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Nesting *House*-designs

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ABSTRACT

A cycle of length 5 with a *chordal*, i.e. an edge joining two non-adjacent vertices of the cycle, is called a graph H_5 or also an *House-graph*. In this paper, the spectrum of House-systems nesting C_3-systems, C_4-systems, C_5-systems and together (C_3, C_4, C_5)-systems, of all admissible indices are completely determined, without exceptions.

1. Introduction

Let λK_v be the complete multigraph defined in a vertex-set X, $|X| = v$. Let G be a subgraph of λK_v. A *G-decomposition* of λK_v, of order v and index λ, is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a partition of the edge-set of λK_v into subsets all of which yield subgraphs isomorphic to G. A *G-decomposition* of λK_v is also called a G-design, of order v and index λ. The classes of the partition $\mathcal{B}$ are said *blocks*. Important and interesting results about G-designs can be found in [5,10,12,13].

A cycle of length 5 with a *chordal*, i.e. an edge joining two not adjacent vertices of the cycle, will be called an *House-graph* and will be denoted by H_5. If $H_5 = (X, E)$, where $X = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{b, c\}, \{c, d\} \{d, e\}, \{e, a\}, \{a, c\}\}$, we will denote such a graph by $[(a), b, (c), d, e]$.

Let $\Sigma = (X, \mathcal{B})$ be H_5-design of order v and index λ or an H_5-decomposition of the complete multigraph λK_v. When a graph $H_5 = [(a), b, (c), d, e]$ is a block of Σ with *multiplicity* n, it will be indicated by $[(a), b, (c), d, e]_{(n)}$. Similar concepts and symbolism are given in [3].

We say that Σ is:

– (1) C_3-*perfect* if the family of all the C_3-cycles having edges $\{a, b\}, \{b, c\}, \{a, c\}$ generates a C_3-design Σ' of order v and index μ;

– (2) C_4-*perfect*, if the family of all the C_4-cycles having edges $\{a, c\}, \{c, d\}, \{d, e\}, \{e, a\}$ generates a C_4-design Σ' of order v and index σ;

– (3) C_5-*perfect*, if the family of all the C_5-cycles having edges $\{a, b\}, \{b, c\}, \{c, d\} \{d, e\}, \{e, a\}$ generates a C_5-design Σ' of order v and index τ.

In the case (1), we say that Σ has indices (λ, μ). Similarly, in (2) its indices are (λ, σ) and in (3) (λ, τ). Similar definitions and symbolism is given in [1,2,6]. For *perfect* G-designs see also [8,11].

In every case, we say that Σ' is a system *nested* into Σ, and also that Σ is nesting Σ'.

We say that an H_5-design Σ, which is C_h-*perfect*, with indices (λ, μ), and C_k-*perfect* with indices (λ, σ), for $h, k = 3, 4, 5$, has indices (λ, μ, σ), and we will say that it is a (C_h, C_k)-perfect. Similarly, if Σ of index λ is C_3-perfect of index μ, C_4-perfect of index σ, and also C_5-perfect of index τ, we will say that Σ is (C_3, C_4, C_5)-perfect, of indices $(\lambda, \mu, \sigma, \tau)$.

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It is known [4] that:

Theorem 1.1. *An H_5-design of order v exists if and only if $v \equiv 0$, or 1, or 4, or 9 (mod 12), $v \geq 9$, with the possible exception of $v = 24$.*

Further, the spectrum of House-designs nesting C_4-systems, for every admissible indices, is determined in [3], where the authors proved that:

Theorem 1.2. *There exists a C_4-perfect H_5-design of order v and indices $(3, 2)$ if and only if $v \equiv 0$ or 1 (mod 4), $v \geq 5$.*

Theorem 1.3. *There exists a C_4-perfect H_5-design of order v and indices $(6, 4)$ if and only if $v \geq 5$.*

Theorem 1.4. *There exists a C_4-perfect H_5-design of order v, $v \geq 5$, and indices (λ, μ) such that $2\lambda = 3\mu$.*

In this paper we study the all possible nestings in House-systems, determining completely the spectrum in all the possible cases.

In what follows, to construct House-systems, we will use often the *difference-method*. This means that we fix as vertex-set $X = Z_v$ and, defined a *base-block* $[(a), b, (c), d, e]$, its *translates* will be all the blocks of type $[(a+i), b+i, (c+i), d+i, e+i]$, for every $i \in \mathbb{Z}_v$. For a given v, it will be $D(v) = \{|x - y| : x, y \in \mathbb{Z}_v, x \neq y\}$.

2. C_3-perfect H_5-designs of index (2, 1)

In this section, the spectrum of C_3-perfect H_5-designs of index (2, 1) is completely determined. We begin with the necessary conditions.

Theorem 2.1. *If $\Sigma = (X, \mathcal{B})$ is a C_3-perfect H_5-design of order v and indices (λ, μ), then:*
(1) $\lambda = 2\mu$;
(2) $|\mathcal{B}| = \mu \frac{v(v-1)}{6}$;
(3) *for $\mu = 1$, it is $v \equiv 1, 3$ (mod 6).*

Proof. Let $\Sigma = (X, B)$ be a C_3-perfect H_5-design of order v and indices (λ, μ). If $\Sigma' = (X, B')$ is the C_3-system nested in Σ, necessarily: $\mathcal{B} = \mathcal{B}'$. Since
$|\mathcal{B}| = \lambda \frac{v(v-1)}{12}$, $|\mathcal{B}'| = \mu \frac{v(v-1)}{6}$,
(1) and (2) follow easily. For (3), consider that Σ' is a Steiner triple system of index 1. □

Now we determine the spectrum of C_3-perfect H_5-designs of index (2, 1), examining at first the case $v = 6h + 1$ and after the case $v = 6h + 3$.

Theorem 2.2. *For $\lambda = 2$, $\mu = 1$ and for every $v \equiv 1$ (mod 6), $v \geq 7$, there exists a C_3-perfect H_5-design of order v and indices (2, 1).*

Proof. Let $v \equiv 1$ (mod 6), $v \geq 7$. We can consider the following cases:
(1) $v \equiv 7$ (mod 18);
(2) $v \equiv 13$, (mod 18);
(3) $v \equiv 1$ (mod 18), $v \geq 19$.

