



# Multiple solutions to a Robin problem with indefinite weight and asymmetric reaction



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## ABSTRACT

The existence of two nontrivial smooth solutions to a semilinear Robin problem with indefinite unbounded potential and asymmetric nonlinearity  $f$  is established. Both crossing and resonance are allowed. A third nonzero solution exists provided  $f$  is  $C^1$ . Proofs exploit variational methods, truncation techniques, and Morse theory.

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## 1. Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  having a  $C^2$ -boundary  $\partial\Omega$ , let  $a \in L^s(\Omega)$  for appropriate  $s \geq 1$ , and let  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function. The semilinear elliptic equation with indefinite unbounded potential

$$-\Delta u + a(x)u = f(x, u) \quad \text{in } \Omega$$

has by now been widely investigated under Dirichlet or Neumann boundary conditions; see [10,19] and [20,22], respectively, besides the references given there. If  $a(x) \equiv 0$  then the case of asymmetric nonlinearities  $f$ , meaning that  $t \mapsto f(x, t)t^{-1}$  crosses at least the principal eigenvalue of the relevant differential operator as  $t$  goes from  $-\infty$  to  $+\infty$ , was also studied; cf. [6,7,24]. From a technical point of view, the Fučík spectrum is often exploited [2], which entails that the limits  $\lim_{t \rightarrow \pm\infty} f(x, t)t^{-1}$  do exist.

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This work treats equations having both difficulties under Robin boundary conditions. Hence, for  $a(x)$  bounded only from above,  $s > N$ , and  $\beta \in W^{1,\infty}(\partial\Omega)$  nonnegative, we consider the problem

$$\begin{cases} -\Delta u + a(x)u = f(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(x)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where  $\frac{\partial u}{\partial n} := \nabla u \cdot n$ , with  $n(x)$  being the outward unit normal vector to  $\partial\Omega$  at its point  $x$ . As usual,  $u \in H^1(\Omega)$  is called a (weak) solution of (1.1) provided

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \beta uv \, d\sigma + \int_{\Omega} auv \, dx = \int_{\Omega} f(x, u)v \, dx \quad \forall v \in H^1(\Omega).$$

Our assumptions on the reaction  $f$  at infinity are essentially the following.

- There exists  $k \geq 2$  such that  $\hat{\lambda}_k \leq \liminf_{t \rightarrow -\infty} \frac{f(x, t)}{t} \leq \limsup_{t \rightarrow -\infty} \frac{f(x, t)}{t} \leq \hat{\lambda}_{k+1}$ ,
- $\limsup_{t \rightarrow +\infty} \frac{f(x, t)}{t} \leq \hat{\lambda}_1$ , and  $\lim_{t \rightarrow +\infty} \left[ f(x, t)t - 2 \int_0^t f(x, \tau) d\tau \right] = +\infty$

uniformly in  $x \in \Omega$ . Here,  $\hat{\lambda}_n$  denotes the  $n$ th-eigenvalue of the problem

$$-\Delta u + a(x)u = \lambda u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} + \beta(x)u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

It should be noted that a possible interaction (resonance) with eigenvalues is allowed. If an additional condition on the behavior of  $t \mapsto f(x, t)t^{-1}$  as  $t \rightarrow 0$  holds then we obtain at least two nontrivial  $C^1$ -solutions to (1.1), one of which is positive; see [Theorems 3.1–3.3](#) for precise statements. As an example, [Theorem 3.1](#) applies when

$$f(x, t) := \begin{cases} bt & \text{if } t \leq 1, \\ \hat{\lambda}_1 t - \sqrt{t} + (b - \hat{\lambda}_1 + 1)t^{-1} & \text{otherwise,} \end{cases}$$

with  $\hat{\lambda}_k \leq b \leq \hat{\lambda}_{k+1}$  and  $k > 2$  large enough, or

$$f(x, t) := \begin{cases} b(t + 1) - c & \text{if } t < -1, \\ ct & \text{if } |t| \leq 1, \\ \hat{\lambda}_1(t - 1) + c & \text{otherwise,} \end{cases}$$

where  $c > \hat{\lambda}_2$ . Let us point out that, unlike previous results, the nonlinearities treated by [Theorem 3.3](#) turn out to be concave near zero. Finally, [Theorem 3.4](#) gives a third nontrivial  $C^1$ -solution once

$$f(x, \cdot) \in C^1(\mathbb{R}) \quad \text{and} \quad \sup_{t \in \mathbb{R}} |f'_t(\cdot, t)| \in L^\infty(\Omega).$$

Our arguments are patterned after those of [\[13\]](#) (cf. also [\[12\]](#)) where, however, the Dirichlet problem is investigated,  $a(x) \equiv 0$ , but the  $p$ -Laplace operator  $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  appears. Moreover, the hypotheses on  $f$  made there do not permit resonance at any eigenvalue. The approach we adopt exploits variational and truncation techniques, as well as results from Morse theory. Regularity of solutions basically arises from [\[26\]](#).

Problem (1.1) evidently falls inside the more general one

$$-\Delta_p u + a(x)|u|^{p-2}u = f(x, u) \quad \text{in } \Omega, \quad |\nabla u|^{p-2} \frac{\partial u}{\partial n} + \beta(x)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega.$$

It will be addressed in a future work but, mainly due to the limited information available at present on the associated eigenvalue problem when  $p \neq 2$ , the matter looks non-standard.

