a combinatorial geometry as just being a collection of subsets of the set $\{1, \ldots, n\}$, where each subset has at least two elements and two subsets which have two elements in common must be equal. We say that a given combinatorial geometry is *representable* over a field k (in this paper the field k will be assumed to be algebraically closed, but not necessarily of characteristic 0) if there is a collection of distinct points $p_1, \ldots, p_n \in \mathbf{P}_k^2$ such that a subset $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$ is an element of the combinatorial geometry if and only if $\{p_{i_1}, \ldots, p_{i_r}\}$ is a maximal collinear subset of $\{p_1, \ldots, p_n\}$.

Combinatorial geometries as matrices. We can think of a combinatorial geometry of rank up to 3 and size n with g elements as specifying a correspondence between a set of g lines and n points, where each line is defined by a maximal collinear subset of the points. If we enumerate the lines, then specifying the combinatorial geometry is equivalent to giving a 0–1 matrix M where the entry in row i and column j is a 1 if and only if the ith line contains the jth point. (Although any two points determine a line, it is convenient to ignore any row with exactly two 1s, and so this is what we will do. This does no harm, since if we know all maximal collinear subsets containing more than two points, we can recover all of the maximal two point subsets. Since there typically are a lot of maximal two point subsets, it is impractical to include them in the matrix M.)

Thus we can regard combinatorial geometries on n points in the plane as being matrices M with n columns, such that each entry of each row is either a 0 or a 1, the sum of the entries for a row is always at least 3, and the dot product of two different rows is either 0 or 1. (If no three points are collinear, then the matrix would have no rows.)

An algebraic geometric perspective on combinatorial geometries. A combinatorial geometry can be regarded as telling us when more than the expected number of points are to lie on a given line, the expected number being 2. But in algebraic geometry we are also interested in the possibility of points being special with respect to curves of higher (and lower) degrees. Thus 6 points on a conic are special (in which case the points are special with respect to a curve of degree 2). Similarly, it is special to have one point be infinitely near another (in which case we can regard the points as being special with respect to a curve of degree 0).

For the purposes of this paper, we now want to introduce an alternative, algebraic geometric, approach to combinatorial geometries, which we will then generalize in the case of $n = 6$ points to sets of points being special with respect to curves other than lines. Consider a set of distinct points $p_1, \ldots, p_n \in \mathbf{P}_k^2$ which represents a given combinatorial geometry, and let $\pi : X \to \mathbf{P}_k^2$ be the morphism obtained by blowing up the points. The divisor class group $\mathrm{Cl}(X)$ is the free abelian group on the divisors classes $L, E_1, \ldots, E_n$, where L is the pullback to X of the class of a line on $\mathbf{P}_k^2$ and E_i is the class of $\pi^{-1}(p_i)$. The group $\mathrm{Cl}(X)$ supports an intersection form; it is the bilinear form defined by requiring that the classes $L, E_1, \ldots, E_n$ be orthogonal with $L^2 = 1$ and $E_i^2 = -1$ for all i.

To ignore trivial cases, assume that $n > 1$. If the points $p_{i_1}, \ldots, p_{i_r}$, with $r > 1$, are collinear and none of the other points p_i are on the same line, then the class of the proper transform of the line through those points is $C = L - E_{i_1} - \cdots - E_{i_r}$, so the classes of the proper transforms of the lines corresponding to the elements of the combinatorial geometry are precisely the classes of all prime divisors C on X with $C^2 < 0$ and $C \cdot L = 1$. If we construct a matrix M' by changing the sign of each nonzero entry of M and then prepending a column of 1s to the left side of M, we obtain a matrix M' whose rows specify (in terms of the basis $L, E_1, \ldots, E_n$) the set of all classes of prime divisors C on X such that $C^2 < 0$ and $C \cdot L = 1$ (with classes having $C^2 = -1$ suppressed, corresponding to suppression in M of rows with exactly two 1s in them).

For example, given 5 points $p_1, \ldots, p_5$ such that the maximal collinear subsets (ignoring two point subsets) are points 1, 4 and 5 and points 2, 3 and 4, the matrices M and M' are:

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \qquad M' = \begin{pmatrix} 1 & -1 & 0 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 & -1 & 0 \end{pmatrix}.$$

The rows of M' specify the classes $L - E_1 - E_4 - E_5$ and $L - E_2 - E_3 - E_4$.

Thus we now can regard combinatorial geometries on n points in the plane as being matrices M' with $n + 1$ columns, where the first entry in each row is a 1, each remaining entry is either a 0 or a -1, the sum of the entries of each row (i.e., the intersection product of a row with itself, with respect to the bilinear form defined above) is always at most -2, and the intersection product of two different rows is either 1 or 0.

Infinitely near points, blow ups, the intersection form and exceptional configurations. We now recall the notion of points being infinitely near. Let $\pi : X \to \mathbf{P}^2$ be the morphism obtained as a sequence of blow ups of points in the following way. Let $p_1 \in X_0 = \mathbf{P}^2$, and let $p_2 \in X_1, \ldots, p_n \in X_{n-1}$, where, for $0 \leqslant i \leqslant n - 1$, $\pi_{i+1} : X_{i+1} \to X_i$ is the blow up of p_{i+1}. We will denote X_n by X and the composition $X = X_n \to \cdots \to X_0 = \mathbf{P}^2$ by π. We say that the points $p_1, \ldots, p_n$ are *essentially distinct* points of $\mathbf{P}^2$ [H4]; note for $j > i$ that we may have $\pi_i \circ \cdots \circ \pi_{j-1}(p_j) = p_i$, in which case we say p_j is infinitely near p_i. (If no point is infinitely near another, the points are just distinct points of $\mathbf{P}^2$ and X is just the surface obtained by blowing the points up in a particular order, but the order does not matter. If the points are only essentially distinct, then p_i needs to be blown up before p_j whenever p_j is infinitely near p_i.)

We denote by E_i the class of the 1-dimensional scheme-theoretic fiber of $X = X_n \to X_{i-1}$ over p_i and the pullback to X of the class of a line in $\mathbf{P}^2$ by L. As before, the classes $L, E_1, \ldots, E_n$ form a basis over the integers of the divisor class group $\mathrm{Cl}(X)$, which is a free abelian group of rank $n + 1$. We call such a basis an *exceptional configuration*, which as before is an orthogonal basis for $\mathrm{Cl}(X)$ with respect to the intersection form.

Ordered and unordered configuration types. We saw above that lines through two or more points give rise to classes of prime divisors of negative self-intersection. Similarly, if instead the points $p_{i_1}, \ldots, p_{i_r}$ lie on an irreducible conic and none of the other points lie on that same conic, then the class of the proper transform of that conic is $D = 2L - E_{i_1} - \cdots - E_{i_r}$ and D is the class of a prime divisor of self-intersection $D^2 = 4 - r$, hence negative if $r > 4$. Instead of just lines through 2 or more points, in the context of algebraic geometry what is of interest is more generally the set of all prime divisors of negative self-intersection.

We can formalize this generalized notion as a *configuration type* of points. Up to equivalence, an *ordered* configuration type of n points in the plane is a matrix T with $n + 1$ columns whose

rows satisfy two conditions: negative self-intersection and pairwise nonnegativity (explained below). Two matrices satisfying these two conditions will be regarded as giving the same ordered configuration type if one matrix can be obtained from the other by permuting its rows. (We will say that two matrices satisfying the two conditions will be regarded as giving the same *unordered* configuration type if one matrix can be obtained from the other by permuting either its rows or columns or both.)

The two conditions come from our wanting the rows to specify the coefficients, with respect to the basis $L, E_1, \ldots, E_n$, of classes of prime divisors of negative self-intersection. (Although in general there can be infinitely many classes of prime divisors of negative self-intersection, if $n \leqslant 8$, it is known there are only finitely many. See Lemma 2.1 for the case of interest here, $n = 6$; the case for any $n \leqslant 8$ is similar.)