(1) Let $v = 7$. It is: $D(7) = \{1, 2, 3\}$. Therefore, consider the block: $B = [(0), 3, (1), 4, 6]$. If $\mathcal{B}$ is the collection of all the translates of B, we can verify that $\Sigma = (\mathbb{Z}_7, \mathcal{B})$ is an H_5-design of order 7 and indices (2, 1). Further, since in B the differences $\{1, 2, 3\}$ cover, exactly one time, the edges of the C_3-cycle, it follows that Σ is C_3-perfect.

Let $v = 18k + 7$, for $k \geq 1$. Since $D = \{1, 2, \dots, 9k + 3\}$, it is possible to define the following $3k + 1$ base-blocks:
$B_{1,h} = [(0), 8k + 2h + 4, (3h + 1), 3k + 2, 3h + 3]$, for $h \in \{0, \dots, k - 1\}$;
$B_{2,h} = [(0), 6k + h + 3, (3h + 2), 9k + 2, 3k + 3h + 2]$, for $h \in \{0, \dots, k - 1\}$;
$B_{3,h} = [(0), 4k + 2h + 4, (3h + 3), 12k + 5, 6k + 3h + 4]$, for $h \in \{0, \dots, k - 1\}$;
$B_4 = [(0), 7k + 3, (3k + 1), 9k + 3, 18k + 6]$.

If $\mathcal{B}$ is the collection of all the translates of these base-blocks, we can verify that $\Sigma = (\mathbb{Z}_{18k+7}, \mathcal{B})$ is an H_5-design having indices (2, 1). Observe that, in the base-blocks, the differences $1, 2, \dots, 9k + 3$ cover, exactly one time, the edges of the C_3-cycles. Further, the number of base-blocks is $3k + 1$ and every of them generates $18k + 7$ translates. It follows that $|\mathcal{B}| = (3k + 1)(18k + 7)$ and Σ is C_3-perfect.

(2) Let $v = 13$. It is: $D = \{1, 2, \dots, 6\}$. Therefore, it is possible to define the two base-blocks: $B_1 = [(0), 4, (1), 7, 3]$, $B_2 = [(0), 7, (2), 4, 5]$. If $\mathcal{B}$ is the collection of all the translates of B_1 and B_2, we can verify that $\Sigma = (\mathbb{Z}_{13}, \mathcal{B})$ is an H_5-design having indices (2, 1). Further, since in B_1 and B_2 the differences $\{1, 3, 4\}$ and $\{2, 5, 6\}$ cover, exactly one time, respectively the edges of the two C_3-cycles, it follows that Σ is C_3-perfect.

Let $v = 18k + 13$, for $k \geq 1$. Since $D = \{1, 2, \dots, 9k + 6\}$, it is possible to define the following $3k + 2$ base-blocks:
$B_{1,h} = [(0), 4k + 2h + 4, (3h + 1), 3k + 2, 3h + 3]$, for $h \in \{0, \dots, k - 1\}$;
$B_{2,h} = [(0), 6k + h + 5, (3h + 2), 9k + 8, 3k + 3h + 5]$, for $h \in \{0, \dots, k - 1\}$;
$B_{3,h} = [(0), 8k + 2h + 8, (3h + 3), 12k + 8, 6k + 3h + 7]$, for $h \in \{0, \dots, k - 1\}$;

$B_4 = [(0), 6k+4, (3k+1), 9k+6, 3k+2]$;
$B_5 = [(0), 10k+7, (3k+2), 6k+5, 6k+6]$.

If $\mathcal{B}$ is the collection of all the translates of these base-blocks, we can verify that $\Sigma = (\mathbb{Z}_{18k+13}, \mathcal{B})$ is an H_5-design having indices $(2, 1)$. Observe that, in the base-blocks, the differences $1, 2, \dots, 9k+6$ cover, exactly one time, the edges of the C_3-cycles. Further, the number of base-blocks is $3k+2$ and every of them generates $18k+13$ translates. It follows that $|\mathcal{B}| = (3k+2)(18k+13)$ and Σ is C_3-perfect.

(3) Let $v = 19$. It is: $D = \{1, 2, \dots, 9\}$. Therefore, it is possible to define the two base-blocks: $B_1 = [(0), 6, (1), 9, 18]$, $B_2 = [(0), 10, (2), 5, 7]$, $B_3 = [(0), 7, (3), 9, 5]$. If $\mathcal{B}$ is the collection of all the translates of B_1, B_2, B_3, we can verify that $\Sigma = (\mathbb{Z}_{19}, \mathcal{B})$ is an H_5-design having indices $(2, 1)$. Further, since in B_1, B_2, B_3, the differences $\{1, 5, 6\}$, $\{2, 8, 10\}$, $\{3, 4, 7\}$ cover, exactly one time, respectively the edges of the three C_3-cycles, it follows that Σ is C_3-perfect.

Let $v = 18k+1$, for $k \geq 2$. Since $D = \{1, 2, \dots, 9k\}$, it is possible to define the following $3k$ base-blocks:
$B_{1,h} = [(0), 4k+2h+2, (3h+1), 3k+2, 3h+3]$, for $h \in \{0, \dots, k-1\}$;
$B_{2,h} = [(0), 8k+2h+2, (3h+2), 9k+2, 3k+3h+2]$, for $h \in \{0, \dots, k-1\}$;
$B_{3,h} = [(0), 6k+h+2, (3h+3), 12k+2, 6k-3h-2]$, for $h \in \{0, \dots, k-2\}$;
$B_4 = [(0), 6k+1, (3k), 9k+1, 18k]$.

If $\mathcal{B}$ is the collection of all the translates of these base-blocks, we can verify that $\Sigma = (\mathbb{Z}_{18k+1}, \mathcal{B})$ is an H_5-design having indices $(2, 1)$. Observe that, in the base-blocks, the differences $1, 2, \dots, 9k$ cover, exactly one time, the edges of the C_3-cycles. Further, the number of base-blocks is $3k$ and every of them generates $18k+1$ translates. It follows that $|\mathcal{B}| = (3k)(18k+1)$ and Σ is C_3-perfect. □

Theorem 2.3. *For $\lambda = 2$, $\mu = 1$ and for every $v \equiv 3 \pmod 6$, $v \geq 9$, there exists a C_3-perfect H_5-design of order v and indices* $(2, 1)$.

