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Hölder regularity for non-divergence-form elliptic equations with discontinuous coefficients

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ABSTRACT

In this note, we study the global Hölder regularity for the gradient of the strong solutions of non-variational elliptic equations.

1. Introduction

The aim of this note is to study the global Morrey regularity for the second derivatives of the strong solutions of non-variational elliptic equations. Namely, given a bounded domain $\Omega \subset \mathbb{R}^n$, we consider the linear equation

$$a_{ij}u_{x_ix_j} + b_iu_{x_i} + cu = f, \tag{1}$$

where the lower order coefficients b_i, c and f are assumed in the Morrey space $L^{p,\lambda}$ $(2 \le p < n, n - p < \lambda < n)$ and the coefficients of the leading part are assumed in the class $VMO \cap L^\infty$. We obtain Morrey estimates for second derivatives for the strong solutions of Eq. (1) (see Theorem 3.2) and, as a consequence of our estimate, we obtain the Hölder continuity of the gradient of the solutions.

The technique we use is quite simple; it is based on a multiplicative inequality for functions in Morrey classes combined with an iterative procedure, and it exploits the Morrey bounds proved in [8] for the principal part operator.

The same problem has been studied in several papers by many Authors. Among them, we cite [1], where Caffarelli considers the equation

$$a_{ij}u_{x_ix_j} = f, \tag{2}$$

proving that, if f belongs to the Morrey space $L^{n,n\alpha}$, with $0 < \alpha < 1$, then every $W^{2,p}$-viscosity solution u is of class $C^{1,\alpha}$.

Subsequently, in [9,8], Caffarelli's result was improved, obtaining for Eq. (2) gradient regularity for any $p < n$. In [9], the interior Morrey space regularity was obtained via a representation formula for the second derivatives of the solutions used in [3] and the study of some non-convolution-type integral operators. In [8], the Authors extend the result to the boundary, obtaining global Morrey regularity.

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2. Preliminaries

Let Ω be a bounded open set in $\mathbb{R}^n$ ($n \geq 3$). If $f \in L^1(\Omega)$ and $E \subset \Omega$, we set $f_E = \frac{1}{|E|}\int_E f dx$.

We recall some classical definitions (see [2,14]).

Definition 2.1. Let Ω be a bounded domain in $\mathbb{R}^n$, $1 \leq p < +\infty$ and $0 \leq \lambda \leq n$. A function $f \in L^p(\Omega)$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ if

$$\|f\|^p_{L^{p,\lambda}(\Omega)} = \sup r^{-\lambda}\int_{B_r(x_0)\cap\Omega} |f|^p dx < +\infty,$$

the supremum being taken over $x_0 \in \Omega$ and $0 < r \leq \operatorname{diam}\Omega$.

It is well known that $L^{p,\lambda}(\Omega)$ is a Banach space endowed with the above norm. Morrey and Campanato spaces are very useful in the study of regularity properties for generalized solutions of elliptic equations (see, e.g., [15,16,10,12,11,13,5–7]).

Definition 2.2. Let Ω be a bounded domain in $\mathbb{R}^n$, $1 \leq p < +\infty$ and $0 \leq \lambda \leq n + p$. A function $f \in L^p(\Omega)$ belongs to the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$ if

$$[f]^p_{\mathcal{L}^{p,\lambda}(\Omega)} = \sup r^{-\lambda}\int_{B_r(x_0)\cap\Omega} |f - f_{B_r(x_0)\cap\Omega}|^p dx < +\infty, \tag{3}$$

the supremum being taken over $x_0 \in \Omega$ and $0 < r \leq \operatorname{diam}\Omega$.

If $\lambda = n$, the definition gives back the definition of the space *BMO*.

Definition 2.3. Let Ω be a bounded domain in $\mathbb{R}^n$. A locally integrable function f of class $BMO(\Omega)$ belongs to $VMO(\Omega)$ if

$$\eta(r) = \sup \fint_{B_\rho(x_0)\cap\Omega} |f - f_{B_\rho(x_0)\cap\Omega}| dx$$

vanishes as $r \to 0^+$. Here, the supremum is taken over $x_0 \in \Omega$ and $0 < \rho < r$.

