Libertas Mathematica (new series)
Volume 33(2013), No. 1, 125-134

A survey of certain problems in analysis and their status

Biagio Ricceri

Abstract: In this paper, we give a survey of certain problems highlighting their current status.

Keywords: Mathematical analysis, variational inequalities, Gevrey spaces, geometry of normed spaces, integral functionals on Sobolev spaces, nonlinear elliptic equations.

MSC2010: 47J20, 46E10, 46B20, 49J45, 35J20.

Dedicated to the memory of Professor Francesco S. De Blasi

In this paper, I wish to offer an overview of some of the numerous problems that I have proposed in the past years. For each one of such problems, I will recall the motivation for studying it and point out its status, in the sense of saying what is known up to date.

1 Variational inequalities

Given a real Hausdorff topological vector space E, a closed convex set $X \subseteq E$, and an operator $\Phi : X \to E^*$, the problem of finding $\hat{x} \in X$ in such a way that

$$\sup_{y \in X} \Phi(\hat{x})(\hat{x} - y) \leq 0$$

is said to be the *variational inequality* associated with X and Φ, denoted by $\mathrm{VI}(X, \Phi)$.

The first basic result, due to P. Hartman and G. Stampacchia [9], is as follows:

Theorem 1.1. *Assume that X is compact and finite-dimensional, and that Φ is weakly-star continuous.*
Then, $VI(X, \Phi)$ has a solution.

Theorem 1.1 is no longer true when X is infinite-dimensional. In this connection, M. Frasca and A. Villani [7], proved the following very interesting result:

Theorem 1.2. *Let $(E, \langle \cdot, \cdot \rangle)$ be any infinite-dimensional Hilbert space. Then, for each closed ball $X \subset E$, there exists some (strongly) continuous affine operator $\Phi : E \to E$ such that, for every $x \in X$, one has*

$$\sup_{y \in X} \langle \Phi(x), x - y \rangle > 0.$$

It is clear how Theorem 1.2 serves to the purpose: one considers E endowed with the weak topology and identify E^* with E. So, X is weakly compact and, at the same time, Φ is weakly continuous, being affine and strongly continuous.

In [10], I proved the following:

Theorem 1.3. *Assume that the relative interior of X (that is the interior of X in its affine hull) is non-empty, and that Φ is weakly-star continuous. Moreorer, let K, K_1 be two non-empty compact subsets of X, with $K_1 \subseteq K$ and K_1 finite-dimensional, such that, for each $x \in X \setminus K$, one has*

$$\sup_{y \in K_1} \langle \Phi(x), x - y \rangle > 0.$$

Then, VI(X, Φ) has a solution lying in K.

In [13], I asked the following:

Problem 1.1. In Theorem 1.3, can one drop the finite-dimensionality assumption on K_1 ?

Up to date, the only (partial) answer to Problem 1.1 was provided by N. D. Yen in [21] who proved

Theorem 1.4. *Let $(E, \langle \cdot, \cdot \rangle)$ be a Hilbert space, let $K \subset E$ be a closed ball and let $\Phi : E \to E$ be an affine (not necessarily continuous) operator such that*

$$\sup_{y \in K} \langle \Phi(x), x - y \rangle > 0$$

for all $x \in E \setminus K$.
Then, there exists $\hat{x} \in K$ such that $\Phi(\hat{x}) = 0$.

2 A special Gevrey space

In [12], in connection with linear partial differential equations of infinite order, I introduced a special Gevrey space. Namely, let m, n be two positive integers. Denote by $V(\mathbf{R}^n)$ the space of all functions $u \in C^\infty(\mathbf{R}^n)$ such that, for each bounded subset $\Omega \subset \mathbf{R}^n$, one has

$$\sup_{\alpha \in \mathbf{N}_0^n} \sup_{x \in \Omega} |D^\alpha u(x)| < +\infty,$$

where $D^\alpha u = \partial^{\alpha_1 + ... + \alpha_n} u / \partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}$, $\alpha = (\alpha_1, ..., \alpha_n)$ and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

Also, for each $\alpha \in \mathbf{N}_0^n$, with $|\alpha| = \alpha_1 + ... + \alpha_n \leq m$, let $a_\alpha \in \mathbf{R}$ be given. Let $P : V(\mathbf{R}^n) \to V(\mathbf{R}^n)$ be the differential operator defined by putting

$$P(u) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha u$$

for all $u \in V(\mathbf{R}^n)$.