Proof. Let $v \equiv 3 \pmod 6$, $v \geq 9$. We can consider the following cases:

(1) $v \equiv 9 \pmod{12}$;

(2) $v \equiv 3 \pmod{12}$, $v \geq 15$.

(1) Let $v = 9$. Consider the system $\Sigma = (\mathbb{Z}_9, \mathcal{B})$, where $\mathcal{B}$ is the following collection of blocks:

$$\Big\{[(0), 2, (1), 4, 3],\ [(3), 5, (4), 2, 1],\ [(7), 6, (8), 2, 5],\ [(0), 6, (3), 7, 4],$$
$$[(1), 7, (4), 8, 5],\ [(2), 8, (5), 0, 7],\ [(0), 4, (8), 7, 1],\ [(1), 5, (6), 3, 8],$$
$$[(3), 2, (7), 6, 5],\ [(0), 7, (5), 4, 6],\ [(1), 8, (3), 2, 6],\ [(2), 4, (6), 8, 0]\Big\}.$$

It is possible to verify that Σ is a C_3-perfect H_5-design of order 9 and indices $(2, 1)$.

Let $v = 12k+9$ for $k \geq 1$. Let us consider the system $\Sigma = (\mathbb{Z}_{4k+3} \times \mathbb{Z}_3, \mathcal{B})$ having as blocks the following:
$A_{i,r} = [((i, 0)), (i+r, 0), ((i+\frac{r}{2}, 1)), (i, 1), (i+\frac{r}{2}, 0)]$, for $i, r \in \mathbb{Z}_{4k+3}$ and $r \in \{1, \dots, 2k+1\}$;
$B_i = [((i, 0)), (i, 2), ((i, 1)), (i+4k+2, 0), (i+2k+2, 1)]$, for $i \in \mathbb{Z}_{4k+3}$;
$C_{i,j} = [((i, 1)), (\frac{i+j}{2}, 2), ((j, 1)), (i, 2), (j, 2)]$, for $i, j \in \mathbb{Z}_{4k+3}$, with $i \not\equiv j$;
$D_{i,r} = [((i+r, 2)), (i, 2), ((i+\frac{r}{2}, 0)), (i+r-2k-2, 2), (i+2r, 0)]$, for $i, r \in \mathbb{Z}_{4k+3}$ and $r \in \{1, \dots, k+1\}$;
$E_i = [((i, 2)), (i+3k+2, 0), ((i+2k+1, 2)), (i, 0), (i, 1)]$, for $i \in \mathbb{Z}_{4k+3}$;
$F_{i,r} = [((i+r, 2)), (i, 2), ((i+\frac{r}{2}, 0)), (i+\frac{3}{2}r+k+2, 1), (i+\frac{r}{2}+2k+2, 0)]$, for $i, r \in \mathbb{Z}_{4k+3}$ and $r \in \{k+2, \dots, 2k\}$ if $k \geq 2$.

Examining all the blocks, we can verify that Σ is a C_3-perfect H_5-design of order $12k+9$ and indices $(2, 1)$.

(2) Let $v = 12k+3$ for $k \geq 1$. Let us consider the system $\Sigma = (\mathbb{Z}_{4k+1} \times \mathbb{Z}_3, \mathcal{B})$ having the following base blocks:
$A_{i,r} = [((i+r, 2)), (i, 2), ((i+\frac{r}{2}, 0)), (i+r-2k-1, 2), (i+2r, 0)]$, for $i, r \in \mathbb{Z}_{4k+1}$ and $r = 1, \dots, k$;
$B_{i,r} = [((i+r, 2)), (i, 2), ((i+\frac{r}{2}, 0)), (i+\frac{3}{2}r+2k+1, 1), (i+\frac{r}{2}+2k+1, 0)]$, for $i, r \in \mathbb{Z}_{4k+1}$ and $r = k+1, \dots, 2k$;
$C_{i,j} = [((i, 1)), (\frac{i+j}{2}, 2), ((j, 1)), (i, 2), (j, 2)]$, for $i, j \in \mathbb{Z}_{4k+1}$ and $i \neq j$;
$D_i = [((i, 0)), (i, 2), ((i, 1)), (i+2k, 1), (i+2k, 2)]$;
$E_i = [((i, 0)), (i+k, 1), ((i+2k, 0)), (i, 1), (i+k, 0)]$;
$F_{i,r} = [((i, 0)), (i+r, 0), ((i+\frac{r}{2}, 1)), (i-k, 1), (i+\frac{r}{2}, 0)]$, for $i, r \in \mathbb{Z}_{4k+1}$ and $r = 1, \dots, 2k-1$.

Examining all the blocks, we can verify that Σ is a C_3-perfect H_5-design of order $12k+3$ and indices $(2, 1)$. □

Collecting together the results of this section, it follows that:

Theorem 2.4. *A C_3-perfect H_5-design of indices* $(2, 1)$ *exists if and only if* $v \equiv 1$ *or* $3 \pmod 6$.

3. C_3-perfect H_5-design with $\mu > 1$

In this section we consider C_3-perfect H_5-design of indices (λ, μ), with $\mu > 1$, determining all the possible v of their spectrum. We recall that a *transversal* T of a *latin square* of order n is a set of n cells, exactly one cell from each row and column, such that each of the elements of Z_n occurs in a cell of T. Further, remember that [7,9]:

Lemma. (1) *An idempotent latin square, defined in* $\mathbb{Z}_n$, *exists for any integer* $n \neq 2$.

(2) *An idempotent commutative latin square, defined in* $\mathbb{Z}_n$, *exists if and only if* n *is odd.*

Latin squares, which are almost equivalent to the concept of finite quasigroups, will be used in the constructions given in Theorems 3.1 and 3.5. They are a common tool, since, given a quasigroup $(\mathbb{Z}_n, \circ)$, all the edges on the complete graph defined on $\mathbb{Z}_n$ are of type $\{i, i \circ j\}$, for any $i, j \in \mathbb{Z}_n$, $i \neq j$.