The space $\mathcal{L}^{p,\lambda}$ is a Banach space endowed with the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{\mathcal{L}^{p,\lambda}(\Omega)},$$

where $[f]_{\mathcal{L}^{p,\lambda}(\Omega)}$ is given by (3).

Definition 2.4. Let Ω be a bounded domain in $\mathbb{R}^n$. We say that Ω satisfies condition K if there exists a constant $K > 0$ such that

$$|B_r(x_0) \cap \Omega| \geq K r^n,$$

$\forall x_0 \in \Omega$ and $0 < r \leq \operatorname{diam}\Omega$.

Remark 2.1. Any convex domain satisfies condition K.

We state some useful results we need in what follows.

Lemma 2.1. *Let $1 \leq q \leq p < +\infty$ and $0 \leq \lambda, \lambda_1 \leq n$. If $q(n - \lambda) \leq p(n - \lambda_1)$, then $L^{p,\lambda}(\Omega)$ is continuously imbedded in $L^{q,\lambda_1}(\Omega)$.*

Proof. The proof is a simple application of Hölder inequality. □

Theorem 2.1 ([2, Theorem 2.I]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying condition K.*

1. *If $0 \leq \lambda < n$ then $\mathcal{L}^{p,\lambda}(\Omega) = L^{p,\lambda}(\Omega)$ and there exist two positive constants C_1 and C_2 such that*

$$C_1[f]_{\mathcal{L}^{p,\lambda}} \leq \|f\|_{L^{p,\lambda}} \leq C_2\|f\|_{\mathcal{L}^{p,\lambda}}.$$

2. *If $n < \lambda \leq n + p$ then $\mathcal{L}^{p,\lambda}(\Omega) = C^{0,\gamma}(\overline{\Omega})$, with $\gamma = \frac{\lambda - n}{p}$, and there exist two positive constants C_3 and C_4 such that*

$$C_3[f]_{\mathcal{L}^{p,\lambda}} \leq [f]_\gamma \leq C_4[f]_{\mathcal{L}^{p,\lambda}},$$

where $[f]_\gamma = \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma}$.

Theorem 2.2 (*Poincaré Inequality*). *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and let $u \in W^{1,p}(\Omega)$. Then there exists a positive constant C such that*

$$\int_\Omega |u - u_\Omega|^p dx \leq C\,(\operatorname{diam}\Omega)^p \int_\Omega |\nabla u|^p dx. \tag{4}$$

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Let $u \in W^{1,p}(\Omega)$ be such that $\nabla u \in L^{p,\mu}(\Omega)$, $0 \le \mu < n+p$. Then $u \in \mathcal{L}^{p,\mu+p}(\Omega)$ and there exists a positive constant C such that*

$$[u]_{\mathcal{L}^{p,\mu+p}(\Omega)} \le C\|\nabla u\|_{L^{p,\mu}(\Omega)}. \tag{5}$$

In particular, if $\mu + p < n$ then

$$\|u\|_{L^{p,\mu+p}(\Omega)} \le C(\|\nabla u\|_{L^{p,\mu}(\Omega)} + \|u\|_{L^p(\Omega)}). \tag{6}$$

Proof. Let $B_r(x_0)$ be a ball. Since $u \in W^{1,p}(B_r(x_0)\cap\Omega)$, from the Poincaré inequality (4) we have

$$\int_{B_r(x_0)\cap\Omega} |u - u_{B_r(x_0)\cap\Omega}|^p dx \le C\, r^p \int_{B_r(x_0)\cap\Omega} |\nabla u|^p dx,$$

from which we have

$$r^{-(\mu+p)} \int_{B_r(x_0)\cap\Omega} |u - u_{B_r(x_0)\cap\Omega}|^p dx \le C\, r^{-\mu} \int_{B_r(x_0)\cap\Omega} |\nabla u|^p dx,$$

which, since $\nabla u \in L^{p,\mu}(\Omega)$, implies that $u \in \mathcal{L}^{p,\mu+p}(\Omega)$ and (5). Moreover, from Theorem 2.1, we obtain (6). □