## 2. Preliminaries

Let  $(X, \|\cdot\|)$  be a real Banach space. Given a set  $V \subseteq X$ , write  $\bar{V}$  for the closure of  $V$ ,  $\partial V$  for the boundary of  $V$ , and  $\text{int}(V)$  for the interior of  $V$ . If  $x \in X$  and  $\delta > 0$  then  $B_\delta(x) := \{z \in X : \|z - x\| < \delta\}$  while  $B_\delta := B_\delta(0)$ . The symbol  $(X^*, \|\cdot\|_{X^*})$  denotes the dual space of  $X$ ,  $\langle \cdot, \cdot \rangle$  indicates the duality pairing between  $X$  and  $X^*$ , while  $x_n \rightarrow x$  (respectively,  $x_n \rightharpoonup x$ ) in  $X$  means ‘the sequence  $\{x_n\}$  converges strongly (respectively, weakly) in  $X$ ’. We say that  $\Phi : X \rightarrow \mathbb{R}$  is coercive iff  $\lim_{\|x\| \rightarrow +\infty} \Phi(x) = +\infty$ .  $\Phi$  is called weakly sequentially lower semi-continuous when  $x_n \rightharpoonup x$  in  $X$  implies

$$\Phi(x) \leq \liminf_{n \rightarrow \infty} \Phi(x_n).$$

Let  $\Phi \in C^1(X)$ . The classical Cerami compactness condition for  $\Phi$  reads as follows.

(C) *Every sequence  $\{x_n\} \subseteq X$  such that  $\{\Phi(x_n)\}$  is bounded and*

$$\lim_{n \rightarrow +\infty} (1 + \|x_n\|) \|\Phi'(x_n)\|_{X^*} = 0$$

*has a convergent subsequence.*

Define, provided  $c \in \mathbb{R}$ ,

$$\begin{aligned} \Phi_c &:= \{x \in X : \Phi(x) \geq c\}, & \Phi^c &:= \{x \in X : \Phi(x) \leq c\}, \\ K(\Phi) &:= \{x \in X : \Phi'(x) = 0\}, & K_c(\Phi) &:= K(\Phi) \cap \Phi^{-1}(c). \end{aligned}$$

Given a topological pair  $(A, B)$  fulfilling  $B \subset A \subseteq X$ , the symbol  $H_q(A, B)$ ,  $q \in \mathbb{N}_0$ , indicates the  $q$ th-relative singular homology group of  $(A, B)$  with integer coefficients. If  $x_0 \in K_c(\Phi)$  is an isolated point of  $K(\Phi)$  then

$$C_q(\Phi, x_0) := H_q(\Phi^c \cap V, \Phi^c \cap V \setminus \{x_0\}), \quad q \in \mathbb{N}_0,$$

are the critical groups of  $\Phi$  at  $x_0$ . Here,  $V$  stands for any neighborhood of  $x_0$  such that  $K(\Phi) \cap \Phi^c \cap V = \{x_0\}$ . By excision, this definition does not depend on the choice of  $V$ . Suppose  $\Phi$  satisfies Condition (C),  $\Phi|_{K(\Phi)}$  is bounded below, and  $c < \inf_{x \in K(\Phi)} \Phi(x)$ . Put

$$C_q(\Phi, \infty) := H_q(X, \Phi^c), \quad q \in \mathbb{N}_0.$$

The Second Deformation Lemma [8, Theorem 5.1.33] implies that this definition does not depend on the choice of  $c$ . If  $K(\Phi)$  is finite, then setting

$$M(t, x) := \sum_{q=0}^{+\infty} \text{rank } C_q(\Phi, x) t^q, \quad P(t, \infty) := \sum_{q=0}^{+\infty} \text{rank } C_q(\Phi, \infty) t^q \quad \forall (t, x) \in \mathbb{R} \times K(\Phi),$$

the Morse relation below holds:

$$\sum_{x \in K(\Phi)} M(t, x) = P(t, \infty) + (1 + t)Q(t), \tag{2.1}$$

where  $Q(t)$  denotes a formal series with nonnegative integer coefficients; see for instance [17, Theorem 6.62].

**Proposition 2.1.** *Let  $h \in C^1([0, 1] \times X)$ . Assume that:*

- (i<sub>1</sub>)  $h$  maps bounded sets into bounded sets.
- (i<sub>2</sub>)  $h(0, \cdot)$  and  $h(1, \cdot)$  satisfy Condition (C).
- (i<sub>3</sub>)  $t \mapsto h'_t(t, x)$  is locally Lipschitz continuous. Moreover, there exist  $\alpha > 0$ ,  $p \in (1, +\infty)$  such that  $|h'_t(t, x)| \leq \alpha \|x\|^p$  in  $[0, 1] \times X$ .
- (i<sub>4</sub>)  $x \mapsto h'_x(t, x)$  is locally Lipschitz continuous and with appropriate  $a, \delta > 0$  one has

$$h(t, x) \leq a \implies (1 + \|x\|)|h'_x(t, x)| \geq \delta \|x\|^p.$$

Then  $C_q(h(0, \cdot), \infty) = C_q(h(1, \cdot), \infty)$  for all  $q \in \mathbb{N}_0$ .

This result represents a slight generalization of [11, Proposition 3.2]. Therefore, we omit the proof.

Now, let  $X$  be a Hilbert space, let  $x \in K(\Phi)$ , and let  $\Phi$  be  $C^2$  in a neighborhood of  $x$ . If  $\Phi''(x)$  turns out to be invertible, then  $x$  is called non-degenerate. The Morse index  $d$  of  $x$  is the supremum of the dimensions of the vector subspaces of  $X$  on which  $\Phi''(x)$  turns out to be negative definite. When  $x$  is non-degenerate and with Morse index  $d$  one has

$$C_q(\Phi, x) = \delta_{q,d} \mathbb{Z}, \quad q \in \mathbb{N}_0. \tag{2.2}$$

The monographs [15,17] represent general references on the subject.