Now we prove the following results:

Theorem 3.1. *A* C_3*-perfect* H_5*-design of indices* $(4, 2)$ *exists if and only if* $v \equiv 0$ *or* $1 \pmod 3$.

Proof. It is known that a 2-fold triple system of order v exists if and only if $v \equiv 0, 1 \pmod 3$ [7,9]. Since, for every $v \equiv 1$ or $3 \pmod 6$, there exist C_3-perfect H_5-design of indices $(2, 1)$ (Theorem 2.2, Theorem 2.3), for such values of v, we can obtain C_3-perfect H_5-design of indices $(4, 2)$ by a repetition of blocks, giving to each of them multiplicity 2.

Therefore, to prove the statement, it remains to examine the cases $v \equiv 0$ or $4 \pmod 6$. We study at first the case (1) $v = 6k$ and after the case (2) $v = 6k + 4$.

(1) Let $v = 6$. Let us consider the system $\Sigma = (\mathbb{Z}_6, \mathcal{B})$ such that:

$$\mathcal{B} = \{[(2), 1, (4), 5, 0], [(4), 2, (5), 3, 0], [(5), 3, (1), 4, 0], [(1), 4, (3), 2, 0], [(3), 5, (2), 1, 0], [(1), 0, (2), 4, 5], [(2), 0, (3), 4, 1], [(3), 0, (4), 2, 5], [(4), 0, (5), 1, 3], [(5), 0, (1), 3, 2]\}.$$

We can verify that Σ is a C_3-perfect H_5-design of order 6 and indices $(4, 2)$.

Let $v = 6k$ for $k \geq 2$. Let us consider an idempotent quasigroup $(\mathbb{Z}_{2k}, \circ)$ and the system $\Sigma = (\mathbb{Z}_{2k} \times \mathbb{Z}_3, \mathcal{B})$ having the following blocks:

$A_i = [((i, 0)), (i, 1), ((i, 2)), (-i + 1, 1), (-i + 1, 2)]$, for $i \in \mathbb{Z}_{2k}$;

$B_i = [((i, 0)), (i, 1), ((i, 2)), (-i + 1, 0), (-i + 1, 1)]$, for $i \in \mathbb{Z}_{2k}$;

$C_{i,j} = [((i, 0)), (i \circ j, 1), ((j, 0)), (-i + 1, 2), (-j + 1, 2)]$, for $i, j \in \mathbb{Z}_{2k}$, $i < j$;

$D_{i,j} = [((i, 0)), (j \circ i, 1), ((j, 0)), (-i + 1, 2), (-j + 1, 2)]$, for $i, j \in \mathbb{Z}_{2k}$, $i < j$;

$E_{i,j} = [((i, 1)), (i \circ j, 2), ((j, 1)), (i, 0), (j, 0)]$, for $i, j \in \mathbb{Z}_{2k}$, $i < j$;

$F_{i,j} = [((i, 1)), (j \circ i, 2), ((j, 1)), (-i + 1, 0), (-j + 1, 0)]$, for $i, j \in \mathbb{Z}_{2k}$, $i < j$;

$G_{i,j} = [((i, 2)), (i \circ j, 0), ((j, 2)), (i, 1), (j, 1)]$, for $i, j \in \mathbb{Z}_{2k}$, $i < j$;

$H_{i,j} = [((i, 2)), (j \circ i, 0), ((j, 2)), (-i + 1, 1), (-j + 1, 1)]$, for $i, j \in \mathbb{Z}_{2k}$ $i < j$.

Examining these blocks, we can verify that Σ is a C_3-perfect H_5-design of order $6k$ and indices $(4, 2)$.

(2) Let $v = 6k + 4$ for $k \geq 1$. Let us consider a quasigroup $(\mathbb{Z}_{2k+1}, \circ)$, idempotent, not necessarily commutative, such that $\{(i, i + 1) \mid i \in \mathbb{Z}_{2k+1}\}$ is a transversal. Define the system $\Sigma = (\{\infty\} \cup \mathbb{Z}_{2k+1} \times \mathbb{Z}_3, \mathcal{B})$ having the following blocks:

$A_i = [((i, 0)), (i, 1), ((i, 2)), \infty, (i + 2, 2)]$, for $i \in \mathbb{Z}_{2k+1}$;

$B_i = [((i, 0)), (i, 1), (\infty), (i + 1, 0), (i + 1, 2)]$, for $i \in \mathbb{Z}_{2k+1}$;

$C_i = [((i, 1)), (i, 2), (\infty), (i + 1, 0), (i + 1, 2)]$, for $i \in \mathbb{Z}_{2k+1}$;

$D_i = [((i, 0)), \infty, ((i, 2)), (i + 1, 2), (i + 1, 1)]$, for $i \in \mathbb{Z}_{2k+1}$;

$E_{i,j} = [((i, 0)), (i \circ j, 1), ((j, 0)), (i + 1, 2), (j + 1, 2)]$, for $i, j \in \mathbb{Z}_{2k+1}$, with $i < j$ and $i - j \not\equiv \pm 1$;

$F_i = [((i, 0)), (i + 1, 0), ((i \circ (i + 1), 1)), \infty, (i + 1, 1)]$, for $i \in \mathbb{Z}_{2k+1}$;

$G_{i,j} = [((i, 0)), (j \circ i, 1), ((j, 0)), (i, 2), (j, 2)]$, for $i, j \in \mathbb{Z}_{2k+1}$, with $i < j$;

$H_{i,j} = [((i, 1)), (i \circ j, 2), ((j, 1)), (i - 1, 0), (j - 1, 0)]$, for $i, j \in \mathbb{Z}_{2k+1}$, with $i < j$;

$I_{i,j} = [((i, 1)), (j \circ i, 2), ((j, 1)), (i - 1, 0), (j - 1, 0)]$, for $i, j \in \mathbb{Z}_{2k+1}$, with $i < j$;

$L_{i,j} = [((i, 2)), (i \circ j, 0), ((j, 2)), (i - 1, 1), (j - 1, 1)]$, for $i, j \in \mathbb{Z}_{2k+1}$, with $i < j$;

$M_{i,j} = [((i, 2)), (j \circ i, 0), ((j, 2)), (i, 1), (j, 1)]$ for, $i, j \in \mathbb{Z}_{2k+1}$, with $i < j$.