In what follows we will use also the following multiplicative inequality.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Let $u \in W^{1,p}(\Omega)$ and $f \in L^{p,\lambda}(\Omega)$, with $2 \le p < n$, $n-p < \lambda < n$. If $\nabla u \in L^{p,\eta}(\Omega)$ for some $\eta \in [0, n-p[$, then*

$$fu \in L^{p,\lambda+\eta-n+p}(\Omega).$$

Moreover, there exists a positive constant C, independent of u and f, such that

$$\|fu\|_{L^{p,\lambda+\eta-n+p}(\Omega)} \le C\|f\|_{L^{p,\lambda}(\Omega)}(\|\nabla u\|_{L^{p,\eta}(\Omega)} + \|u\|_{L^p(\Omega)}).$$

The lemma has been proved in [4, Lemma 4.1] for $p = 2$. The extension to the case $p \ne 2$ is straightforward.

3. Hölder regularity

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let us consider the following linear second order elliptic equation in non-divergence form:

$$a_{ij}u_{x_ix_j} + b_iu_{x_i} + cu = f, \tag{7}$$

where we assume that

$$a_{ij}(x) = a_{ji}(x) \quad a.e.\, x \in \Omega,\, i,j = 1,\dots,n, \tag{8}$$

$$\exists\, \mu > 0 : \mu|\xi|^2 \le \sum_{ij=1}^n a_{ij}(x)\xi_i\xi_j \le \frac{1}{\mu}|\xi|^2, \quad a.e.\, x \in \Omega,\ \forall \xi \in \mathbb{R}^n, \tag{9}$$

$$a_{ij} \in L^\infty(\Omega) \cap VMO(\Omega) \tag{10}$$

and

$$b_i\,, c\,, f \in L^{p,\lambda}(\Omega), \qquad 2 \le p < n, \qquad n-p < \lambda < n. \tag{11}$$

Definition 3.1. A function u in $W^{2,p}(\Omega)$ is a strong solution of Eq. (7) if u satisfies (7) a.e. $x \in \Omega$.

We recall Theorem 3.3 in [8] regarding Eq. (7) when $b_i = 0$ and $c = 0$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Let (8)–(11) hold true. Let $u \in W^{2,p} \cap W_0^{1,p}(\Omega)$ be a strong solution of (7), with $b_i = 0$ and $c = 0$. Then $D^2u \in L^{p,\lambda}(\Omega)$ and there exists a positive constant C such that*

$$\|D^2u\|_{L^{p,\lambda}(\Omega)} \le C\left(\|u\|_{L^{p,\lambda}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\right).$$

Our main Theorem is the following.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Let (8)–(11) hold true and let $u \in W^{2,p} \cap W_0^{1,p}(\Omega)$ be a strong solution of (7). Then there exists $0 < \gamma < 1$ such that $\nabla u \in C^{0,\gamma}(\overline{\Omega})$.*

We prove Theorem 3.2 in two special cases. First, we consider the case $b_i = 0$, $i = 1, 2, \dots, n$, that is the equation

$$a_{ij}u_{x_ix_j} + cu = f, \tag{12}$$

and later we study the case $c = 0$, that is the equation

$$a_{ij}u_{x_ix_j} + b_iu_{x_i} = f. \tag{13}$$

The general case can be proved arguing in the same way as for the special cases. We have chosen to not give the detailed proof of the general case because it is very intricate and the proof does not contain anything new.