A natural problem arises:

Problem 2.1. Find necessary and sufficient conditions in order that

$$P(V(\mathbf{R}^n)) = V(\mathbf{R}^n) \ .$$

Up to date, the only (very partial) answer to Problem 2.1 is provided by the following result from [12]:

Theorem 2.1. *Let $a, b \in \mathbf{R} \setminus \{0\}$ and $h, k \in \mathbf{N}$. For each $u \in V(\mathbf{R}^2)$, put*

$$P(u) = a \frac{\partial^h u}{\partial x^h} + b \frac{\partial^k u}{\partial y^k}.$$

Then, one has

$$P(V(\mathbf{R}^2)) = V(\mathbf{R}^2)$$

if and only if $|a| \neq |b|$.

In [16], I asked the following

Problem 2.2. Let $f : \mathbf{R} \to \mathbf{R}$ be a function such that, for each $u \in V(\mathbf{R}^n)$, the composite function $x \to f(u(x))$ belongs to $V(\mathbf{R}^n)$.
Then, must f necessarily be affine ?

In [20], by means of an ingenious proof based on the Baire category, M. Romeo gave a complete positive answer to Problem 2.2.

3 Geometry of normed spaces

Let E be a real vector space. A non-empty set $A \subset E$ is said to be *totally antiproximinal* if, for every norm $\|\cdot\|$ on E, every $x \in E \setminus A$ and every $y \in A$, one has

$$\|x - y\| > \inf_{z \in A} \|x - z\| \ .$$

Applying Theorem 4 of [10], one can prove the following

Theorem 3.1. *Let X, E be two real vector spaces, C a non-empty convex subset of X, F a multifunction from C onto E, with non-empty values and convex graph. Then, for every non-empty convex set $A \subseteq C$ which is open with respect to the relativization to C of the strongest vector topology on X, the set $F(A)$ is totally antiproximinal.*

In view of Theorem 3.1, in [14], I formulated the following conjecture:

Conjecture 3.1. There exists a non-complete real normed space whose totally antiproximinal convex subsets are not rare.

Recently, in [8], F. J. Garcia-Pacheco gave a partial answer to Conjecture 3.1 proving what follows

Theorem 3.2. *There exists a non-complete real normed space whose totally antiproximinal absolutely convex subsets are not rare.*

Let E be a normed space and let S be the unit sphere of E. The space E is said to have the Kadec-Klee property if, for every sequence $\{x_n\}$ in S weakly converging to some $x \in S$, one has

$$\lim_{n \to \infty} \|x_n - x\| = 0 \ .$$

In [19], I proved the following

Theorem 3.3. *Let E be an infinite-dimensional reflexive real Banach space and let $f : S \to E^*$ be a compact function such that*

$$\inf_{x \in S} \|f(x)\|_{E^*} > 0 \ .$$

Then, there exists $\hat{x} \in S$ such that

$$f(\hat{x})(\hat{x}) = \|f(\hat{x})\|_{E^*} \ .$$

In [19] again, I also proposed the following

Problem 3.1. Let E be an infinite-dimensional reflexive real Banach space such that, for each compact function $f : S \to E^*$ satisfying

$$\inf_{x \in S} \|f(x)\|_{E^*} > 0 \ ,$$

there exists $\hat{x} \in S$ for which

$$f(\hat{x})(\hat{x}) = \|f(\hat{x})\|_{E^*} \ .$$

Then, does E possess the Kadec-Klee property ?

Very recently, J. Saint Raymond informed me that in a forthcoming paper he will give a complete positive answer to Problem 3.1.

4 Integral functionals on Sobolev spaces and nonlinear elliptic equations

In the sequel, $\Omega \subset \mathbf{R}^n$ $(n \geq 3)$ is a bounded domain with smooth boundary and X will stand for $H^1_0(\Omega)$, with the usual norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^2 dx \right)^{\frac{1}{2}} \ .$$

For $q > 0$, denote by $\mathcal{A}_q$ the class of all continuous functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$\sup_{\xi \in \mathbf{R}} \frac{|f(\xi)|}{1 + |\xi|^q} < +\infty \ .$$

For $0 < q \leq \frac{n+2}{n-2}$ and $f \in \mathcal{A}_q$, put

$$J_f(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx - \int_\Omega F(u(x)) dx$$

for all $u \in X$, where

$$F(\xi) = \int_0^\xi f(t) dt \ .$$

So, the functional J_f is of class C^1 on X and one has

$$J'_f(u)(v) = \int_\Omega \nabla u(x) \nabla v(x) dx - \int_\Omega f(u(x)) v(x) dx$$

for all $u, v \in X$.