Note that in the blocks in 6, thanks to the hypothesis that $\{(i, i + 1) \mid i \in \mathbb{Z}_{2k+1}\}$ is a transversal, any vertex $(j, 1)$ with $j \in \mathbb{Z}_{2k+1}$ is of the type $(i \circ (i + 1), 1)$ for some $i \in \mathbb{Z}_{2k+1}$. So, examining the system, we can verify that Σ is a C_3-perfect H_5-design of order $6k + 4$ and indices $(4, 2)$.

This completes the proof. □

Theorem 3.2. *A C_3-perfect H_5-design of indices* $(6, 3)$ *exists if and only if* v *odd,* $v \geq 5$.

Proof. It is known that a 3-fold triple system of order v exists if and only if v is odd [7,9].

At first, we consider the two cases $v = 5$ and $v = 9$.

Let $v = 5$. Define in $\mathbb{Z}_5$ the following two base-blocks:
$B_1 = [(0), 4, (1), 3, 2]$, $B_2 = [(0), 3, (2), 1, 4]$.

If $\mathcal{B}$ is the collection of all the translates of B_1, B_2, then $\Sigma = (\mathbb{Z}_5, \mathcal{B})$ is a C_3-perfect H_5-design of order $v = 5$ and indices $(6, 3)$.

Let $v = 9$. Define in $\mathbb{Z}_9$ the following four base-blocks:
$C_1 = [(0), 1, (4), 5, 2]$, $C_2 = [(0), 2, (3), 4, 1]$,
$C_3 = [(0), 2, (4), 8, 3]$, $C_4 = [(0), 3, (4), 6, 2]$.

If $\mathcal{C}$ is the collection of all the translates of C_1, C_2, C_3, C_4, then $\Sigma = (\mathbb{Z}_9, \mathcal{C})$ is a C_3-perfect H_5-design of order $v = 9$ and indices $(6, 3)$.

Let $v = 2k + 1$, for $k \geq 3$, $v \neq 9$. Let us consider the cyclic system $\Sigma = (\mathbb{Z}_{2k+1}, \mathcal{B})$ having as base blocks:

$$\left[(0), \frac{r}{2}, (r), 3r, 2r\right],$$

for every $r \in Z_{2k+1}$, $r \in \{1, \dots, k\}$. It is possible to verify that Σ is a C_3-perfect H_5-design of order $2k + 1$ and indices $(6, 3)$. □

Theorem 3.3. *A C_3-perfect H_5-design of indices* $(8, 4)$ *exists if and only if* $v \equiv 0$ *or* $1 \pmod 3$.

Proof. A 4-fold triple system of order v exists if and only if $v \equiv 0$ or $1 \bmod 3$ [7,9]. So a C_3-perfect H_5-design of indices $(8, 4)$ and order v is such that $v \equiv 0$ or $1 \bmod 3$.

Conversely, given $\Sigma = (X, \mathcal{B})$ a C_3-perfect H_5-design of indices $(4, 2)$ (Theorem 3.1), the system $\Sigma' = (X, \mathcal{B}')$, where the blocks of $\mathcal{B}'$ are those of $\mathcal{B}$, each repeated twice, is a C_3-perfect H_5-design of indices $(8, 4)$. □

Theorem 3.4. *A C_3-perfect H_5-design of indices* $(10, 5)$ *there exists if and only if* $v \equiv 1$ *or* $3 \pmod 6$.

Proof. A 5-fold triple system of order v exists if and only if $v \equiv 1$ or $3 \bmod 6$ [7,9]. So a C_3-perfect H_5-design of indices $(10, 5)$ and order v is such that $v \equiv 1$ or $3 \pmod 6$.

Conversely, given $\Sigma = (X, \mathcal{B})$ a C_3-perfect H_5-design of indices $(2, 1)$ (Theorem 2.4), the system $\Sigma' = (X, \mathcal{B}')$, where the blocks of $\mathcal{B}'$ are those of $\mathcal{B}$, each repeated five times, is a C_3-perfect H_5-design of indices $(10, 5)$. □

Theorem 3.5. *A C_3-perfect H_5-design of indices* $(12, 6)$ *exists if and only if* $v \geq 5$.

Proof. Let v be odd. Consider a C_3-perfect H_5-design of indices $(6, 3)$ (Theorem 3.2) $\Sigma = (X, \mathcal{B})$. The system $\Sigma' = (X, \mathcal{B}')$, where the blocks of $\mathcal{B}'$ are those of $\mathcal{B}$, each repeated twice, is a C_3-perfect H_5-design of indices $(12, 6)$.

Let $v \equiv 0, 4 \pmod 6$. Consider a C_3-perfect H_5-design of indices $(4, 2)$ $\Sigma = (X, \mathcal{B})$ (Theorem 3.1). The system $\Sigma' = (X, \mathcal{B}')$, where the blocks of $\mathcal{B}'$ are those of $\mathcal{B}$, each repeated three times, is a C_3-perfect H_5-design of indices $(12, 6)$.

Let $v = 6k + 2$, for some $k \geq 1$. Let us consider an idempotent quasigroup $(\mathbb{Z}_{2k}, \circ)$ and the system $\Sigma = (\{\infty_1, \infty_2\} \cup \mathbb{Z}_{2k} \times \mathbb{Z}_3, \mathcal{B})$ having the following blocks:

$A_{i,r,s} = [((i, 0)), (i, 1), (\infty_r), \infty_s, (i, 2)]$, for $i \in \mathbb{Z}_{2k}$ and $r, s \in \{1, 2\}$, $r \neq s$;
$B_{i,r,s} = [((i, 1)), (i, 2), (\infty_r), \infty_s, (i, 0)]$, for $i \in \mathbb{Z}_{2k}$ and $r, s \in \{1, 2\}$, $r \neq s$;
$C_{i,r,s} = [((i, 2)), (i, 0), (\infty_r), \infty_s, (i, 1)]$, for $i \in \mathbb{Z}_{2k}$ and $r, s \in \{1, 2\}$, $r \neq s$;
$D_{i,r,s} = [((i, 0)), (i, 1), (\infty_r), (i, 2), \infty_s]$, for $i \in \mathbb{Z}_{2k}$ and $r, s \in \{1, 2\}$, $r \neq s$;
$E_{i,r,s} = [((i, 1)), (i, 2), (\infty_r), (i, 0), \infty_s]$, for $i \in \mathbb{Z}_{2k}$ and $r, s \in \{1, 2\}$, $r \neq s$;
$F_{i,r,s} = [((i, 2)), (i, 0), (\infty_r), (i, 1), \infty_s]$, for $i \in \mathbb{Z}_{2k}$ and $r, s \in \{1, 2\}$, $r \neq s$;
$G_i = [(\infty_1), (i, 0), (\infty_2), (i, 2), (i, 1)]_{(2)}$, for $i \in \mathbb{Z}_{2k}$;
$H_i = [(\infty_1), (i, 1), (\infty_2), (i, 0), (i, 2)]_{(2)}$, for $i \in \mathbb{Z}_{2k}$;
$I_i = [(\infty_1), (i, 2), (\infty_2), (i, 1), (i, 0)]_{(2)}$, for $i \in \mathbb{Z}_{2k}$;
$L_{i,j} = [((i, 0)), (i \circ j, 1), ((j, 0)), (i + 1, 2), (j + 1, 2)]$, for $i, j \in \mathbb{Z}_{2k}$, $i < j$;
$K_{i,j} = [((i, 0)), (i \circ j, 1), ((j, 0)), (i, 2), (j, 2)]_{(2)}$, for $i, j \in \mathbb{Z}_{2k}$, $i < j$;
$M_{i,j} = [((i, 0)), (j \circ i, 1), ((j, 0)), (i, 2), (j, 2)]_{(3)}$, for $i, j \in \mathbb{Z}_{2k}$, $i < j$;
$N_{i,j} = [((i, 1)), (i \circ j, 2), ((j, 1)), (i + 1, 0), (j + 1, 0)]$, for $i, j \in \mathbb{Z}_{2k}$, $i < j$;
$O_{i,j} = [((i, 1)), (i \circ j, 2), ((j, 1)), (i - 1, 0), (j - 1, 0)]$, for $i, j \in \mathbb{Z}_{2k}$, $i < j$;
$P_{i,j} = [((i, 1)), (i \circ j, 2), ((j, 1)), (i, 0), (j, 0)]$, for $i, j \in \mathbb{Z}_{2k}$, $i < j$;
$Q_{i,j} = [((i, 1)), (j \circ i, 2), ((j, 1)), (i, 0), (j, 0)]_{(3)}$, for $i, j \in \mathbb{Z}_{2k}$, $i < j$;
$R_{i,j} = [((i, 2)), (i \circ j, 0), ((j, 2)), (i - 1, 1), (j - 1, 1)]$, for $i, j \in \mathbb{Z}_{2k}$, $i < j$;

$S_{i,j} = [((i, 2)), (i \circ j, 0), ((j, 2)), (i, 1), (j, 1)]_{(2)}$, for $i, j \in \mathbb{Z}_{2k}$, $i < j$;

$T_{i,j} = [((i, 2)), (j \circ i, 0), ((j, 2)), (i, 1), (j, 1)]_{(3)}$, for $i, j \in \mathbb{Z}_{2k}$, $i < j$;

$U_i = [((i, 0)), (i, 2), ((i, 1)), (i + 1, 2), (i + 1, 1)]$, for $i \in \mathbb{Z}_{2k}$;

$V_i = [((i, 0)), (i, 2), ((i, 1)), (i + 1, 0), (i + 1, 2)]$, for $i \in \mathbb{Z}_{2k}$.

Examining these blocks, we can verify that Σ is a C_3-perfect H_5-design of indices $(12, 6)$. This completes the proof. □

Collecting together all the previous results, with the condition $v \equiv 1$ or $3 \pmod 6$ [9], we have that:

Theorem 3.6. *A C_3-perfect H_5-design of indices $(2\mu, \mu)$ exists if and only if:*
$v \equiv 1$ *or* $3 \pmod 6$, *if* $\mu \equiv 1$ *or* $5 \pmod 6$*;*
$v \equiv 0$ *or* $1 \pmod 3$, *if* $\mu \equiv 2$ *or* $4 \pmod 6$*;*
v *odd,* $v \geq 5$, *if* $\mu \equiv 3 \pmod 6$*;*
$v \geq 5$, *if* $\mu \equiv 0 \pmod 6$.

4. C_5-perfect H_5-designs

In this section, we examine C_5-perfect H_5-designs, determining the spectrum completely, without exceptions, in all the cases.

At first, we see possible necessary conditions.

Theorem 4.1. *If $\Sigma = (X, \mathcal{B})$ is a C_5-perfect H_5-design of order v and indices (λ, τ), then:*
(1) $5\lambda = 6\tau$*;*
(2) $|\mathcal{B}| = \lambda \frac{v(v-1)}{12}$.

Proof. Let $\Sigma = (X, B)$ be a C_5-perfect H_5-design of order v and indices (λ, τ). If $\Sigma' = (X, B')$ is the C_5-system nested in Σ, necessarily: $\mathcal{B} = \mathcal{B}'$. Since
$|\mathcal{B}| = \lambda \frac{v(v-1)}{12}$, $|\mathcal{B}'| = \tau \frac{v(v-1)}{6}$,
both (1), (2) follow easily. □

From Theorem 4.1 it follows that for every positive integers v, $v \geq 5$, the existence of C_5-perfect H_5-designs of order v and indices (λ, τ), with $5\lambda = 6\tau$, is possible. At first we examine the possible existence for systems having odd order v, after we see what happens for v even.