Thus if $(d, m_1, \ldots, m_n)$ is a row of the matrix T, we require $d^2 - m_1^2 - \cdots - m_n^2 < 0$ (negative self-intersection), and if $(d', m_1', \ldots, m_n')$ is another row of the matrix, we require $dd' - m_1 m_1' - \cdots - m_n m_n' \geqslant 0$ (pairwise nonnegativity), corresponding to an intersection theoretic version of Bezout's theorem, saying that $C \cdot D \geqslant 0$ if C and D are prime divisors with $C \neq D$. We will say a configuration type T is *representable* if there is a set of essentially distinct points $p_1, \ldots, p_n$ giving a surface X such that the rows of T are (in terms of the exceptional configuration $L, E_1, \ldots, E_n$ for X) the classes of all prime divisors of negative self-intersection on X.

Goals and motivation. The goal of this paper is to classify all of the configuration types for $n = 6$ essentially distinct points of $\mathbf{P}^2$ which when blown up give a surface X for which $-K_X$ is nef, and to determine representability for each configuration type. In order to formally write down possible matrices, we must have a set S of possible vectors $(d, m_1, \ldots, m_n)$ to draw from. In principle, S should consist of all coefficient vectors which occur for prime divisors of negative self-intersection for any prime divisor that occurs for any choice of the points p_i. Then we can attempt to write down all possible matrices satisfying the two given conditions (of negative self-intersection and pairwise nonnegativity) where each row is chosen from S. Having written down all possible matrices, we can consider representability: i.e., for which matrices is there an algebraically closed field k and an actual set of points p_i in $\mathbf{P}_k^2$ such that the set of prime divisors on X is exactly that specified by the matrix.

The underlying motivation for carrying out this classification is that, if $n \leqslant 8$, then two sets of points, $p_1, \ldots, p_n$ and $p_1', \ldots, p_n'$, have the same ordered configuration type if and only if, for all choices of nonnegative integers $m_1, \ldots, m_n$, the Hilbert functions of the fat point subschemes $m_1 p_1 + \cdots + m_n p_n$ and $m_1 p_1' + \cdots + m_n p_n'$ are the same [GH]. (We recall the notions of fat points, their ideals and their Hilbert functions in Section 2, and their graded Betti numbers in Section 4.)

Previous work. We classified the configuration types for sets of $n = 6$ distinct points of $\mathbf{P}^2$ in [GH], and we also showed that if $p_1, \ldots, p_n$ and $p_1', \ldots, p_n'$ have the same ordered configuration type, then for any nonnegative integers $m_1, \ldots, m_n$, the graded Betti numbers of the ideals $I(m_1 p_1 + \cdots + m_n p_n)$ and $I(m_1 p_1' + \cdots + m_n p_n')$ defining the fat point subschemes are the same. (In [GH] for efficiency we listed only the unordered configuration types, of which there are 11. These 11 comprise 353 ordered configuration types, but two ordered types with the same unordered type differ only in the indexation of the points. For example, one of the 11 is the situation where 3 points in a set of 6 points is collinear, and otherwise no more than 2 of the 6 points is collinear. There are $\binom{6}{3} = 20$ essentially different ways to number the 6 points, so this

one unordered type comprises 20 ordered types. Thus there is little reason to explicitly list the ordered types, and we will normally only explicitly list unordered types, as was done in [GH].)

Since the graded Betti numbers determine the Hilbert function, and since knowing the Hilbert functions of $m_1 p_1 + \cdots + m_6 p_6$ for all choices of the m_i allows one to determine the set of prime divisors on X of negative self-intersection and hence to recover the configuration type of the points, this shows that a classification of configuration types of 6 distinct points in the plane is actually a classification of the points up to graded Betti numbers (i.e., where we regard two sets of 6 points $p_1, \ldots, p_6$ and $p'_1, \ldots, p'_6$ as equivalent if the graded Betti numbers of $I(m_1 p_1 + \cdots + m_6 p_6)$ and $I(m_1 p'_1 + \cdots + m_6 p'_6)$ are the same for all choices of nonnegative integers m_i).

Results. In this paper we consider the classification of 6 essentially distinct points, but for both technical and practical reasons we do so only under the restriction that the anticanonical divisor $-K_X$ on X is nef. With this restriction, we show that every type is representable over every algebraically closed field k and we show that a classification by type is equivalent to a classification up to graded Betti numbers. We also give an explicit procedure for determining the graded Betti numbers for the ideal $I(Z)$ for any fat point subscheme $Z = m_1 p_1 + \cdots + m_6 p_6$ supported at the six points, given only the coefficients m_i and the ordered configuration type of the points. While this procedure can easily be carried out by hand, an awk script automating the procedure can be run over the web at http://www.math.unl.edu/~bharbourne1/6ptsNef-K/6reswebsite.html. For some examples, see Section 5.

The problem of determining all possible Hilbert functions and graded Betti numbers for arbitrary fat point subschemes $2p_1 + \cdots + 2p_n$, and of determining the configurations of the points that give rise to the different possibilities, was raised in [GMS]. Thus [GH] completely solves the problem for $n = 6$ in the original context of distinct points, not only for double points but for fat point schemes $m_1 p_1 + \cdots + m_6 p_6$ with m_i arbitrary. What we do here likewise completely solves the problem for arbitrary m_i, in the case of 6 essentially distinct points under the condition that $-K_X$ is nef. Indeed, what we find is that there are 90 different unordered configuration types, corresponding to equivalence classes of matrices whose rows are drawn from a certain set S as discussed above and given explicitly in Lemma 2.1. (If we were to remove the restriction that $-K_X$ is nef, we would, in addition to what is specified in Lemma 2.1, also have to include in S the coefficient vectors of all classes of the form $E_i - E_{j_1} - \cdots - E_{j_r}$ for all subsets $\{j_1, \ldots, j_r\} \subsetneq \{1, \ldots, 6\}$ with $r > 1$ and $i < j_1 < \cdots < j_r$, and also all classes of the form $L - E_{i_1} - \cdots - E_{i_l}$ for all $0 < i_1 < \cdots < i_l \leqslant 6$ with $l > 3$. This results in many more configuration types. Having $-K_X$ be nef also affords technical simplifications in computing generators for dual cones given generators for a cone, which we need to do for our method of proving that the graded Betti numbers of $I(m_1 p_1 + \cdots + m_6 p_6)$ depend only on the coefficients m_i and the configuration type of the points p_i.)

The condition that $-K_X$ be nef is fairly reasonable, both algebraically and geometrically. Algebraically, one of the cases of most interest is the uniform case, i.e., cases where the fat point subscheme Z is of the form $Z = mp_1 + \cdots + mp_6$. Also, one typically considers schemes Z only which satisfy the proximity inequalities (see Section 4), and if $-K_X$ is nef, then a uniform Z satisfies the proximity inequalities if and only if $m \geqslant 0$.

Geometrically, the surfaces obtained by blowing up 6 essentially distinct points of $\mathbf{P}^2$ such that $-K_X$ is nef are precisely the surfaces which occur by resolving the singularities of normal cubic surfaces in $\mathbf{P}^3$. Thus this paper can be regarded as a contribution to the long history of work on cubic surfaces. The classification of normal cubic surfaces up to the types of their singularities

(as given by the Dynkin diagrams of the singular points) is classical, at least in characteristic 0 (see [BW]). What is new here is first the (relatively easy) classification of the corresponding configuration types of points in $\mathbf{P}^2$. (Resolving the singularities of a normal cubic surface gives a surface X for which $-K_X$ is nef, but each X typically has several birational morphisms to $\mathbf{P}^2$, and each such morphism gives a set of 6 points in $\mathbf{P}^2$ which when blown up give X. Thus typically several configuration types occur for each Dynkin diagram.) It is much harder to show that the configuration type of the points p_i is enough together with the coefficients m_i to determine the graded Betti numbers of $I(m_1 p_1 + \cdots + m_6 p_6)$. When the points are distinct we showed this in [GH] without requiring $-K_X$ be nef. What is new here is that we show this for points that can be infinitely near, but under the assumption that $-K_X$ is nef. (It was already known, as a consequence of Theorem 8 of [H2], that the configuration type of the points p_i is enough, together with the coefficients m_i, to determine the Hilbert function of $I(m_1 p_1 + \cdots + m_n p_n)$ for any $n \leqslant 8$ essentially distinct points of $\mathbf{P}^2$, whether $-K_X$ is nef or not.)