Hence, the critical points of J_f in X are exactly the weak solutions of the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ \\ u_{|\partial\Omega} = 0 \ . \end{cases} \tag{P_f}$$

One of the classical basic existence results for Problem (P_f) was provided by A. Ambrosetti and P. H. Rabinowitz who in [1] proved

Theorem 4.1. *Let $f \in \mathcal{A}_q$, with $q < \frac{n+2}{n-2}$, and assume that the following conditions are satisfied:*

(a) *there are constants $r \geq 0$ and $c > 2$ such that*

$$0 < cF(\xi) \leq \xi f(\xi)$$

for all $\xi \in \mathbf{R}$ with $|\xi| \geq r$;

(b) $\lim_{\xi \to 0} \frac{f(\xi)}{\xi} = 0$.

Then, Problem (P_f) has a non-zero weak solution.

In [15], I obtained the following

Theorem 4.2. *Assume that the assumptions of Theorem 4.1 hold, but (b). Then, for each $\rho > 0$ and each μ satisfying*

$$\mu > \inf_{\|u\|^2 < \rho} \frac{\sup_{\|v\|^2 < \rho} \int_\Omega F(v(x))dx - \int_\Omega F(u(x))dx}{\rho - \|u\|^2} \tag{4.1}$$

Problem $(P_{\frac{1}{2\mu}f})$ has at least two weak solutions one of which has norm less that $\sqrt{\rho}$.

Comparing Theorems 4.1 and 4.2, the following problem naturally arises

Problem 4.1. Is there some $\rho > 0$ such that the infimum appearing in 4.1 is less than $\frac{1}{2}$?

A complete answer to Problem 4.1 was provided by G. Anello in [2] who proved

Theorem 4.3. *The answer to Problem 4.1 is, in general, "no" if $r > 0$, while is "yes" if $r = 0$. Moreover, there are f satisfying condition (a) with $r = 0$, but not satisfying condition (b).*

Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. Set

$$\Lambda_f = \{\lambda > 0 : (P_{\lambda f}) \text{ has a non-zero classical solution}\} \ .$$

The Pohozaev identity tells us that, if u is a classical solution of $(P_{\lambda f})$, then one has

$$\frac{2-n}{2}\int_\Omega |\nabla u(x)|^2 dx + n\lambda \int_\Omega F(u(x))dx = \frac{1}{2}\int_{\partial\Omega} |\nabla u(x)|^2 x \cdot \nu(x) ds \tag{4.2}$$

where ν denotes the unit outward normal to $\partial\Omega$.

From 4.2, in particular, it follows that, if Ω is star-shaped with respect to 0 (so $x \cdot \nu(x) \geq 0$ on $\partial\Omega$), then the set Λ_f is empty in the two following cases:

(α) $f(\xi) = |\xi|^{p-2}\xi$ with $n \geq 3$ and $p \geq \frac{2n}{n-2}$;

(β) $\sup_{\xi\in\mathbf{R}} F(\xi) = 0$.

For $L > 0$, denote by $\mathcal{C}_L$ the class of all Lipschitzian functions $f : \mathbf{R} \to \mathbf{R}$, with Lipschitz constant L, such that $\sup_{\xi\in\mathbf{R}} F(\xi) = 0$.

In [18], I proved

Theorem 4.4. *One has*

$$\inf_{f\in\mathcal{C}_L} \inf \Lambda_f \geq \frac{3\lambda_1}{L} \ .$$

I also proposed the following

Conjecture 4.1. If Ω is not star-shaped with respect to 0, one has

$$\inf_{f\in\mathcal{C}_L} \inf \Lambda_f = \frac{3\lambda_1}{L} \ .$$

X. L. Fan disproved Conjecture 4.1 in [4] and, at the same time, he surprisingly proved, in [5], that a non-autonomous version of it holds with any Ω.

We now recall a problem from [6] which still is completely open:

Problem 4.1. If $\Omega = \{x \in \mathbf{R}^n : a < |x| < b\}$ with $0 < a < b$, is there some $f \in \mathcal{C}_1$ for which the set Λ_f is non-empty ?

In particular, what about this problem when $f(\xi) = -\sin\xi$?

In [6], it only was remarked what follows

Theorem 4.5. *Let* $\Omega = \{x \in \mathbf{R}^n : a < |x| < b\}$ *with* $0 < a < b$.
Then, for every continuous function $f : \mathbf{R} \to \mathbf{R}$, *satisfying* $\sup_{\xi\in\mathbf{R}} F(\xi) = 0$, *and every* $\lambda > 0$, *problem* $(P_{\lambda f})$ *has no radially symmetric non-zero classical solutions.*

If $f \in \mathcal{A}_q$, with $q < \frac{n+2}{n-2}$, by the Rellich-Kondrachov theorem, the functional $\Phi : X \to \mathbf{R}$ defined by

$$\Phi_f(u) = \int_\Omega F(u(x))dx$$

is sequentially weakly continuous in X.