Theorem 4.2. *For $\lambda = 6$, $\tau = 5$, and for every v odd, $v \geq 5$, there exists a C_5-perfect H_5-design of order v and indices $(6, 5)$.*

Proof. Let $v = 2k + 1$, for $k \geq 2$. Consider the following base-blocks, constructed by difference method and defined in $X = \mathbb{Z}_{2k+1}$, where $D = \{1, 2, \ldots, k\}$:
$B_i = [(0), i + 1, (2i + 1), 2k, i]$, for $i \in \{1, \ldots, k - 1\}$
$B = [(0), 2, (1), k + 1, k]$.
If $\mathcal{B}$ is the collection of all the translates of these base-blocks, we can verify that $\Sigma = (X, \mathcal{B})$ is a C_5-perfect H_5-design of order v and indices $(6, 5)$. □

In conclusion, for v odd, we have that:

Theorem 4.3. *For every λ, τ, such that $5\lambda = 6\tau$, and for every v odd, $v \geq 5$, there exists a C_5-perfect H_5-design of order v and indices (λ, τ).*

Proof. The statement follows from Theorem 4.2. Indeed, if $\lambda = 6h$, $\tau = 5h$, a C_5-perfect H_5-design of order v and indices $(6k, 5k)$ can be obtained from a C_5-perfect H_5-design of order v and indices $(6, 5)$, by a repetition of blocks, giving to every block multiplicity h. □

Now, we examine the case v even. At first, we observe that:

Theorem 4.4. *If $\Sigma = (X, \mathcal{B})$ is a C_5-perfect H_5-design of order $v \geq 6$ even and indices $(6h, 5h)$, then h is even.*

Proof. In a C_5-design of order v and index τ, every vertex is contained in exactly $\tau(v-1)/2$ blocks. Indeed, if we fix a vertex x, the number of pairs containing x is $\tau(v - 1)$. Since in every block C_5, every vertex has degree two, the number $\tau(v - 1)$ must be even. But v even implies $\tau = 5h$ even, hence h even. □

Theorem 4.5. *For $\lambda = 12$, $\tau = 10$, and for every v even, $v \geq 6$, there exists a C_5-perfect H_5-design of order v and indices $(12, 10)$.*

Proof. Let $v = 2k$, for $k \geq 3$. Further, let $X = \{\infty\} \cup \mathbb{Z}_{2k-1}$, where ∞ is a fixed point, $\infty \notin \mathbb{Z}_{2k-1}$. Consider the system $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is the collection of all the translates of the following base-blocks, constructed by difference method:
$B_{1,i} = [(0), 2i, (i), 2i + 1, i + 1]_{(2)}$, for any $i \in \{2, \ldots, k-2\}$, with $k \geq 4$;
$B_2 = [(0), 2, (1), 2k-2, 2k-3]$;
$B_3 = [(0), 1, (k), k-1, \infty]_{(2)}$;
$B_4 = [(0), k, (1), -1, \infty]$;
$B_5 = [(\infty), 0, (1), 3, 2]_{(2)}$.
Examining all the blocks, we can verify that Σ is a C_5-perfect H_5-design of order v and indices $(12, 10)$. □

Collecting together all the results of this section, we have that:

Theorem 4.6. *A C_5-perfect H_5-design of order v and indices $(6h, 5h)$ there exists if and only if:*

(1) *v odd, h odd, $v \geq 5$;*

(2) *h even, $v \geq 5$.*

Proof. The statement follows from the previous results. □

5. (C_3, C_4, C_5)-perfect H_5-designs

In this section, we determine completely the spectrum of (C_3, C_4, C_5)-perfect H_5-designs.
At first, we prove the following three Theorems.

Theorem 5.1. *An H_5-design, which is (C_3, C_4)-perfect, is also C_5-perfect.*

Proof. Suppose that $\Sigma = (X, \mathcal{B})$ is a (C_3, C_4)-perfect H_5-design, of order v and indices (λ, μ, σ). Since a C_3-design of order v and index μ has $b' = \mu v(v-1)/6$ blocks, a C_4-design of order v and index σ has $b'' = \sigma v(v-1)/8$ blocks, $|\mathcal{B}| = b = \lambda v(v-1)12$, and $b = b' = b''$, it follows that:

$$\lambda \frac{v(v-1)}{12} = \mu \frac{v(v-1)}{6} = \sigma \frac{v(v-1)}{8}.$$

From which: $\lambda/12 = \mu/6 = \sigma/8$. Hence, for some $h \geq 1$: $\lambda = 6h$, $\mu = 3h$ and $\sigma = 4h$.
Given an edge $\{x, y\}$ we denote by $U(\{x, y\})$ the number of blocks of $\mathcal{B}$ in which $\{x, y\}$ appears as one of the edges $\{a, b\}$ and $\{b, c\}$ of H_5; we denote by $M(\{x, y\})$ the number of blocks of $\mathcal{B}$ in which $\{x, y\}$ appears as the edge $\{a, c\}$; we denote by $L(\{x, y\})$ the number of blocks of $\mathcal{B}$ in which $\{x, y\}$ appears as one of the edges $\{c, d\}$, $\{d, e\}$ and $\{e, a\}$. For any edge $\{x, y\}$ it must be:

$$U(\{x, y\}) + M(\{x, y\}) = 3h$$
$$M(\{x, y\}) + L(\{x, y\}) = 4h$$
$$U(\{x, y\}) + M(\{x, y\}) + L(\{x, y\}) = 6h,$$

so that $U(\{x, y\}) = 2h$, $M(\{x, y\}) = h$ and $L(\{x, y\}) = 3h$ for any $\{x, y\}$. This implies that Σ is C_5-perfect of index $5h$. □

Theorem 5.2. *An H_5-design, which is (C_3, C_5)-perfect, is also C_4-perfect.*

Proof. Suppose that $\Sigma = (X, \mathcal{B})$ is a (C_3, C_5)-perfect H_5-design, of order v and indices (λ, μ, τ). Since a C_3-design of order v and index μ has $b' = \mu v(v-1)/6$ blocks, a C_5-design of order v and index τ has $b''' = \tau v(v-1)/10$ blocks, $|\mathcal{B}| = b = \lambda v(v-1)12$, and $b = b' = b'''$, it follows that:

$$|\mathcal{B}| = \lambda \frac{v(v-1)}{12} = \mu \frac{v(v-1)}{6} = \tau \frac{v(v-1)}{10}.$$

So $\lambda = 6h$, $\mu = 3h$ and $\tau = 5h$ for some $h \geq 1$. Keeping the previous notation, for any edge $\{x, y\}$ it must be:

$$U(\{x, y\}) + M(\{x, y\}) = 3h$$
$$U(\{x, y\}) + L(\{x, y\}) = 5h$$
$$U(\{x, y\}) + M(\{x, y\}) + L(\{x, y\}) = 6h,$$

so that $U(\{x, y\}) = 2h$, $M(\{x, y\}) = h$ and $L(\{x, y\}) = 3h$ for any $\{x, y\}$. This implies that Σ is C_4-perfect of index $4h$. □