The result related to Eq. (12) is contained in the following Theorem.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Let (8)–(11) hold true and let $u \in W^{2,p} \cap W_0^{1,p}(\Omega)$ be a strong solution of (12). Then, for all $0 < \epsilon < \min\{n-p, p+\lambda-n\}$, we have that $\nabla u \in C^{0,\gamma}(\overline{\Omega})$, where $\gamma = 1 - \frac{n-\lambda+\epsilon}{p}$. Moreover there exists a positive constant C such that*

$$[\nabla u]_\gamma \leq C\left(\|c\|_{L^{p,\lambda}(\Omega)}\|\nabla u\|_{L^{p,n-p-\epsilon}(\Omega)} + (\|c\|_{L^{p,\lambda}(\Omega)} + 1)\|u\|_{L^{p,\lambda-\epsilon}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\right).$$

Proof. We start by noting that, since $\nabla u \in W^{1,p}(\Omega)$, from Lemma 2.2, $\nabla u \in L^{p,p}(\Omega)$.

If $p \geq n-p$, then, from Lemma 2.1, $\nabla u \in L^{p,n-p-\epsilon}(\Omega)$, with $0 < \epsilon < \min\{n-p, p+\lambda-n\}$, and we apply Lemma 2.3 to obtain $cu \in L^{p,\lambda-\epsilon}(\Omega)$, and from Theorem 3.1 we obtain $D^2u \in L^{p,\lambda-\epsilon}(\Omega)$. Then, from Lemma 2.2, $\nabla u \in \mathcal{L}^{p,p+\lambda-\epsilon}(\Omega)$, and, from Theorem 2.1, since $p+\lambda-\epsilon > n$, we obtain that $\nabla u \in C^{0,\gamma}(\overline{\Omega})$.

If $p < n-p$, from Lemma 2.3, we obtain that $cu \in L^{p,\lambda+2p-n}(\Omega)$, and from Theorem 3.1, since $\lambda+2p-n < \lambda$, we obtain that $D^2u \in L^{p,\lambda+2p-n}(\Omega)$ and, using Lemma 2.2, consequently $\nabla u \in \mathcal{L}^{p,\lambda+3p-n}(\Omega)$.

If $\lambda+3p-n > n$, from Theorem 2.1, ∇u is Hölder continuous in $\overline{\Omega}$.

If $\lambda+3p-n < n$, from Theorem 2.1, $\nabla u \in L^{p,\lambda+3p-n}(\Omega)$. If also $\lambda+3p-n \geq n-p$ then, from Lemma 2.1, $\nabla u \in L^{p,n-p-\epsilon}(\Omega)$, with $0 < \epsilon < \min\{n-p, p+\lambda-n\}$, and we apply Lemma 2.3 to obtain $cu \in L^{p,\lambda-\epsilon}(\Omega)$, from which and Theorem 3.1 we obtain that $D^2u \in L^{p,\lambda-\epsilon}(\Omega)$. Then $\nabla u \in \mathcal{L}^{p,p+\lambda-\epsilon}(\Omega)$ and since $p+\lambda-\epsilon > n$, we obtain that $\nabla u \in C^{0,\gamma}(\overline{\Omega})$.

If $\lambda+3p-n < n-p$, then, from Lemma 2.3, we obtain that $cu \in L^{p,2\lambda+4p-2n}(\Omega)$, from which and Theorem 3.1, since $2\lambda+4p-n < \lambda$, we obtain that $D^2u \in L^{p,2\lambda+4p-2n}(\Omega)$ and $\nabla u \in \mathcal{L}^{p,2\lambda+5p-2n}(\Omega)$.

If $2\lambda+5p-2n > n$, then ∇u is Hölder continuous in $\overline{\Omega}$.

If $2\lambda+5p-2n < n$, $\nabla u \in L^{p,2\lambda+5p-2n}(\Omega)$ and we can proceed as in the case $\lambda+3p-n < n$.