In [17], I asked the following

Problem 4.2. Are there non-constant $f \in \mathcal{A}_q$, with $q < \frac{n+2}{n-2}$, for which the functional Φ_f is weakly continuous ?

A complete negative answer to Problem 4.2 was provided by R. Černý, S. Hencl and J. Kolář in [3].

I conclude proposing the following

Conjecture 4.2. Let $f \in \mathcal{A}_q$, with $q < \frac{n+2}{n-2}$. Assume also that the functional J_f is weakly lower semicontinuous in $H_0^1(\Omega)$.
Then, one has

$$\lim_{\|u\| \to +\infty} J_f(u) = +\infty \ .$$

References

[1] A. AMBROSETTI and P. H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349-381.

[2] G. ANELLO, *A note on a problem by Ricceri on the Ambrosetti-Rabinowitz condition*, Proc. Amer. Math. Soc., **135** (2007), 1875-1879.

[3] ČERNÝ, S. HENCL and J. KOLÁŘ, *Integral functionals that are continuous with respect to the weak topology on $W_0^{1,p}(\Omega)$*, Nonlinear Anal., **71** (2009), 2753-2763.

[4] X. L. FAN, *On Ricceri's conjecture for a class of nonlinear eigenvalue problems*, Appl. Math. Lett., **22** (2009), 1386-1389.

[5] X. L. FAN, *A remark on Ricceri's conjecture for a class of nonlinear eigenvalue problems*, J. Math. Anal. Appl., **349** (2009), 436-442.

[6] X. L. FAN and B. RICCERI, *On the Dirichlet problem involving nonlinearities with non-positive primitive: a problem and a remark*, Appl. Anal., **89** (2010), 189-192.

[7] M. FRASCA and A.VILLANI, *A property of infinite-dimensional Hilbert spaces*, J. Math. Anal. Appl., **139** (1989), 352-361.

[8] F. J. GARCÍA-PACHECO, *An approach to a Ricceri's conjecture*, Topology Appl., **159** (2012), 3307-3313.

[9] P. HARTMAN and G. STAMPACCHIA, *On some nonlinear elliptic differential equations*, Acta Math., **115** (1966).

[10] B. RICCERI, *Un théorème d'existence pour les inéquations variationnelles*, C. R. Acad. Sci. Paris, Série I, **301** (1985), 885-888.

[11] B. RICCERI, *Images of open sets under certain multifunctions*, Rend. Accad. Naz. Sci. XL, **10** (1986), 33-37.

[12] B. RICCERI, *On the well-posedness of the Cauchy problem for a class of linear partial differential equations of infinite order in Banach spaces*, J. Fac. Sci. Univ. Tokyo, Sec. IA, **38** (1991), 623-640.

[13] B. RICCERI, *Basic existence theorems for generalized variational and quasi-variational inequalities*, in "Variational inequalities and network equilibrium problems", F.Giannessi and A.Maugeri eds., 251-255, Plenum Press, 1995.

[14] B. RICCERI, *On some motivated conjectures and problems*, Matematiche, **51** (1996), 369-373.

[15] B. RICCERI, *On a classical existence theorem for nonlinear elliptic equations*, in "Experimental, constructive and nonlinear analysis", M. Théra ed., 275-278, CMS Conf. Proc. **27**, Canad. Math. Soc., 2000.

[16] B. RICCERI, *Some research perspectives in nonlinear functional analysis*, Seminar on fixed point theory, Cluj-Napoca, **3** (2002), 99-110.

[17] B. RICCERI, *Three topological problems about integral functionals on Sobolev spaces*, J. Global Optim., **28** (2004), 401-404.

[18] B. RICCERI, *A remark on a class of nonlinear eigenvalue problems*, Nonlinear Anal., **69** (2008), 2964-2967.

[19] B. RICCERI, *Existence of fixed points for a particular multifunction*, Fixed Point Theory, **11** (2010), 125-128.

[20] M. ROMEO, *Superposition operator in a space of infinitely differentiable functions*, Z. Anal. Anwend., **27** (2008), 463-467.

[21] N. D. YEN, *On a problem of Ricceri on variational inequalities*, Fixed point theory and applications. Vol. 5, 163-173, Nova Sci. Publ., 2004.

Biagio Ricceri
Department of Mathematics and Informatics,
University of Catania,
Viale A. Doria 6,
95125, Catania,
Italy
E-mail: ricceri@dmi.unict.it