2. Background

We recall here some of the background we will need on fat points and on surfaces obtained by blowing up essentially distinct points of $\mathbf{P}^2$. We work over an algebraically closed field k of arbitrary characteristic.

A fat point subscheme $Z = m_1 p_1 + \cdots + m_n p_n$ usually is considered in the case that the points $\{p_i\}$ are distinct points. In particular, let $p_1, \ldots, p_n$ be distinct points of $\mathbf{P}^2$. Given nonnegative integers m_i, the fat point subscheme $Z = m_1 p_1 + \cdots + m_n p_n \subset \mathbf{P}^2$ is defined to be the subscheme defined by the ideal $I(Z) = I(p_1)^{m_1} \cap \cdots \cap I(p_n)^{m_n}$, where $I(p_i) \subseteq R = k[\mathbf{P}^2]$ is the ideal generated by all forms (in the polynomial ring R in three variables over the field k) vanishing at p_i. The *support* of Z consists of the points p_i for which m_i is positive. For another perspective, let $\mathcal{I}_Z$ be the sheaf of ideals defining Z as a subscheme of $\mathbf{P}^2$. Now let X be obtained by blowing up the points p_i. Given a divisor F we will denote the corresponding line bundle by $\mathcal{O}_X(F)$. With this convention, $\mathcal{I}_Z = \pi_*(\mathcal{O}_X(-m_1 E_1 - \cdots - m_n E_n))$ and the stalks of $\mathcal{I}_Z$ are complete ideals (as defined in [Z] and [ZS]) in the local rings of the structure sheaf of $\mathbf{P}^2$. We can recover $I(Z)$ from $\mathcal{I}_Z$ since the homogeneous component $I(Z)_t$ of $I(Z)$ of degree t is just $H^0(X, \mathcal{I}_Z(t))$.

We can just as well consider essentially distinct points $p_1, \ldots, p_n \in \mathbf{P}^2$. Again let $\pi : X \to \mathbf{P}^2$ be given by blowing up the points p_i, in order. We define the fat point subscheme $Z = m_1 p_1 + \cdots + m_n p_n$ to be the subscheme whose ideal sheaf is the coherent sheaf of ideals $\pi_*(\mathcal{O}_X(-m_1 E_1 - \cdots - m_n E_n))$. Note that the stalks of $\pi_*(\mathcal{O}_X(-m_1 E_1 - \cdots - m_n E_n))$ are again complete ideals in the stalks of the local rings of the structure sheaf of $\mathbf{P}^2$, and, conversely, if $\mathcal{I}$ is a coherent sheaf of ideals on $\mathbf{P}^2$ whose stalks are complete ideals and if $\mathcal{I}$ defines a 0-dimensional subscheme, then there are essentially distinct points $p_1, \ldots, p_n$ of $\mathbf{P}^2$ and integers m_i such that with respect to the corresponding exceptional configuration we have $\mathcal{I} = \pi_*(\mathcal{O}_X(-m_1 E_1 - \cdots - m_n E_n))$ (see [H6,Z,ZS] for more details). As before we define $I(Z)$ to be the ideal in R given as $I(Z) = \bigoplus_{t \geqslant 0} H^0(X, \mathcal{I}_Z(t))$. The Hilbert function of a homogeneous ideal $I \subseteq R$ is just the function $h_I(t) = \dim I_t$ giving the vector space dimension of the homogeneous component I_t of I as a function of the degree t. The Hilbert function of a fat point subscheme Z will be the function $h_Z(t) = \dim(R/I)_t$ giving the vector space dimension of the homogeneous components of the quotient ring R/I as a function of degree. Note that $h_{I(Z)}(t) + h_Z(t) = \binom{t+2}{2}$. (We recall in Section 4 the notions of the minimal free resolution of $I(Z)$ and its graded Betti numbers.)

Every smooth projective surface X with a birational morphism to $\mathbf{P}^2$ arises as a blow up of n essentially distinct points, where n is uniquely determined by X, since $n+1$ is the rank of $\mathrm{Cl}(X)$ as a free abelian group. Since here we are interested in the case $n = 6$, we will always hereafter assume that $n = 6$. We will also mainly be interested in those X for which the anticanonical class is nef. The anticanonical class has an intrinsic definition, but in terms of an exceptional configuration it is always $3L - E_1 - \cdots - E_n$. A divisor (or divisor class) F being *nef* means that $F \cdot D \geqslant 0$ whenever D is the class of an effective divisor (with *effective* meaning that D is a nonnegative integer linear combination of reduced irreducible curves).

We now recall the connection of normal cubic surfaces with blow ups X of $\mathbf{P}^2$ at 6 essentially distinct points such that $-K_X$ is nef. If $-K_X$ is nef, by Lemma 2.1 the linear system $|-K_X|$ has no base points so it defines a morphism $\phi_{|-K_X|} : X \to \mathbf{P}^3$, whose image is a cubic surface. By Proposition 3.2 of [H3], the image of $\phi_{|-K_X|}$ is normal, obtained by contracting to a point every prime divisor orthogonal to $-K_X$ (i.e., every smooth rational curve of self-intersection -2). In fact, the images of the (-2)-curves are rational double points, and the inverse image of each singular point is a minimal resolution of the singularity. It is not hard to check that the subgroup $K_X^\perp \subsetneq \mathrm{Cl}(X)$ of all divisor classes orthogonal to K_X is negative definite. Thus Theorem 2.7 and Fig. 2.8, both of [A], apply; i.e., the intersection graph of a fiber over a singular point is a Dynkin diagram of type A_i, D_i or E_i. The combinations of Dynkin diagrams that occur for the singularities on a single surface are well known. A determination in characteristic 0 is given in [BW]. We recover that result for any characteristic; see Table 3.1.

To state the next result, let $\mathrm{NEG}(X)$ denote the set of classes of prime divisors of negative self-intersection on a surface X obtained by blowing up 6 essentially distinct points of $\mathbf{P}^2$. Let $\mathcal{B} = \{E_i : i > 0\}$ ($\mathcal{B}$ here is for *blow up* of a point), $\mathcal{V} = \{E_i - E_{i_1} - \cdots - E_{i_r} : r \geqslant 1,\ 0 < i < i_1 < \cdots < i_r \leqslant 6\}$ ($\mathcal{V}$ here is for *vertical*), $\mathcal{L} = \{L - E_{i_1} - \cdots - E_{i_r} : r \geqslant 2,\ 0 < i_1 < \cdots < i_r \leqslant 6\}$ ($\mathcal{L}$ here is for points on a *line*), and $\mathcal{Q} = \{2L - E_{i_1} - \cdots - E_{i_r} : r \geqslant 5,\ 0 < i_1 < \cdots < i_r \leqslant 6\}$ ($\mathcal{Q}$ here is for points on a conic, defined by a *quadratic* equation). Also, let $\mathcal{B}' = \mathcal{B}$, $\mathcal{V}' = \{E_i - E_j : 0 < i < j \leqslant 6\}$, $\mathcal{L}' = \{L - E_i - E_j : 0 < i < j \leqslant 6\} \cup \{L - E_i - E_j - E_k : 0 < i < j < k \leqslant 6\}$, and $\mathcal{Q}' = \mathcal{Q}$, and let $\mathcal{V}'' = \mathcal{V}'$, $\mathcal{L}'' = \{L - E_i - E_j - E_k : 0 < i < j < k \leqslant 6\}$, and $\mathcal{Q}'' = \{2L - E_1 - \cdots - E_6\}$.