Throughout this paper,  $\Omega$  denotes a bounded domain of the real euclidean  $N$ -space  $(\mathbb{R}^N, |\cdot|)$  whose boundary is  $C^2$ . On  $\partial\Omega$  we will employ the  $(N-1)$ -dimensional Hausdorff measure  $\sigma$ . The Trace Theorem [3, Theorem 2.79] ensures that there exists a unique completely continuous linear operator  $\gamma : H^1(\Omega) \rightarrow L^2(\partial\Omega)$  such that

$$\gamma(u) = u|_{\partial\Omega} \quad \forall u \in C^1(\bar{\Omega}), \quad \ker(\gamma) = H_0^1(\Omega).$$

To simplify notation, we let  $u$  in place of  $\gamma(u)$  when no confusion can arise. The symbol  $\|\cdot\|_q$  with  $q \geq 1$  indicates the usual norm of  $L^q(\Omega)$  and

$$\begin{aligned} \|u\| &:= (\|\nabla u\|_2^2 + \|u\|_2^2)^{1/2}, \quad u \in H^1(\Omega), \\ C_+ &:= \{u \in C^0(\bar{\Omega}) : u(x) \geq 0 \quad \forall x \in \bar{\Omega}\}. \end{aligned}$$

Write  $2^*$  for the critical exponent of the Sobolev embedding  $H^1(\Omega) \subseteq L^q(\Omega)$ . Recall that  $2^* = 2N/(N-2)$  if  $2 < N$ ,  $2^* = +\infty$  otherwise, and the embedding is compact whenever  $1 \leq q < 2^*$ . Moreover,

$$\text{int}(C_+) = \{u \in C_+ : u(x) > 0 \quad \forall x \in \bar{\Omega}\}.$$

Given  $t \in \mathbb{R}$ ,  $u, v : \Omega \rightarrow \mathbb{R}$ , and  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ , define

$$t^\pm := \max\{\pm t, 0\}, \quad u^\pm(x) := u(x)^\pm, \quad N_f(u)(x) := f(x, u(x)).$$

$u \leq v$  signifies  $u(x) \leq v(x)$  for almost every  $x \in \Omega$ . The meaning of  $u < v$  etc. is analogous.

**Remark 2.1.** If  $u \in H^1(\Omega)$ ,  $w \in L^2(\Omega)$ , and  $\beta \in L^\infty(\partial\Omega)$  then the condition

$$\int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx + \int_{\partial\Omega} \beta(x)u(x)v(x) d\sigma = \int_{\Omega} w(x)v(x) dx, \quad v \in H^1(\Omega),$$

is equivalent to

$$-\Delta u = w \quad \text{a.e. in } \Omega, \quad \frac{\partial u}{\partial n} + \beta(x)u = 0 \quad \text{on } \partial\Omega.$$

This easily comes out from the nonlinear Green’s identity [8, Theorem 2.4.54]; see for instance the proof of [21, Proposition 3].

We shall employ some facts about the spectrum of the operator  $u \mapsto -\Delta u + a(x)u$  in  $H^1(\Omega)$  with homogeneous Robin boundary conditions. So, consider the eigenvalue problem (1.2) where, from now on,

$$a \in L^s(\Omega) \quad \text{for some } s > N, \quad a^+ \in L^\infty(\Omega), \quad \beta \in W^{1,\infty}(\partial\Omega), \quad \text{and } \beta \geq 0. \tag{2.3}$$

Define

$$\mathcal{E}(u) := \|\nabla u\|_2^2 + \int_{\Omega} a(x)u(x)^2 dx + \int_{\partial\Omega} \beta(x)u(x)^2 d\sigma \quad \forall u \in H^1(\Omega). \tag{2.4}$$

**Lemma 2.1.** *There exist  $\hat{a}, \hat{b} > 0$  such that*

$$\mathcal{E}(u) + \hat{a}\|u\|_2^2 \geq \hat{b}\|u\|^2 \quad \forall u \in H^1(\Omega).$$

**Proof.** If the conclusion was false, we could construct a sequence  $\{u_n\} \subseteq H^1(\Omega)$  fulfilling

$$\mathcal{E}(u_n) + n\|u_n\|_2^2 < \frac{1}{n}\|u_n\|^2, \quad n \in \mathbb{N}. \tag{2.5}$$

Set  $v_n := \|u_n\|^{-1}u_n$ . Since

$$\|v_n\| = 1 \quad \forall n \in \mathbb{N}, \tag{2.6}$$

we may assume that

$$v_n \rightharpoonup v \quad \text{in } H^1(\Omega), \quad v_n \rightarrow v \quad \text{in } L^2(\Omega), \quad \text{and } v_n \rightarrow v \quad \text{in } L^2(\partial\Omega). \tag{2.7}$$

Therefore,

$$\mathcal{E}(v) \leq \liminf_{n \rightarrow +\infty} \mathcal{E}(v_n). \tag{2.8}$$

From (2.5)–(2.8) it follows  $v = 0$  as well as  $n\|v_n\|_2^2 \rightarrow 0$ , which implies

$$\lim_{n \rightarrow +\infty} \|v_n\| = 0. \tag{2.9}$$

In fact, on account of (2.5),

$$0 = \mathcal{E}(0) \leq \liminf_{n \rightarrow +\infty} \mathcal{E}(v_n) \leq \limsup_{n \rightarrow +\infty} \mathcal{E}(v_n) \leq \lim_{n \rightarrow +\infty} \left( \frac{1}{n} - n\|v_n\|_2^2 \right) = 0$$

and, by (2.7),

$$0 = \lim_{n \rightarrow +\infty} \mathcal{E}(v_n) = \lim_{n \rightarrow +\infty} \|\nabla v_n\|_2^2.$$