Finally, there exists $k \in \mathbb{N}$ such that $n-p \leq (2k+1)p + k(\lambda-n)$and $\nabla u \in L^{p,(2k+1)p+k(\lambda-n)}(\Omega)$, then $\nabla u \in C^{0,\gamma}(\overline{\Omega})$. Moreover, from Theorems 2.1 and 3.1 and Lemma 2.3, we get

$$\begin{aligned}
[\nabla u]_\gamma &\leq C[\nabla u]_{\mathcal{L}^{p,p+\lambda-\epsilon}(\Omega)} \leq C\|D^2u\|_{L^{p,\lambda-\epsilon}(\Omega)} \\
&\leq C\{\|cu\|_{L^{p,\lambda-\epsilon}(\Omega)} + \|u\|_{L^{p,\lambda-\epsilon}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\} \\
&\leq C\{\|c\|_{L^{p,\lambda}(\Omega)}[\|\nabla u\|_{L^{p,n-p-\epsilon}(\Omega)} + \|u\|_{L^p(\Omega)}] + \|u\|_{L^{p,\lambda-\epsilon}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\} \\
&\leq C\{\|c\|_{L^{p,\lambda}(\Omega)}\|\nabla u\|_{L^{p,n-p-\epsilon}(\Omega)} + (\|c\|_{L^{p,\lambda}(\Omega)} + 1)\|u\|_{L^{p,\lambda-\epsilon}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\}. \quad \square
\end{aligned}$$

Now we study Eq. (13). We have the following.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Let (8)–(11) hold true. Let $u \in W^{2,p} \cap W_0^{1,p}(\Omega)$ be a strong solution of (13). Then there exists $k \in \mathbb{N}$ such that $k\lambda - kn + (k+1)p > n$ and $\nabla u \in C^{0,\gamma}(\overline{\Omega})$, with $\gamma = \frac{k\lambda-(k+1)n+(k+1)p}{p}$. Moreover, there exists a positive constant C depending on $\|b\|_{L^{p,\lambda}(\Omega)}$ and k such that*

$$[\nabla u]_\gamma \leq C\left(\|D^2u\|_{L^p(\Omega)} + \|u\|_{L^{p,\lambda-n+p}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\right). \tag{14}$$

Proof. Since $D^2u \in L^p(\Omega)$, from Lemma 2.3, we obtain that $b \cdot \nabla u \in L^{p,\lambda-n+p}(\Omega)$. Then, from Theorem 3.1, $D^2u \in L^{p,\lambda-n+p}(\Omega)$ and we have

$$\begin{aligned}
\|D^2u\|_{L^{p,\lambda-n+p}(\Omega)} &\leq C\{\|b\nabla u\|_{L^{p,\lambda-n+p}(\Omega)} + \|u\|_{L^{p,\lambda-n+p}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\} \\
&\leq C\{\|b\|_{L^{p,\lambda}(\Omega)}[\|D^2u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}] + \|u\|_{L^{p,\lambda-n+p}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\} \\
&\leq C\{\|b\|_{L^{p,\lambda}(\Omega)}\|D^2u\|_{L^p(\Omega)} + (\|b\|_{L^{p,\lambda}(\Omega)} + 1)\|u\|_{L^{p,\lambda-n+p}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\}.
\end{aligned} \tag{15}$$

From Lemma 2.2, $\nabla u \in \mathcal{L}^{p,\lambda-n+2p}(\Omega)$, where $0 < \lambda-n+2p < n+p$.

If $\lambda-n+2p > n$, from Theorem 2.1, we have that $\nabla u \in C^{0,\gamma}(\overline{\Omega})$, with $\gamma = \frac{\lambda-2n+2p}{p}$. So, from Theorem 2.1, inequalities (5) and (15),

$$\begin{aligned}
[\nabla u]_\gamma &\leq [\nabla u]_{\mathcal{L}^{p,\lambda-n+2p}(\Omega)} \leq C\|D^2u\|_{L^{p,\lambda-n+p}(\Omega)} \\
&\leq C\{\|b\|_{L^{p,\lambda}(\Omega)}\|D^2u\|_{L^p(\Omega)} + (\|b\|_{L^{p,\lambda}(\Omega)} + 1)\|u\|_{L^{p,\lambda-n+p}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\}.
\end{aligned}$$