Lemma 2.1. *Let X be obtained by blowing up 6 essentially distinct points of $\mathbf{P}^2$. Then the following hold*:

(a) $\mathrm{NEG}(X) \subseteq \mathcal{B} \cup \mathcal{V} \cup \mathcal{L} \cup \mathcal{Q}$, *and every class in $\mathrm{NEG}(X)$ is the class of a smooth rational curve*;

(b) *if moreover $-K_X$ is nef, then $\mathrm{NEG}(X) \subseteq \mathcal{B}' \cup \mathcal{V}' \cup \mathcal{L}' \cup \mathcal{Q}'$;*

(c) *for any nef $F \in \mathrm{Cl}(X)$, F is effective (hence $h^2(X, F) = 0$ by duality), $|F|$ is base point free, $h^0(X, F) = (F^2 - K_X \cdot F)/2 + 1$ and $h^1(X, F) = 0$;*

(d) $\mathrm{NEG}(X)$ *generates the subsemigroup $\mathrm{EFF}(X) \subsetneq \mathrm{Cl}(X)$ of classes of effective divisors*; *and*

(e) *any class F is nef if and only if $F \cdot C \geqslant 0$ for all $C \in \mathrm{NEG}(X)$.*

Proof. This result is well known. A proof of parts (a), (c), (d) and (e) when the points are assumed to be distinct is given in detail in [GH]. The same proof carries over with only minor changes here. Part (b) follows from (a) just by taking into account that each class C in $\mathrm{NEG}(X)$ must satisfy $-K_X \cdot C \geqslant 0$. $\square$

Remark 2.2. In the same way that it is easier to specify a combinatorial geometry of points in the plane by specifying which sets of three or more points are collinear (suppressing mention of all of the pairs of points which define a line going through no other point), it is often easier to work with the set $\text{neg}(X) = \{C \in \text{NEG}(X): C^2 < -1\}$ than with $\text{NEG}(X)$. As shown in Remark 2.2 of [GH], $\text{neg}(X)$ determines $\text{NEG}(X)$. In fact, we have:

$$\text{NEG}(X) = \text{neg}(X) \cup \{C \in \mathcal{B} \cup \mathcal{L} \cup \mathcal{Q} \mid C^2 = -1, \ C \cdot D \geqslant 0 \text{ for all } D \in \text{neg}(X)\}.$$

If $-K_X$ is nef, note that $\text{neg}(X) \subseteq \mathcal{V}'' \cup \mathcal{L}'' \cup \mathcal{Q}''$.

3. Configuration types

In this section we determine the configuration types of 6 essentially distinct points of $\mathbf{P}^2$, under the restriction that $-K_X$ is nef. I.e., we find all *pairwise nonnegative* subsets of $\mathcal{B}' \cup \mathcal{V}' \cup \mathcal{L}' \cup \mathcal{Q}'$ (a pairwise nonnegative subset being a subset such that whenever C and D are distinct elements of the subset, we have $C \cdot D \geqslant 0$). With Remark 2.2 in mind, we actually only do this for subsets of $\mathcal{V}'' \cup \mathcal{L}'' \cup \mathcal{Q}''$. Also, we do this only up to permutations of the classes $E_1, \ldots, E_6$. Thus we find the unordered configuration types, hence only one representative of each orbit under the action of the group of permutations of $E_1, \ldots, E_6$. (Note for example that $\{E_1 - E_3, E_2 - E_4\}$ and $\{E_1 - E_2, E_3 - E_4\}$ are the same up to permutations of the E_i.)

We also show that each configuration type actually occurs over every algebraically closed field (regardless of the characteristic). Both for this latter question of representability and for distinguishing when different pairwise nonnegative subsets T give different configuration types, it is helpful to compute the torsion groups Tors_T for the quotients $\text{Cl}(X)/\langle T \rangle$ of the divisor class group by the subgroup generated by the elements of T (or equivalently, the torsion subgroup of $K_X^{\perp}/\langle T \rangle$). So we include this information in Table 3.1, whenever $\text{Tors}_T \neq 0$.

We can associate a graph (whose connected components are Dynkin diagrams [HDCM]) to each configuration type. If T is a pairwise nonnegative subset of $\mathcal{V}'' \cup \mathcal{L}'' \cup \mathcal{Q}''$, we have the graph G_T, whose vertices are the elements of T and we have $C \cdot D$ edges between each distinct pair of vertices $C, D \in T$. It turns out that there is at most one edge between any two vertices, and, in terms of the standard notation for Dynkin diagrams, the connected components of each G_T are always among the following types: A_i, $1 \leqslant i \leqslant 5$; D_4; D_5 and E_6. (If the graph G_T for a subset T has more than one connected component, say an A_1 and two of type A_2, we write this as $A_1 2A_2$.) Different configuration types can have the same graph, but the torsion subgroup for each configuration type turns out to be determined by the graph. Since different configuration types can have the same graph, the Dynkin diagram (such as $A_1 2A_2$) is not by itself enough to uniquely identify a configuration type, so we distinguish different configuration types with the same graph by appending a lower case letter (for example, $A_1 2A_2 a$ or $A_1 2A_2 b$) when there is more than one configuration type with a given graph.

The 90 different configuration types (i.e., the classification, up to permutations, of the pairwise nonnegative subsets of $\mathcal{V}'' \cup \mathcal{L}'' \cup \mathcal{Q}''$) are shown in Table 3.1. For each configuration type T we give the corresponding graph G_T (using Dynkin notation), we give the set T itself, and, when not 0, we give Tors_T (which is always either 0, $\mathbf{Z}/2\mathbf{Z}$ or $\mathbf{Z}/3\mathbf{Z}$; we denote the latter two by $\mathbf{Z}_2$ and $\mathbf{Z}_3$ in Table 3.1). We give the set T by listing its elements, following the approach used in [BCH]. We use letters A through F to denote the points p_1 through p_6, and numbers to indicate the degree of the curve. For example, the set T for $3A_1 d$ is given as 0: AB, CD; 2: ABCDEF. Thus T consists of the classes $E_1 - E_2$, $E_3 - E_4$ and $2L - E_1 - \cdots - E_6$.