However, (2.9) contradicts (2.6).  $\square$

Thanks to the above lemma, letting

$$(u, v) := \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Omega} (a(x) + \hat{a}) uv dx + \int_{\partial\Omega} \beta(x) uv d\sigma \quad \forall u, v \in H^1(\Omega)$$

produces a scalar product on  $H^1(\Omega)$  equivalent to the usual one. Further, given  $u \in L^2(\Omega)$ , there exists a unique  $\tilde{u} \in H^1(\Omega)$  such that

$$(\tilde{u}, v) = \int_{\Omega} u(x)v(x) dx, \quad v \in H^1(\Omega).$$

Let  $K : L^2(\Omega) \rightarrow H^1(\Omega)$  be defined by

$$K(u) := \tilde{u} \quad \text{for every } u \in L^2(\Omega)$$

and let  $i : H^1(\Omega) \rightarrow L^2(\Omega)$  be the embedding map. Obviously,  $K \circ i : H^1(\Omega) \rightarrow H^1(\Omega)$  is linear, compact, self-adjoint, while

$$(K \circ i(u), v) = \int_{\Omega} u(x)v(x) dx \quad \forall u, v \in H^1(\Omega).$$

Consequently,

$$(K \circ i(u), u) = \|u\|_2^2, \quad u \in H^1(\Omega).$$

Theorem 3.1.57 in [8] ensures that  $K \circ i$  possesses a decreasing sequence  $\{\mu_n\}$  of positive eigenvalues such that  $\mu_n \rightarrow 0$ . Then

$$\hat{\lambda}_n := \frac{1}{\mu_n} - \hat{a}, \quad n \in \mathbb{N},$$

represent the eigenvalues of (1.2) and there exists a corresponding sequence  $\{\hat{u}_n\} \subseteq H^1(\Omega)$  of eigenfunctions, which turns out to be an orthonormal basis of  $H^1(\Omega)$ . For each  $n \in \mathbb{N}$ , denote by  $E(\hat{\lambda}_n)$  the eigenspace associated with  $\hat{\lambda}_n$ . It is known that:

- (P<sub>1</sub>)  $E(\hat{\lambda}_n)$  is finite dimensional.
- (P<sub>2</sub>) If  $u$  lies in  $E(\hat{\lambda}_n)$  and vanishes on a set of positive Lebesgue measure, then  $u = 0$ .
- (P<sub>3</sub>)  $E(\hat{\lambda}_n) \subseteq C^1(\bar{\Omega})$ .
- (P<sub>4</sub>)  $H^1(\Omega) = \overline{\bigoplus_{n=1}^{\infty} E(\hat{\lambda}_n)}$ . Moreover,

$$\hat{\lambda}_1 = \inf \left\{ \frac{\mathcal{E}(u)}{\|u\|_2^2} : u \in H^1(\Omega), u \neq 0 \right\}, \tag{2.10}$$

$$\hat{\lambda}_n = \inf \left\{ \frac{\mathcal{E}(u)}{\|u\|_2^2} : u \in \hat{H}_n, u \neq 0 \right\} = \sup \left\{ \frac{\mathcal{E}(u)}{\|u\|_2^2} : u \in \bar{H}_n, u \neq 0 \right\}, \quad n \geq 2,$$

where  $\hat{H}_n := \overline{\bigoplus_{i=1}^n E(\hat{\lambda}_i)}$  and  $\bar{H}_n := \overline{\bigoplus_{i=1}^n E(\hat{\lambda}_i)}$ .

- (p<sub>5</sub>) Elements of  $E(\hat{\lambda}_1)$  do not change sign and  $\hat{\lambda}_1$  is simple.
- (p<sub>6</sub>) There exists an  $L^2$ -normalized eigenfunction  $\hat{u}_1 \in \text{int}(C_+)$  associated with  $\hat{\lambda}_1$ .
- (p<sub>7</sub>) Each  $\hat{\lambda}_n$  with  $n \geq 2$  possesses nodal eigenfunctions.

In particular, (p<sub>2</sub>) comes out from [5, Proposition 3], the regularity results of [26, Section 5] imply (p<sub>3</sub>), while (p<sub>5</sub>) is easily verified through Picone’s identity [17, p. 255] besides (2.10). The same holds true for (p<sub>7</sub>); see, e.g., [17, Section 9.3]. Finally, Theorems 2.5.2 and 5.5.1 in [23] basically yield (p<sub>6</sub>).

The next characterization of  $\hat{\lambda}_2$  will be used later. Its proof is analogous to that of [21, Proposition 5].

- (p<sub>8</sub>) Write  $M := \{u \in H^1(\Omega) : \|u\|_2 = 1\}$  as well as

$$\Gamma_1 := \{\gamma \in C^0([-1, 1], M) : \gamma(-1) = -\hat{u}_1, \gamma(1) = \hat{u}_1\}.$$

Then

$$\hat{\lambda}_2 = \inf_{\gamma \in \Gamma_1} \max_{t \in [-1, 1]} \mathcal{E}(\gamma(t)).$$

A simple argument, based on orthogonality, (p<sub>2</sub>), and (p<sub>4</sub>), gives the next result.