If $\lambda-n+2p < n$, since $\lambda-n+p < n-p$, we can apply Lemma 2.3 to obtain $b\nabla u \in L^{p,2\lambda-2n+2p}(\Omega)$. Since $2\lambda-2n+2p < \lambda$, from Theorem 3.1, $D^2u \in L^{p,2\lambda-2n+2p}(\Omega)$, and we get also from (15) that

$$\begin{aligned}
\|D^2u\|_{L^{p,2\lambda-2n+2p}(\Omega)} &\leq C\{\|b\nabla u\|_{L^{p,2\lambda-2n+2p}(\Omega)} + \|u\|_{L^{p,2\lambda-2n+2p}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\} \\
&\leq C\{\|b\|_{L^{p,\lambda}(\Omega)}[\|D^2u\|_{L^{p,\lambda-n+p}(\Omega)} + \|u\|_{L^p(\Omega)}] + \|u\|_{L^{p,2\lambda-2n+2p}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\} \\
&\leq C\{\|b\|^2_{L^{p,\lambda}(\Omega)}\|D^2u\|_{L^p(\Omega)} + (\|b\|^2_{L^{p,\lambda}(\Omega)} + \|b\|_{L^{p,\lambda}(\Omega)} + 1)\|u\|_{L^{p,\lambda-n+p}(\Omega)} \\
&\quad + (\|b\|_{L^{p,\lambda}(\Omega)} + 1)\|f\|_{L^{p,\lambda}(\Omega)}\}.
\end{aligned}$$

Now, from Lemma 2.2, we obtain that $\nabla u \in \mathcal{L}^{p,2\lambda-2n+3p}(\Omega)$, where $0 < 2\lambda - 2n + 3p < n + p$.
If $2\lambda - 2n + 3p > n$ then $\nabla u \in C^{0,\gamma}(\overline{\Omega})$, with $\gamma = \frac{2\lambda-3n+3p}{p}$, and

$$\begin{aligned}[\nabla u]_\gamma &\le C[\nabla u]_{\mathcal{L}^{p,2\lambda-2n+3p}(\Omega)} \le C\|D^2u\|_{L^{p,2\lambda-2n+2p}(\Omega)}\\ &\le C\{\|b\|^2_{L^{p,\lambda}(\Omega)}\|D^2u\|_{L^p(\Omega)} + (\|b\|^2_{L^{p,\lambda}(\Omega)} + \|b\|_{L^{p,\lambda}(\Omega)} + 1)\|u\|_{L^{p,\lambda-n+p}(\Omega)} + (\|b\|_{L^{p,\lambda}(\Omega)} + 1)\|f\|_{L^{p,\lambda}(\Omega)}\}.\end{aligned}$$

If $2\lambda - 2n + 3p < n$ $(2\lambda - 2n + 2p < n - p)$, then we proceed as the previous cases.

Finally, there exists a positive integer k such that $n < k\lambda - kn + (k+1)p$, then $\nabla u \in C^{0,\gamma}(\overline{\Omega})$, with $\gamma = \frac{k\lambda-(k+1)n+(k+1)p}{p}$ and

$$\begin{aligned}[\nabla u]_\gamma &\le C[\nabla u]_{\mathcal{L}^{p,k\lambda-kn+(k+1)p}(\Omega)} \le C\|D^2u\|_{L^{p,k\lambda-kn+(k+1)p}(\Omega)}\\ &\le C\{\|b\|^k_{L^{p,\lambda}(\Omega)}\|D^2u\|_{L^p(\Omega)} + (\|b\|^k_{L^{p,\lambda}(\Omega)} + \|b\|^{k-1}_{L^{p,\lambda}(\Omega)} + \cdots + \|b\|_{L^{p,\lambda}(\Omega)} + 1)\|u\|_{L^{p,\lambda-n+p}(\Omega)}\\ &\quad + (\|b\|^{k-1}_{L^{p,\lambda}(\Omega)} + \cdots + \|b\|_{L^{p,\lambda}(\Omega)} + 1)\|f\|_{L^{p,\lambda}(\Omega)}\},\end{aligned}$$

from which (14) follows. □

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