Table 3.1
Configuration types

G_T	T	Tors$_T$	G_T	T	Tors$_T$
1. $\emptyset$			46. A_1A_3d	0: AB, BC; 1: ABC, ADE	
2. A_1a	0: AB		47. A_1A_3e	0: AF, BC; 1: ABC, ADE	
3. A_1b	1: ABC		48. A_1A_3f	0: BC, CD; 1: ABC, DEF	
4. A_1c	2: ABCDEF		49. A_1A_3g	0: BC, CD, EF; 1: ABC	
5. $2A_1a$	0: AB, CD		50. A_1A_3h	0: AB, BC, CD; 2: ABCDEF	
6. $2A_1b$	0: AB; 1: ABC		51. $2A_2a$	0: AB, BC, DE, EF	
7. $2A_1c$	0: DE; 1: ABC		52. $2A_2b$	0: AB, CF; 1: ABC, ADE	
8. $2A_1d$	0: AB; 2: ABCDEF		53. $2A_2c$	0: AB, BC; 1: ABC, DEF	
9. $2A_1e$	1: ABC, ADE		54. A_4a	0: AB, BC, CD, DE	
10. A_2a	0: AB, BC		55. A_4b	0: AB, CD, DE; 1: ACD	
11. A_2b	0: CD; 1: ABC		56. A_4c	0: AB, BC, EF; 1: ADE	
12. A_2c	1: ABC, DEF		57. A_4d	0: CD, DE; 1: ABC	
13. $3A_1a$	0: AB, CD, EF		58. A_4e	0: BC, CD; 1: ABC, BEF	
14. $3A_1b$	0: AB, DE; 1: ABC		59. A_4f	0: AB, BC, CD; 1: ABC	
15. $3A_1c$	0: BC; 1: ABC, ADE		60. D_4a	0: BC, CD, DE; 1: ABC	
16. $3A_1d$	0: AB, CD; 2: ABCDEF		61. D_4b	0: AB, CD, EF; 1: ACE	
17. $3A_1e$	1: ABC, ADE, BDF		62. A_12A_2a	0: AB, BC, DE, EF; 1: ABC	
18. A_1A_2a	0: AB, BC, DE		63. A_12A_2b	0: AB, CF, DE; 1: ABC, ADE	
19. A_1A_2b	0: AB, BC; 1: ABC		64. A_12A_2c	0: AB, BC, DE; 1: ABC, DEF	
20. A_1A_2c	0: AB, BC; 1: DEF		65. A_12A_2d	0: AB, CD; 1: ABC, AEF, CDE	
21. A_1A_2d	0: AB, CD; 1: ABC		66. A_12A_2e	0: AB, BC, DE, EF; 2: ABCDEF	
22. A_1A_2e	0: CD, EF; 1: ABC		67. $2A_1A_3a$	0: BC, CD, EF; 1: ABC, AEF	$\mathbf{Z}_2$
23. A_1A_2f	0: CD; 1: ABC, AEF		68. $2A_1A_3b$	0: AD, CE; 1: ABC, ADF, CEF	$\mathbf{Z}_2$
24. A_1A_2g	0: AB; 1: ABC, ADE		69. $2A_1A_3c$	0: AB, BC, DE; 1: ABC, ADE	$\mathbf{Z}_2$
25. A_1A_2h	0: AB; 1: ABC, DEF		70. $2A_1A_3d$	0: AF, BC, DE; 1: ABC, ADE	$\mathbf{Z}_2$
26. A_1A_2i	0: AB, BC; 2: ABCDEF		71. $2A_1A_3e$	0: BC, CF, DE; 1: ABC, ADE	$\mathbf{Z}_2$
27. A_3a	0: AB, BC, CD		72. $2A_1A_3f$	0: AB, BC, CD, EF; 2: ABCDEF	$\mathbf{Z}_2$
28. A_3b	0: CD, DE; 1: ABC		73. A_1A_4a	0: AB, BC, CD, EF; 1: ABC	
29. A_3c	0: AB, DE; 1: ACD		74. A_1A_4b	0: AB, CD, DE, EF; 1: ABC	
30. A_3d	0: AF; 1: ABC, ADE		75. A_1A_4c	0: AB, DE, EF; 1: ABC, ADE	
31. A_3e	0: BC, CD; 1: ABC		76. A_1A_4d	0: AB, BF, DE; 1: ABC, ADE	
32. $4A_1a$	0: BC, DE; 1: ABC, ADE	$\mathbf{Z}_2$	77. A_1A_4e	0: AB, BC, EF; 1: ABC, ADE	
33. $4A_1b$	0: AB, CD, EF; 2: ABCDEF	$\mathbf{Z}_2$	78. A_1A_4f	0: AB, BC, CD, DE; 2: ABCDEF	
34. $4A_1c$	1: ABC, ADE, BDF, CEF	$\mathbf{Z}_2$	79. A_5a	0: AB, BC, CD, DE, EF	
35. $2A_1A_2a$	0: AB, BC, DE; 1: ABC		80. A_5b	0: AB, BC, DE, EF; 1: ADE	
36. $2A_1A_2b$	0: AB, CD, EF; 1: ABC		81. A_5c	0: AB, BC, CD; 1: ABC, AEF	
37. $2A_1A_2c$	0: AB, DE; 1: ABC, DEF		82. D_5a	0: BC, CD, DE, EF; 1: ABC	
38. $2A_1A_2d$	0: AB, DE, EF; 1: ABC		83. D_5b	0: AB, CD, DE, EF; 1: ACD	
39. $2A_1A_2e$	0: AB, DE; 1: ABC, ADE		84. D_5c	0: AB, BC, CD, DE; 1: ABC	
40. $2A_1A_2f$	0: BF, DE; 1: ABC, ADE		85. $3A_2a$	0: AB, BC, DE, EF; 1: ABC, DEF	$\mathbf{Z}_3$
41. $2A_1A_2g$	0: AC; 1: ABC, ADE, BDF		86. $3A_2b$	0: AB, CD, EF; 1: ABC, AEF, CDE	$\mathbf{Z}_3$
42. $2A_1A_2h$	0: AB, BC, DE; 2: ABCDEF		87. A_1A_5a	0: AB, BC, DE, EF; 1: ABC, ADE	$\mathbf{Z}_2$
43. A_1A_3a	0: AB, BC, CD, EF		88. A_1A_5b	0: AB, BC, CF, DE; 1: ABC, ADE	$\mathbf{Z}_2$
44. A_1A_3b	0: AB, CD, DE; 1: ABC		89. A_1A_5c	0: AB, BC, CD, DE, EF; 2: ABCDEF	$\mathbf{Z}_2$
45. A_1A_3c	0: AB, DF; 1: ABC, ADE		90. E_6	0: AB, BC, CD, DE, EF; 1: ABC	

We obtained the table by brute force as follows. Start by finding all single element configuration types, which is easy. These are just the single element subsets of $\mathcal{V}'' \cup \mathcal{L}'' \cup \mathcal{Q}''$. Pick a representative for each orbit under the permutation action. We get three singleton sets T, corresponding to items 2, 3 and 4 in Table 3.1. Add to each singleton configuration type T each

element of $\mathcal{V}'' \cup \mathcal{L}'' \cup \mathcal{Q}''$ which meets every class already in T nonnegatively, and again pick a representative set from each orbit. Continue this way for six cycles. (Six is enough since, as shown in the proof of Proposition 3.1, the elements of each T are linearly independent, and so T can have at most 6 elements.)

Proposition 3.1. *Over every algebraically closed field k, each configuration type occurs as* $\mathrm{neg}(X)$ *for some surface X obtained by blowing up 6 essentially distinct points of $\mathbf{P}_k^2$.*

Proof. Let T be the set of classes of a configuration type, and consider the group $K^\perp/\langle T \rangle$. Since $\mathcal{V}'' \cup \mathcal{L}'' \cup \mathcal{Q}''$ is a finite set and since from Table 3.1 we see that the torsion subgroup of $K^\perp/\langle T \rangle$ is either trivial or has prime order, we can pick a squarefree positive integer l and a surjective homomorphism $\phi : K^\perp/\langle T \rangle \to \mathbf{Z}/l\mathbf{Z}$ such that no element $C \in \mathcal{V}'' \cup \mathcal{L}'' \cup \mathcal{Q}''$ not already in $\langle T \rangle$ maps to 0 in $\mathbf{Z}/l\mathbf{Z}$.

Now let C be a non-supersingular smooth plane cubic curve. Since C is not supersingular, $\mathrm{Pic}^0(C)$ has a subgroup isomorphic to $\mathbf{Z}/l\mathbf{Z}$; we identify $\mathbf{Z}/l\mathbf{Z}$ with this subgroup of $\mathrm{Pic}^0(C)$. Thus there is a homomorphism $\Phi : \mathrm{Cl}(X) \to \mathrm{Pic}(C)$ such that the image $\Phi(K_X^\perp)$ is exactly $\mathbf{Z}/l\mathbf{Z}$. Pick any point on C to be p_1. Then pick p_i to be the image of $E_i - E_1$ in $\mathrm{Pic}^0(C)$.

Under the usual identification of $\mathrm{Pic}^0(C)$ with C itself, this gives us six points $p_1, \ldots, p_6$. (It may be that some of the points are formally the same. For example, if $E_1 - E_2 \in T$, then $p_1 = p_2$. This just means that p_2 is the point on the proper transform C' of C on X_1 infinitely near $p_1 \in X_0$. Since restricting the mapping $\pi_1 : X_1 \to \mathbf{P}^2$ to C' gives an isomorphism of C' to C, there is a natural identification of C' with C. Under this identification we can indeed regard p_1 and p_2 as being the same point of C, even though properly speaking $p_1 \in \mathbf{P}^2$ and $p_2 \in X_1$.)