**Lemma 2.2.** *Let  $n \in \mathbb{N}$  and let  $\theta \in L^\infty(\Omega) \setminus \{\hat{\lambda}_n\}$  satisfy  $\theta \geq \hat{\lambda}_n$ . Then there exists a constant  $\bar{c} > 0$  such that*

$$\mathcal{E}(u) - \int_{\Omega} \theta(x)u(x)^2 dx \leq -\bar{c}\|u\|^2 \quad \forall u \in \bar{H}_n.$$

*Let  $n \in \mathbb{N}$  and let  $\theta \in L^\infty(\Omega) \setminus \{\hat{\lambda}_n\}$  satisfy  $\theta \leq \hat{\lambda}_n$ . Then there exists a constant  $\hat{c} > 0$  such that*

$$\mathcal{E}(u) - \int_{\Omega} \theta(x)u(x)^2 dx \geq \hat{c}\|u\|^2 \quad \forall u \in \hat{H}_n.$$

Finally, consider the weighted eigenvalue problem

$$-\Delta u + a(x)u = \lambda\alpha(x)u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} + \beta(x)u = 0 \quad \text{on } \partial\Omega, \tag{2.11}$$

where  $\alpha \in L^\infty(\Omega) \setminus \{0\}$  and  $\alpha \geq 0$ . Arguing as before produces an increasing sequence  $\{\hat{\lambda}_n(\alpha)\}$  of eigenvalues for (2.11), which enjoys similar properties. In particular, via the analogue of (p<sub>2</sub>) we achieve the following (cf. [5, Proposition 1]):

- (p<sub>9</sub>) *If  $\alpha_1, \alpha_2 \in L^\infty(\Omega) \setminus \{0\}$ ,  $0 \leq \alpha_1 \leq \alpha_2$ , and  $\alpha_1 \neq \alpha_2$  then  $\hat{\lambda}_n(\alpha_2) < \hat{\lambda}_n(\alpha_1)$  for all  $n \in \mathbb{N}$ .*

### 3. Existence results

To avoid unnecessary technicalities, ‘for every  $x \in \Omega$ ’ will take the place of ‘for almost every  $x \in \Omega$ ’ and the variable  $x$  will be omitted when no confusion can arise. Define

$$n_0 := \inf\{n \in \mathbb{N} : \hat{\lambda}_n \geq 0\}.$$

Let  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function such that  $f(\cdot, 0) = 0$  and let

$$F(x, \xi) := \int_0^\xi f(x, t)dt, \quad (x, \xi) \in \Omega \times \mathbb{R}. \tag{3.1}$$

We will posit the following assumptions.

(f<sub>1</sub>) *There exists  $a_0 \in L^\infty(\Omega)$  such that*

$$|f(x, t)| \leq a_0(x)(1 + |t|) \quad \forall (x, t) \in \Omega \times \mathbb{R}.$$

(f<sub>2</sub>)  *$\limsup_{t \rightarrow +\infty} \frac{f(x, t)}{t} \leq \hat{\lambda}_1$  and  $\lim_{t \rightarrow +\infty} [f(x, t)t - 2F(x, t)] = +\infty$  uniformly in  $x \in \Omega$ .*

(f<sub>3</sub>) *For some  $k \geq \max\{n_0, 2\}$  one has*

$$\hat{\lambda}_k \leq \liminf_{t \rightarrow -\infty} \frac{f(x, t)}{t} \leq \limsup_{t \rightarrow -\infty} \frac{f(x, t)}{t} \leq \hat{\lambda}_{k+1}$$

*uniformly with respect to  $x \in \Omega$  and*

$$f(x, t)t - 2F(x, t) \geq 0 \quad \forall (x, t) \in \Omega \times \mathbb{R}_0^-. \tag{3.2}$$

(f<sub>4</sub>) *There exist  $a_1, a_2 \in L^\infty(\Omega)$  such that  $\hat{\lambda}_2 < a_1 \leq a_2$  and*

$$a_1(x) \leq \liminf_{t \rightarrow 0} \frac{f(x, t)}{t} \leq \limsup_{t \rightarrow 0} \frac{f(x, t)}{t} \leq a_2(x)$$

*uniformly in  $x \in \Omega$ .*

We start by pointing out the next auxiliary results.

**Lemma 3.1.** *Let (f<sub>2</sub>) be satisfied. Then*

$$\lim_{t \rightarrow +\infty} [\hat{\lambda}_1 t^2 - 2F(x, t)] = +\infty$$

*uniformly with respect to  $x \in \Omega$ .*

**Proof.** Given any  $K > 0$ , one can find  $\delta_K > 0$  such that  $f(x, t)t - 2F(x, t) > K$  for all  $(x, t) \in \Omega \times [\delta_K, +\infty)$ . Hence,

$$\frac{d}{dt} \left( \frac{F(x, t)}{t^2} \right) > \frac{K}{t^3} \quad \text{provided } t \geq \delta_K$$

and, a fortiori,

$$\frac{F(x, \xi)}{\xi^2} - \frac{F(x, t)}{t^2} > -\frac{K}{2} \left( \frac{1}{\xi^2} - \frac{1}{t^2} \right)$$

whenever  $\xi \geq t \geq \delta_K$ . Since

$$\limsup_{\xi \rightarrow +\infty} \frac{2F(x, \xi)}{\xi^2} \leq \hat{\lambda}_1 \quad \text{uniformly in } x \in \Omega,$$

the above inequality produces  $\hat{\lambda}_1 t^2 - 2F(x, t) \geq K$  for every  $(x, t) \in \Omega \times [\delta_K, +\infty)$ . As  $K$  was arbitrary, the conclusion follows.  $\square$



**Lemma 3.2.** *If (2.3), (f<sub>1</sub>), and (f<sub>4</sub>) hold, then every nontrivial solution  $u_0 \geq 0$  of (1.1) belongs to  $\text{int}(C_+)$ .*