Theorem 5.3. *An H_5-design, which is (C_4, C_5)-perfect, is also C_3-perfect.*

Proof. Suppose that $\Sigma = (X, \mathcal{B})$ is a (C_4, C_5)-perfect H_5-design, of order v and indices (λ, σ, τ). Since a C_4-design of order v and index σ has $b'' = \sigma v(v-1)/8$ blocks, a C_5-design of order v and index τ has $b''' = \tau v(v-1)/10$ blocks, $|\mathcal{B}| = b = \lambda v(v-1)12$, and $b = b'' = b'''$, it follows that:

$|\mathcal{B}| = \lambda \frac{v(v-1)}{12} = \sigma \frac{v(v-1)}{8} = \tau \frac{v(v-1)}{10}$.

So $\lambda = 6h$, $\sigma = 4h$ and $\tau = 5h$ for some $h \geq 1$. Keeping the previous notation, for any edge $\{x, y\}$ it must be:

$$M(\{x,y\}) + L(\{x,y\}) = 4h$$
$$U(\{x,y\}) + L(\{x,y\}) = 5h$$
$$U(\{x,y\}) + M(\{x,y\}) + L(\{x,y\}) = 6h,$$

so that $U(\{x,y\}) = 2h$, $M(\{x,y\}) = h$ and $L(\{x,y\}) = 3h$ for any $\{x,y\}$. This implies that Σ is C_3-perfect of index $3h$. □

At this point, we begin to determine the spectrum of (C_3, C_4, C_5)-perfect H_5-designs. At first, we determine some necessary conditions, after we determine the spectrum.

Theorem 5.4. *If $\Sigma = (X, \mathcal{B})$ is a (C_3, C_4, C_5)-perfect H_5-design of order v and indices $(\lambda, \mu, \sigma, \tau)$, then:*

(1) $\frac{\lambda}{6} = \frac{\mu}{3} = \frac{\sigma}{4} = \frac{\tau}{5}$;

(2) $|\mathcal{B}| = \lambda \frac{v(v-1)}{12}$.

Proof. Let $\Sigma = (X, \mathcal{B})$ be a (C_3, C_4, C_5)-perfect H_5-design of order v and indices $(\lambda, \mu, \sigma, \tau)$. Since all the C_k-designs nested in Σ, for $k = 3, 4, 5$, have necessarily the same number of blocks, it follows that:

$$|\mathcal{B}| = \lambda \frac{v(v-1)}{12} = \mu \frac{v(v-1)}{6} = \sigma \frac{v(v-1)}{8} = \tau \frac{v(v-1)}{10};$$

and this proves the statements. □

Theorem 5.5. *For $\lambda = 6$, $\mu = 3$, $\sigma = 4$, $\tau = 5$, and for every v odd, $v \geq 5$, there exists a (C_3, C_4, C_5)-perfect H_5-design of order v and indices $(6, 3, 4, 5)$.*

Proof. Let $v = 2k+1$, for some $k \geq 2$. Consider the system $\Sigma = (\mathbb{Z}_{2k+1}, \mathcal{B})$ having as blocks the translates of the following base blocks:

1. $[(0), 2i, (i), 2k, k+i]$ for any $i \in \{1, \ldots, k-1\}$
2. $[(0), k, (k+1), k+2, 1]$.

We can verify that Σ is a C_3-perfect, C_4-perfect and C_5-perfect H_5-design of order v and indices $(6, 3, 4, 5)$. □

Theorem 5.6. *For every λ, μ, σ, τ, such that $\frac{\lambda}{6} = \frac{\mu}{3} = \frac{\sigma}{4} = \frac{\tau}{5}$, and for every v odd, $v \geq 5$, there exists a (C_3, C_4, C_5)-perfect H_5-design of order v and indices $(\lambda, \mu, \sigma, \tau)$.*

Proof. The statement follows by the previous result of Theorem 5.5. Indeed, let $\lambda = 6h$, $\mu = 3h$, $\sigma = 4$ and $\tau = 5h$, for some $h \in \mathbb{N}$ and let $\Sigma = (X, \mathcal{B})$ a (C_3, C_4, C_5)-perfect H_5-design of order v and indices $(6, 3, 4, 5)$. Then, the system $\Sigma' = (X, \mathcal{B}')$, obtained from Σ, by a repetition of blocks, each repeated h times, is a (C_3, C_4, C_5)-perfect H_5-design of order v and indices $(\lambda, \mu, \sigma, \tau)$. □

Theorem 5.7. *For $\lambda = 12$, $\mu = 6$, $\sigma = 8$, $\tau = 10$, and for every v even, $v \geq 6$, there exists a (C_3, C_4, C_5)-perfect H_5-design of order v and indices $(12, 6, 8, 10)$.*

Proof. Let $v = 2k$, for some $k \geq 3$. Consider the system $\Sigma = (\{\infty\} \cup \mathbb{Z}_{2k-1}, \mathcal{B})$ having as blocks the translates of the following base blocks:

$B_{i,1} = [(0), 2i, (i), 2i+1, i+1]_{(2)}$ for $k \geq 4$ and for any $i \in \{2, \ldots, k-2\}$;
$B_2 = [(0), 2, (1), 3, \infty]_{(2)}$;
$B_3 = [(0), \infty, (k-1), k, 1]$;
$B_4 = [(\infty), 0, (1), 2, 3]$;
$B_5 = [(\infty), k, (0), 1, -1]$;
$B_6 = [(0), 1, (k), k+1, 2]$.

Examining the blocks so obtained, we can verify that Σ is a (C_3, C_4, C_5)-perfect H_5-design of order v and indices $(12, 6, 8, 10)$. □

Theorem 5.8. *A (C_3, C_4, C_5)-perfect H_5-design of order v and indices $(6h, 3h, 4h, 5h)$ there exists if and only if:*

(1) *v odd, $v \geq 5$, h odd;*

(2) *$v \geq 5$, h even.*

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