By construction, the surface X obtained by blowing up the points $p_1, \ldots, p_6$ has the property that an element $D \in \mathcal{V}'' \cup \mathcal{L}'' \cup \mathcal{Q}''$ is in the kernel of Φ if and only if $D \in \langle T \rangle$. By [H1], an element D of $\mathcal{V}'' \cup \mathcal{L}'' \cup \mathcal{Q}''$ is the class of an effective divisor if and only if $D \in \ker(\Phi)$. Thus $D \in \mathcal{V}'' \cup \mathcal{L}'' \cup \mathcal{Q}''$ is effective if and only if $D \in \langle T \rangle$.

In particular, the elements of T are effective and $\mathrm{neg}(X) \subseteq \langle T \rangle$. Let $D \in \mathrm{neg}(X)$. By Lemma 2.1(b), $D^2 = -2$. Now write D as an integer linear combination of elements of T. Thus we can write $D = D_1 - D_2$, where D_1 is a sum of elements of T with positive coefficients and D_2 is either 0 or a sum of different elements of T with positive coefficients. Note that D_1 is not zero, since otherwise D is either 0 or antieffective, neither of which can hold since D is the class of a prime divisor. We claim however that $D_2 = 0$. If not, then, since $K_X^\perp$ is negative definite and even, we have $D_1^2 \leqslant -2$ and $D_2^2 \leqslant -2$. Since D_1 and D_2 involve different elements of T (which therefore meet nonnegatively), we also see $D_1 \cdot D_2 \geqslant 0$. Thus $D^2 = D_1^2 - 2D_1 \cdot D_2 + D_2^2 \leqslant -4$, contradicting $D^2 = -2$. (A similar argument shows that the elements of T are linearly independent. If not, we can find an expression $D_1 - D_2 = 0$ for some nonnegative linear integer combinations D_i of elements of disjoint subsets $T_i \subseteq T$. By pairwise nonnegativity, we have $D_1 \cdot D_2 \geqslant 0$, but $-K_X^\perp$ is negative definite, so $0 \geqslant D_i^2 = D_1 \cdot D_2$, hence $D_i^2 = 0$, so $D_i = 0$. But $T \subseteq \mathcal{V}'' \cup \mathcal{L}'' \cup \mathcal{Q}''$, and every element of $\mathcal{V}'' \cup \mathcal{L}'' \cup \mathcal{Q}''$ meets $A = 14L - 6E_1 - 5E_2 - 4E_3 - 3E_4 - 2E_5 - E_6$ positively, so if D_i is not a linear integer combination of elements of T_i with 0 coefficients, then we have $0 < A \cdot D_i = A \cdot 0 = 0$, which is impossible. Thus each D_i is the trivial linear combination, hence T is linearly independent.)

Thus every element of $\mathrm{neg}(X)$ is a nonnegative sum of elements of T, each of which is effective. But the elements of $\mathrm{neg}(X)$ are prime divisors of negative self-intersection, hence each can

be written as a nonnegative sum of classes of effective divisors only one way; i.e., every element of neg(X) is an element of T.

By Lemma 2.1(d), every element of T is a nonnegative integer linear combination of elements of NEG(X). But $T \subseteq K_X^\perp$, so in fact every element of T is a nonnegative integer linear combination of elements of neg(X). Since neg(X) $\subseteq T$, and since T is linearly independent, this is possible only if neg(X) = T.   □

As a check on our list of configuration types as given in Table 3.1, we have the following well-known result, Theorem 3.2. (See [BW] for a version of the result in characteristic 0, or see Theorem IV.1 of the arXiv version math.AG/0506611 of the paper [GH] for a proof in general. The proof is to study the morphisms $X \to \mathbf{P}^2$ obtained by mapping X to $\mathbf{P}^3$ using the linear system $|-K_X|$, and then mapping the image $\bar{X}$ to $\mathbf{P}^2$ by projecting from a singular point.) Thus we get the same Dynkin diagrams from Theorem 3.2 as we found by a brute force determination of configuration types. Moreover, one can (as we did in fact do) find all exceptional configurations for each of the 20 graphs listed in Theorem 3.2, and for each exceptional configuration one can write down the corresponding (representable) configuration type. Since by Proposition 3.1 every type is representable over every algebraically closed field, it follows that the types obtained this way should be (and in fact are) the same types we found by brute force.

Theorem 3.2. *Let X be a blow up of $\mathbf{P}^2$ at 6 essentially distinct points of $\mathbf{P}^2$, such that $-K_X$ is nef. Assume that X has at least one (-2)-curve. Then the intersection graph of the set of (-2)-curves is one of the following 20 graphs: A_1, $2A_1$, A_2, $3A_1$, A_1A_2, A_3, $4A_1$, $2A_1A_2$, A_1A_3, $2A_2$, A_4, D_4, A_12A_2, $2A_1A_3$, A_1A_4, A_5, D_5, $3A_2$, A_1A_5, and E_6. Each of these graphs occurs as the graph of the set of (-2)-curves on some X, and in a unique way (unique in the sense that if the same graph occurs on two surfaces X and X', then there are exceptional configurations $L, E_1, \ldots, E_6$ on X and $L', E'_1, \ldots, E'_6$ on X', such that a class $a_0L + \sum_i a_i E_i$ is the class of a (-2)-curve on X if and only if $a_0L' + \sum_i a_i E'_i$ is the class of a (-2)-curve on X').*

4. Resolutions

Let $p_1, \ldots, p_6$ be essentially distinct points of $\mathbf{P}^2$. Let $Z = m_1 p_1 + \cdots + m_6 p_6$ be a fat point subscheme of $\mathbf{P}^2$, and let $F(Z, i) = iL - m_1 E_1 - \cdots - m_6 E_6$ on the surface X obtained by blowing up the points p_i. As explained in Section 2, the ideal $I(Z)$ is obtained as follows. Let $\pi : X \to \mathbf{P}^2$ be the morphism to $\mathbf{P}^2$ given by blowing up the points p_i, and let $L, E_1, \ldots, E_6$ be the corresponding exceptional configuration. Let $F = -(m_1 E_1 + \cdots + m_6 E_6)$. Then $\mathcal{I}_Z = \pi_*(\mathcal{O}_X(-m_1 E_1 - \cdots - m_6 E_6))$ is a sheaf of ideals on $\mathbf{P}^2$, and $I(Z) = \bigoplus_{i \geqslant 0} H^0(\mathbf{P}^2, \mathcal{I}_Z \otimes \mathcal{O}_{\mathbf{P}^2}(i))$. Also, we may as well assume that the coefficients m_i satisfy the proximity inequalities. If they do not, there is another choice of coefficients m'_i which do satisfy them, giving a 0-cycle Z' for which $I(Z) = I(Z')$. (The proximity inequalities are precisely the conditions on the m_i given by the inequalities $F \cdot C \geqslant 0$ for each divisor class C which is the class of a component of the curves whose classes are $E_1, \ldots, E_6$. In the case that the points p_i are distinct, the proximity inequalities are merely that $m_i \geqslant 0$ for all i. If p_2 is infinitely near p_1, then we would have the additional requirement that $m_1 \geqslant m_2$. This corresponds to the fact that a form cannot vanish at p_2 without already vanishing at p_1. If the m_i do not satisfy the proximity inequalities, then $F(Z, i)$ will never be nef: no matter how large i is, some component C of some E_j, $j > 0$, will have $F(Z, i) \cdot C < 0$. Thus C will be a fixed component of $|F(Z, i)|$ for all i. By subtracting off such fixed components one obtains a class $iL - (m'_1 E_1 + \cdots + m'_6 E_6)$, which also gives a 0-cycle

$Z' = m'_1 p_1 + \cdots + m'_6 p_6$ satisfying the proximity inequalities and which gives the same ideal $I(Z) = I(Z')$. See [H6] for more details.)