**Proof.** Using (f<sub>1</sub>) and (f<sub>4</sub>) we get  $c_0 > 0$  such that  $|f(x, t)| \leq c_0|t|$  in  $\Omega \times \mathbb{R}$ . Therefore, the function  $b : \Omega \rightarrow \mathbb{R}$  defined by

$$b(x) := \frac{f(x, u_0(x))}{u_0(x)} \quad \text{if } u_0(x) \neq 0, \quad b(x) := 0 \quad \text{otherwise,}$$

is essentially bounded. Since  $u_0$  turns out to be a weak solution of the problem

$$-\Delta u = [b(x) - a(x)]u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} + \beta(x)u = 0 \quad \text{on } \partial\Omega,$$

where, because of (2.3),  $b - a \in L^s(\Omega)$  for some  $s > N$ , Lemma 5.1 in [26] and the Moser iteration technique yield  $u_0 \in L^\infty(\Omega)$ . Through [26, Lemma 5.2] we achieve  $u_0 \in C^{1,\alpha}(\bar{\Omega})$ . So, in particular,  $u_0 \in C_+ \setminus \{0\}$ . Finally, from

$$\Delta u_0(x) \leq (\|a^+\|_\infty + \|b\|_\infty) u_0(x) \quad \text{for every } x \in \Omega$$

and the Boundary Point Lemma [23, p. 120] it follows  $u_0 \in \text{int}(C_+)$ , as desired.  $\square$

To simplify the notation, write  $X := H^1(\Omega)$ . The energy functional  $\varphi : X \rightarrow \mathbb{R}$  stemming from Problem (1.1) is

$$\varphi(u) := \frac{1}{2}\mathcal{E}(u) - \int_{\Omega} F(x, u(x)) \, dx, \quad u \in X, \tag{3.3}$$

with  $\mathcal{E}$  and  $F$  given by (2.4) and (3.1), respectively. One clearly has  $\varphi \in C^1(X)$ . Moreover,

**Proposition 3.1.** *Under (2.3) and (f<sub>1</sub>)–(f<sub>3</sub>), the functional  $\varphi$  satisfies Condition (C).*

**Proof.** Let  $\{u_n\}$  be a sequence in  $X$  such that

$$|\varphi(u_n)| \leq c_1 \quad \forall n \in \mathbb{N}, \tag{3.4}$$

$$\lim_{n \rightarrow +\infty} (1 + \|u_n\|) \|\varphi'(u_n)\|_{X^*} = 0. \tag{3.5}$$

We first show that  $\{u_n\}$  is bounded. This evidently happens once the same holds for both  $\{u_n^+\}$  and  $\{u_n^-\}$ .

**Claim 1.** *The sequence  $\{u_n^+\}$  is bounded.*

If the assertion was false then, up to subsequences,  $\|u_n^+\| \rightarrow +\infty$ . Write  $v_n := \|u_n^+\|^{-1}u_n^+$ . From  $\|v_n\| = 1$  it follows, along a subsequence when necessary,

$$v_n \rightharpoonup v \quad \text{in } X, \quad v_n \rightarrow v \quad \text{in } L^{2N/(N-1)}(\Omega) \quad \text{and in } L^2(\partial\Omega). \tag{3.6}$$

Through (3.5) one has  $\langle \varphi'(u_n), u_n^+ \rangle \rightarrow 0$ , which, dividing by  $\|u_n^+\|^2$ , easily entails

$$\mathcal{E}(v_n) \leq \varepsilon_n + \int_{\Omega} \frac{f(x, u_n^+(x))}{\|u_n^+\|} v_n(x) \, dx \quad \forall n \in \mathbb{N}, \tag{3.7}$$

where  $\varepsilon_n \rightarrow 0^+$ . Because of  $(f_1)$  the sequence  $\{\|u_n^+\|^{-1}N_f(u_n^+)\} \subseteq L^2(\Omega)$  is bounded. Via the same reasoning made in [17, pp. 317–318] we thus get a function  $\theta \in L^\infty(\Omega)$  such that  $\theta \leq \hat{\lambda}_1$  and

$$\frac{1}{\|u_n^+\|}N_f(u_n^+) \rightharpoonup \theta v \quad \text{in } L^2(\Omega).$$

Thanks to (3.6)–(3.7) this produces, as  $n \rightarrow +\infty$ ,

$$\mathcal{E}(v) - \int_{\Omega} \theta(x)v(x)^2 dx \leq 0. \tag{3.8}$$

If  $\theta \neq \hat{\lambda}_1$  then, by Lemma 2.2,  $v = 0$ . Consequently, on account of (3.6)–(3.7) again,  $v_n \rightarrow 0$  in  $X$ , which contradicts  $\|v_n\| \equiv 1$ . Otherwise, from (3.8), (2.10),  $(p_5)$ , and  $(p_6)$  it follows  $v = t\hat{u}_1$  for some  $t > 0$ . So,  $u_n^+ \rightarrow +\infty$  a.e. in  $\Omega$ . Using  $(f_2)$  and Fatou's Lemma we thus obtain

$$\lim_{n \rightarrow +\infty} \int_{\Omega} [f(x, u_n^+(x))u_n^+(x) - 2F(x, u_n^+(x))] dx = +\infty. \tag{3.9}$$

On the other hand, (3.4) forces

$$\mathcal{E}(u_n) - \int_{\Omega} 2F(x, u_n(x)) dx \leq 2c_1$$

while (3.5) easily yields

$$-\mathcal{E}(u_n) + \int_{\Omega} f(x, u_n(x))u_n(x) dx \leq c_2.$$

Therefore, on account of (3.2),

$$\int_{\Omega} [f(x, u_n^+)u_n^+ - 2F(x, u_n^+)] dx \leq \int_{\Omega} [f(x, u_n)u_n - 2F(x, u_n)] dx \leq 2c_1 + c_2 \quad \forall n \in \mathbb{N},$$

which contradicts (3.9).

**Claim 2.** *The sequence  $\{u_n^-\}$  is bounded.*

If the assertion was false then, up to subsequences,  $\|u_n^-\| \rightarrow +\infty$ . Write, like before,  $w_n := \|u_n^-\|^{-1}u_n^-$ . From  $\|w_n\| \equiv 1$  it follows, along a subsequence when necessary,

$$w_n \rightharpoonup w \quad \text{in } X, \quad w_n \rightarrow w \quad \text{in } L^{2N/(N-1)}(\Omega) \quad \text{and in } L^2(\partial\Omega), \quad w \geq 0. \tag{3.10}$$

Through (3.5) one has

$$\left| \frac{1}{2} \langle \mathcal{E}'(u_n), v \rangle - \int_{\Omega} f(x, u_n)v dx \right| \leq \varepsilon_n \|v\| \quad \forall v \in X, \tag{3.11}$$

where  $\varepsilon_n \rightarrow 0^+$ . A simple computation based on  $(f_1)$  and the boundedness of  $\{u_n^+\}$  immediately leads to

$$\left| \frac{1}{2} \langle \mathcal{E}'(u_n^+), v \rangle - \int_{\Omega} f(x, u_n^+) v \, dx \right| \leq c_3 \|v\|. \tag{3.12}$$

Since  $u_n = u_n^+ - u_n^-$ , inequalities (3.11)–(3.12) produce, after dividing by  $\|u_n^-\|$ ,

$$\left| \frac{1}{2} \langle \mathcal{E}'(-w_n), v \rangle - \frac{1}{\|u_n^-\|} \int_{\Omega} f(x, -u_n^-) v \, dx \right| \leq \varepsilon'_n \|v\|, \quad v \in X, \tag{3.13}$$

with  $\varepsilon'_n \rightarrow 0^+$ . Observe next that, by (f<sub>1</sub>) besides (3.10),

$$\lim_{n \rightarrow +\infty} \frac{1}{\|u_n^-\|} \int_{\Omega} f(x, -u_n^-) (w_n - w) \, dx = 0.$$

So, (3.13) written for  $v := w_n - w$  provides

$$\lim_{n \rightarrow +\infty} \langle \mathcal{E}'(-w_n), w_n - w \rangle = 0,$$

whence

$$\lim_{n \rightarrow +\infty} w_n = w \quad \text{in } X \tag{3.14}$$

because, on account of (3.10) and (2.3),

$$\int_{\Omega} a(x) w_n(x) (w_n(x) - w(x)) \, dx \rightarrow 0, \quad \int_{\partial\Omega} \beta(x) w_n(x) (w_n(x) - w(x)) \, d\sigma \rightarrow 0.$$

Thanks to (f<sub>1</sub>) the sequence  $\{\|u_n^-\|^{-1} N_f(-u_n^-)\} \subseteq L^2(\Omega)$  is bounded. Using the arguments made in [17, pp. 317–318] we thus obtain a function  $\alpha \in L^\infty(\Omega)$  such that  $\hat{\lambda}_k \leq \alpha \leq \hat{\lambda}_{k+1}$  and

$$\frac{1}{\|u_n^-\|} N_f(-u_n^-) \rightharpoonup -\alpha w \quad \text{in } L^2(\Omega).$$

By (3.13)–(3.14) this implies, as  $n \rightarrow +\infty$ ,

$$\frac{1}{2} \langle \mathcal{E}'(w), v \rangle = \int_{\Omega} \alpha(x) w(x) v(x) \, dx \quad \forall v \in X,$$

i.e.,  $w$  turns out to be a weak solution of the problem

$$-\Delta u + a(x)u = \alpha(x)u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} + \beta(x)u = 0 \quad \text{on } \partial\Omega. \tag{3.15}$$

If  $\alpha \neq \hat{\lambda}_k$  and  $\alpha \neq \hat{\lambda}_{k+1}$  then (p<sub>9</sub>) yields

$$\hat{\lambda}_k(\alpha) < \hat{\lambda}_k(\hat{\lambda}_k) = 1 = \hat{\lambda}_{k+1}(\hat{\lambda}_{k+1}) < \hat{\lambda}_{k+1}(\alpha).$$

Therefore  $w = 0$ , which contradicts  $\|w\| = 1$ ; cf. (3.14). Otherwise, either  $\alpha = \hat{\lambda}_k$  or  $\alpha = \hat{\lambda}_{k+1}$ . In both cases, via (3.15) one sees that  $w$  has to be nodal, against (3.10).

Summing up, the sequence  $\{u_n\} \subseteq X$  is bounded. Along a subsequence when necessary, we may thus assume

$$u_n \rightharpoonup u \text{ in } X, \quad u_n \rightarrow u \text{ in } L^{2N/(N-1)}(\Omega) \text{ and in } L^2(\partial\Omega),$$

whence, like before,  $u_n \rightarrow u$  in  $X$ . This completes the proof.  $\square$

### 3.1. Existence of at least two nontrivial solutions

Define, provided  $x \in \Omega$  and  $t, \xi \in \mathbb{R}$ ,

$$\hat{f}_+(x, t) := f(x, t^+) + \hat{a}t^+, \quad \hat{F}_+(x, \xi) := \int_0^\xi \hat{f}_+(x, t) dt. \quad (3.16)$$

It is evident that the corresponding truncated functional

$$\hat{\varphi}_+(u) := \frac{1}{2} (\mathcal{E}(u) + \hat{a}\|u\|_2^2) - \int_\Omega \hat{F}_+(x, u(x)) dx, \quad u \in X,$$

belongs to  $C^1(X)$  also.