The minimal free resolution of $I(Z)$ is an exact sequence of the form

$$0 \to F_1 \to F_0 \to I(Z) \to 0$$

where each F_i is a free graded R-module, with respect to the usual grading of R by degree, and all nonzero entries of the matrix defining the homomorphism $F_1 \to F_0$ are homogeneous polynomials in R of degree at least 1. Since F_0 and F_1 are free graded R-modules, we know that there are integers g_i and s_j such that $F_0 \cong \bigoplus_i R[-i]^{g_i}$ and $F_1 \cong \bigoplus_j R[-j]^{s_j}$. These integers are the graded Betti numbers of $I(Z)$. To determine the modules F_1 and F_0 up to graded isomorphism (or, equivalently, to determine the graded Betti numbers of the minimal free resolution of $I(Z)$), it is enough, as for distinct points (as explained in [GH]), to determine $h^0(X, F(Z, i))$ and the ranks for all $i \geqslant 0$ of the multiplication maps $\mu_{Z,i} : I(Z)_i \otimes R_1 \to I(Z)_{i+1}$ for each $i \geqslant 0$, where, given a graded R-module M, M_t denotes the graded component of degree t. Since (see [GH]) the rank of $\mu_{Z,i}$ is the same as the rank of $\mu_{F(Z,i)} : H^0(X, F(Z, i)) \otimes H^0(X, L) \to H^0(X, L + F(Z, i))$, it is enough to determine the rank of $\mu_{F(Z,i)}$.

As explained in [GH], we can compute $h^0(X, F(Z, i))$ if we know $\mathrm{NEG}(X)$ (or therefore even just $\mathrm{neg}(X)$), and we can compute the rank of $\mu_{F(Z,i)}$ if we can compute the rank of μ_F whenever F is nef (which the main result of this section, Theorem 4.3, says we can do).

The method we use to prove Theorem 4.3 is precisely the method used in [GH]. It involves the quantities $q(F) = h^0(X, F - E_1)$ and $l(F) = h^0(X, F - (L - E_1))$, and bounds on the dimension of the cokernel of μ_F, defined in terms of quantities $q^*(F) = h^1(X, F - E_1)$ and $l^*(F) = h^1(X, F - (L - E_1))$, introduced in [H5] and [FHH]. A version of Lemma 4.1 for distinct points is given in [H5] and [FHH], but with only trivial changes the proof for essentially distinct points is the same.

Lemma 4.1. *Let X be obtained by blowing up essentially distinct points $p_i \in \mathbf{P}^2$, and let F be the class of an effective divisor on X with $h^1(X, F) = 0$. Then $l(F) \leqslant \dim \ker \mu_F \leqslant q(F) + l(F)$ and $\dim \mathrm{cok} \, \mu_F \leqslant q^*(F) + l^*(F)$.*

Remark 4.2. *The quantities $q(F)$ and $l(F)$ are defined in terms of E_1 and $L - E_1$, but in fact E_j, $j > 0$, can often be used in place of $j = 1$. This is always true if the points p_i are distinct, since one can reindex the points. Likewise, if the points are only essentially distinct, any j can be used so long as p_j is a point on $\mathbf{P}^2$, and not only infinitely near a point of $\mathbf{P}^2$.*

Theorem 4.3. *Let X be obtained by blowing up 6 essentially distinct points of $\mathbf{P}^2$. Let $L, E_1, \ldots, E_6$ be the corresponding exceptional configuration. Assume that $-K_X$ is nef, and let F be a nef divisor. Then μ_F has maximal rank.*

Proof. The case of general points (i.e., that $\mathrm{neg}(X)$ is empty) is done in [F1] (but it can be recovered by the methods we use here). This handles one of the 90 cases of Table 3.1. Also, for 28 of the cases of Table 3.1, a conic goes through the six points (i.e., $h^0(X, 2L - E_1 - \cdots - 2E_6) > 0$); these cases are configuration types 4, 8, 12, 16, 25, 26, 30, 33, 37, 42, 47, 48, 50, 53, 58, 61, 64, 66, 70, 72, 76, 78, 81, 83, 85, 88, 89 and 90. The result holds for these cases by Theorem 3.1.2 of [H4] (also see Lemma 2.11 of [GH]).

Four of the remaining 61 cases correspond to distinct points, and were handled in [GH]. These cases are 3, 9, 17 and 34. The remaining cases are handled by the same method as these four. The basic idea is this. If F is a nef divisor such that $l(F) > 0$, $q(F) > 0$, and $l^*(F) = 0 = q^*(F)$, then not only is it true that μ_F is surjective (by Lemma 4.1), but $l(F + G) > 0$ and $q(F + G) > 0$ by Lemma 2.1, and $l^*(F + G) = 0 = q^*(F + G)$ holds for all nef G (the proof of this fact is the main and final step in the proof of Corollary 2.8 of [GH]), hence μ_{F+G} is surjective for all nef G.

Using Lemma 2.5 of [GH] one can easily give an explicit list of generators of the nef cone for each configuration type. In the best of all worlds, what would happen is that we would find that $l(F) > 0$, $q(F) > 0$, $l^*(F) = 0 = q^*(F)$, for every F in our set of generators, and the result would be proved. But our world is not the best of all imaginable worlds, so some additional work is needed. In [GH] this is done, applying Corollary 2.8, Lemmas 2.9 and 2.10 of [GH]. These are all stated for 6 distinct points or $\mathbf{P}^2$, but it is easy to check that the proofs continue to hold for 6 essentially distinct points if $-K_X$ is nef.

We now describe what this additional work is. Let $\Gamma(X)$ be a set of generators of the nef cone for X. (For practical purposes of actually carrying out the calculations, it is best to choose a minimal set of generators.) Let $\Gamma_i(X)$ be the set of all sums with exactly i terms, where each term is an element (with coefficient 1) of $\Gamma(X)$. Let $S(X)$ be the set of all nef classes F such that either $q(F) = 0$, $l(F) = 0$ or $l^*(F) + q^*(F) > 0$. Then let $S_i(X) = S(X) \cap \Gamma_i(X)$; by Corollary 2.8 [GH], we have $S_{i+1}(X) \subseteq S_i(X) + S_1(X)$. Typically the subsets $S_i(X)$ are nonempty. But for $i \geqslant 3$, it always turns out that Lemma 2.9 [GH] applies. This lemma involves a parameter k which we can always take to be $k = 2$. It also involves a particular choice of class $C_F \in S_1(X)$ for each $F \in S_i(X)$. The result is that $S_i(X) \subseteq \{F + (i - 3)C_F \colon F \in S_3(X),\ C_F \in S_1(X)\}$.

First one verifies directly that maximal rank holds for μ_F for all $F \in S_i(X)$ for $i \leqslant 3$, using Lemma 4.1. An induction (applying Lemma 2.10 [GH]) then verifies maximal rank for the strings $F + (i - 3)C_F$, and hence for all nef F. There is one case that must be handled ad hoc (as was done by [F1] and as we demonstrate below). If $F = 5L - 2(E_1 + \cdots + E_6)$, then $C_F = F$, but $l(iF) > 0$ for $i \geqslant 3$ (so μ_{iF} is not injective by Lemma 4.1) while $l^*(iF) > 0$ for all i (so the bounds in Lemma 4.1 never force surjectivity). We now treat one case in detail, as an example. The remaining cases are similar.

Consider configuration type 2, so $\mathrm{neg}(X) = \{N\}$, where $N = E_1 - E_2$. Then $S_1(X)$ has 58 elements, $S_2(X)$ has 140, and $S_3(X)$, $S_4(X)$ and $S_5(X)$ have 150. Moreover, μ_H has maximal rank (by a case by case application of Lemma 4.1) for each element H of $S_i(X)$, $1 \leqslant i \leqslant 5$, except possibly mH when $H = 5L - 2(E_1 + \cdots + E_6)$ for $m > 1$ (since $q(mH) + l(mH) > 0$ and $q^*(mH) + l^*(mH) > 0$ in these cases). To show μ_H is onto for $H = 2(5L - 2(E_1 + \cdots + E_6))$, let $C = 2L - E_1 - \cdots - E_5$, and consider $F = H - C$. Then μ_F is onto (by Lemma 4.1, since $q^*(F) + l^*(F) = 0$) hence μ_H is onto (by Lemma 2.10 [GH]), and now μ_{H+iC} is onto for all $i \geqslant 0$ (also by Lemma 2.10 [GH], taking F to be mH and $C = 5E_0 - 2(E_1 + \cdots + E_6)$ for the induction in Lemma 2.10 [GH]). By brute force check, applying Lemma 2.9 [GH] (with $k = 2$ and $j = 2$) and Lemma 2.10 [GH], it follows that μ_F has maximal rank for every F in each $S_i(X)$. $\square$

5. Examples

Given only the configuration type and multiplicities $m_1, \ldots, m_6$ satisfying the proximity inequalities, Lemma 2.1 and Theorem 4.3 allow us to determine the Hilbert function and graded Betti numbers for $I(Z)$ for any fat point subscheme $Z = m_1 p_1 + \cdots + m_6 p_6$ supported at 6 es-

Table 5.1
Resolutions and Hilbert functions

Scheme		Resolution		Hilbert function
		F_1	F_0	$h_{R/I(mZ)}, m = 1, 2$
1:	Z	$R[-5]$	$R[-3] \oplus R[-2]$	1, 3, 5, 6
(a):	$2Z$	$R[-8] \oplus R[-7]$	$R[-6] \oplus R[-5] \oplus R[-4]$	1, 3, 6, 10, 14, 17, 18
2:	Z	$R[-4]^3$	$R[-3]^4$	1, 3, 6
(a):	$2Z$	$R[-7]^4$	$R[-6]^4 \oplus R[-4]$	1, 3, 6, 10, 14, 18
(b1):	$2Z$	$R[-7]^3$	$R[-6]^1 \oplus R[-5]^3$	1, 3, 6, 10, 15, 18
(b2):	$2Z$	$R[-7]^3 \oplus R[-6]$	$R[-6]^2 \oplus R[-5]^3$	1, 3, 6, 10, 15, 18
(b3):	$2Z$	$R[-7]^3 \oplus R[-6]^2$	$R[-6]^3 \oplus R[-5]^3$	1, 3, 6, 10, 15, 18

sentially distinct points p_i of $\mathbf{P}^2$ which when blown up give a surface X for which $-K_X$ is nef (i.e., give a surface isomorphic to the desingularization of a normal cubic surface).

The procedure for doing so is exactly the same as described in detail in [GH]. We briefly recall the procedure. Given $Z = m_1 p_1 + \cdots + m_6 p_6$, to determine $h_{I(Z)}(t)$, compute $h^0(X, F(Z, t))$, where $F(Z, t) = tL - m_1 E_1 - \cdots - m_6 E_6$. To do this, let $D = F(Z, t)$ and check $D \cdot C$ for all prime divisors C with $C^2 < 0$. (Knowing the configuration type tells us the list of these divisors C.) Whenever $D \cdot C < 0$, replace D by $D - C$ and again check $D \cdot C$ with this new D against all C. Eventually either $D \cdot L < 0$ (in which case $h^0(X, F(Z, t)) = h^0(X, D) = 0$), or $D \cdot C \geqslant 0$ for all C (in which case, by Lemma 2.1, D is nef and $h^0(X, F(Z, t)) = h^0(X, D) = (D^2 - K_X \cdot D)/2 + 1$).

To determine the graded Betti numbers, note that it suffices to compute the Betti numbers g_i for all i, since the exact sequence $0 \to F_1 \to F_0 \to I(Z) \to 0$ allows one to determine F_1 up to graded isomorphism if one knows the graded Betti numbers for F_0 and also the Hilbert function for $I(Z)$. To determine g_{t+1}, note that $g_{t+1} = h^0(X, F(Z, t + 1))$ if $h^0(X, F(Z, t)) = 0$. If $h^0(X, F(Z, t)) > 0$, obtain the nef divisor D from $F(Z, t)$ as above. Then $g_{t+1} = (h^0(X, F(Z, t + 1)) - h^0(X, D + L)) + \max(0, h^0(X, D + L) - 3h^0(X, D))$.

The procedure thus involves nothing more than taking dot products of integer vectors, and can easily be done by hand. An awk script which automates the steps is available at http://www.math.unl.edu/~bharbourne1/6ptsNef-K/Res6pointNEF-K. It can be run over the web at http://www.math.unl.edu/~bharbourne1/6ptsNef-K/6reswebsite.html.

Using this script we determined all possible Hilbert functions and graded Betti numbers for fat points of the form $Z = p_1 + \cdots + p_6$ and $2Z = 2p_1 + \cdots + 2p_6$ for essentially distinct points p_i such that $-K_X$ is nef on the resulting surface X. For 6 essentially distinct points with nef $-K_X$, this completely answers the questions raised in [GMS]. We show what happens in Table 5.1. (The table regards a Hilbert function $h = h_{R/I(mZ)}$ as the sequence $h(0), h(1), h(2), \ldots$. But any such h reaches a maximum value at the regularity; i.e., for all t greater than or equal to the regularity of $I(mZ)$, we have $h(t) = h(t + 1)$. Thus Table 5.1 gives h only up to this maximum value.)

In Table 5.1, case 1 occurs for the following configuration types (as denoted in Table 3.1): 4, 8, 12, 16, 25, 26, 30, 33, 37, 42, 47, 48, 50, 53, 58, 61, 64, 66, 70, 72, 76, 78, 81, 83, 85, 88, 89, 90. For each of these types, only one Hilbert function occurs for $2Z = 2p_1 + \cdots + 2p_6$, the one given as 1(a). These all have the same graded Betti numbers too.

Case 2 occurs for the remaining configuration types: 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 27, 28, 29, 31, 32, 34, 35, 36, 38, 39, 40, 41, 43, 44, 45, 46, 49, 51, 52, 54, 55, 56, 57, 59, 60, 62, 63, 65, 67, 68, 69, 71, 73, 74, 75, 77, 79, 80, 82, 84, 86, 87. For

these, two different Hilbert functions occur for $2Z = 2p_1 + \cdots + 2p_6$, given as 2(a) and 2(b). Case 2(a) occurs for types 34, 68 and 87, and these three all have the same graded Betti numbers. Case 2(b) occurs for types 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 27, 28, 29, 31, 32, 35, 36, 38, 39, 40, 41, 43, 44, 45, 46, 49, 51, 52, 54, 55, 56, 57, 59, 60, 62, 63, 65, 67, 69, 71, 73, 74, 75, 77, 79, 80, 82, 84, and 86. These all have the same Hilbert function, but three different possibilities occur for the graded Betti numbers, which we distinguish in the table by cases 2(b1), 2(b2) and 2(b3). Case 2(b1) occurs for types 1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 18, 19, 20, 21, 22, 27, 28, 31, 35, 36, 38, 43, 44, 49, 51, 54, 57, 59, 62, 73, 74 and 79. Case 2(b2) occurs for types 9, 15, 23, 24, 29, 32, 39, 40, 46, 52, 55, 56, 60, 63, 67, 69, 71, 82 and 84, and case 2(b3) occurs for the remaining types 17, 41, 45, 65, 75, 77, 80 and 86.

We close with one final example. The Hilbert functions that occur for Z or $2Z$ for every choice of 6 essentially distinct points $Z = p_1 + \cdots + p_6 \subset \mathbf{P}^2$ all already occur for distinct points. The first case of a Hilbert function that occurs for 6 essentially distinct points mZ of multiplicity m that does not occur for any 6 distinct points of multiplicity m is for $m = 3$, and in this case there is only one, this being the Hilbert function for the ideal $I(Z)$ of 6 essentially distinct points of multiplicity 3 with configuration type 86, which is $h_{I(Z)}(t) = 0$ for $t < 6$, $h_{I(Z)}(6) = 1$, $h_{I(Z)}(7) = 3$, and, for $t > 7$, $h_{I(Z)}(t) = \binom{t+2}{2} - 36$. Applying the results of [GH], we see this Hilbert function does not occur for any configuration of 6 distinct points. The graded Betti numbers for $I(Z)$ are such that $F_0 \cong R[-9]^3 \oplus R[-8]^3 \oplus R[-6]$ and $F_1 \cong R[-10]^3 \oplus R[-9]^3$.

Acknowledgments

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