**Proposition 3.2.** *Let (f<sub>1</sub>)–(f<sub>2</sub>) be satisfied. Then  $\hat{\varphi}_+$  is coercive.*

**Proof.** If the conclusion was false, we may construct a sequence  $\{u_n\} \subseteq X$  such that  $\|u_n\| \rightarrow +\infty$  but

$$\frac{1}{2} (\mathcal{E}(u_n) + \hat{a}\|u_n\|_2^2) - \int_\Omega \hat{F}_+(x, u_n(x)) dx \leq c_3 \quad \forall n \in \mathbb{N}. \quad (3.17)$$

Write  $v_n := \|u_n\|^{-1}u_n$ . From  $\|v_n\| = 1$  it follows, along a subsequence when necessary, (3.6). Moreover, by (3.16)–(3.17),

$$\frac{1}{2} (\mathcal{E}(v_n) + \hat{a}\|v_n^-\|_2^2) - \frac{1}{\|u_n\|^2} \int_\Omega F(x, u_n^+(x)) dx \leq \frac{c_3}{\|u_n\|^2}, \quad n \in \mathbb{N}. \quad (3.18)$$

Because of (f<sub>1</sub>) the sequence  $\{\|u_n\|^{-2}N_F(u_n^+)\} \subseteq L^1(\Omega)$  is uniformly integrable. Via the Dunford–Pettis Theorem and the same reasoning made in [17, pp. 317–318] we thus get a function  $\theta \in L^\infty(\Omega)$  such that  $\theta \leq \hat{\lambda}_1$  and, up to subsequences,

$$\frac{1}{\|u_n\|^2} N_F(u_n^+) \rightharpoonup \frac{1}{2}\theta(v^+)^2 \text{ in } L^1(\Omega).$$

Using (3.18), besides (3.6), this produces, as  $n \rightarrow +\infty$ ,

$$\mathcal{E}(v) + \hat{a}\|v^-\|_2^2 - \int_\Omega \theta(x)v^+(x)^2 dx \leq 0, \quad (3.19)$$

whence, in view of Lemma 2.1,

$$\mathcal{E}(v^+) \leq \int_\Omega \theta(x)v^+(x)^2 dx. \quad (3.20)$$

If  $\theta \neq \hat{\lambda}_1$ , then Lemma 2.2 and (3.20) force  $v^+ = 0$ . From (3.19) and Lemma 2.1 again it follows  $v^- = 0$ , namely  $v = 0$ . Since, like before,

$$\frac{1}{\|u_n\|^2} N_{\hat{F}_+}(u_n) \rightharpoonup 0 \quad \text{in } L^1(\Omega),$$

inequality (3.17) combined with Lemma 2.1 yield  $\|v_n\| \rightarrow 0$ . However, this is impossible. So, suppose  $\theta = \hat{\lambda}_1$ . Gathering (3.20) and (p<sub>4</sub>)–(p<sub>6</sub>) together lead to  $v^+ = t\hat{u}_1$  for some  $t \geq 0$ . The above argument shows that  $t > 0$ . Hence,  $v^+ > 0$  in  $\Omega$ , which actually means  $v = v^+ > 0$ . Recalling the definition of  $\{v_n\}$  we thus have  $u_n(x) = u_n^+(x)$  as well as  $u_n^+(x) \rightarrow +\infty$  for every  $x \in \Omega$ . Consequently, by Lemma 3.1 and Fatou's Lemma,

$$\lim_{n \rightarrow +\infty} \int_{\Omega} [\hat{\lambda}_1 u_n(x)^2 - 2F(x, u_n(x))] dx = +\infty. \tag{3.21}$$

On the other hand, (3.16)–(3.17), besides (2.10), easily give rise to

$$\int_{\Omega} [\hat{\lambda}_1 u_n(x)^2 - 2F(x, u_n(x))] dx \leq 2c_3 \quad \forall n \in \mathbb{N},$$

against (3.21).  $\square$

**Theorem 3.1.** Under (2.3), (f<sub>1</sub>)–(f<sub>4</sub>), and the assumption that, for appropriate  $a_3 > \hat{\lambda}_1$ ,

$$2F(x, \xi) \geq a_3 \xi^2 \quad \text{in } \Omega \times \mathbb{R}_0^-, \tag{3.22}$$

Problem (1.1) possesses at least two nontrivial solutions  $u_0 \in \text{int}(C_+)$  and  $u_1 \in C^1(\bar{\Omega})$ .

**Proof.** A standard argument, which exploits the Sobolev Embedding Theorem and the compactness of the trace operator, ensures that  $\hat{\varphi}_+$  is weakly sequentially lower semi-continuous. Since, due to Proposition 3.2, it is coercive, we have

$$\inf_{u \in X} \hat{\varphi}_+(u) = \hat{\varphi}_+(u_0) \tag{3.23}$$

for some  $u_0 \in X$ . Fix  $\varepsilon > 0$ . Assumption (f<sub>4</sub>) yields  $\delta > 0$  small such that

$$F(x, \xi) \geq \frac{a_1(x) - \varepsilon}{2} \xi^2 \quad \forall (x, \xi) \in \Omega \times [-\delta, \delta]. \tag{3.24}$$

If  $\tau \in (0, 1)$  complies with  $\tau\hat{u}_1 \leq \delta$ , then by (3.16), the above inequality, (p<sub>6</sub>), and (f<sub>4</sub>),

$$\begin{aligned} \hat{\varphi}_+(\tau\hat{u}_1) &\leq \frac{\tau^2}{2} \left( \mathcal{E}(\hat{u}_1) - \int_{\Omega} (a_1 - \varepsilon) \hat{u}_1^2 dx \right) = \frac{\tau^2}{2} \left( \hat{\lambda}_1 \|\hat{u}_1\|_2^2 - \int_{\Omega} (a_1 - \varepsilon) \hat{u}_1^2 dx \right) \\ &= \frac{\tau^2}{2} \left( \int_{\Omega} (\hat{\lambda}_1 - a_1) \hat{u}_1^2 dx + \varepsilon \right) < 0 \end{aligned}$$

as soon as  $\varepsilon < \int_{\Omega} (a_1 - \hat{\lambda}_1) \hat{u}_1^2 dx$ . Hence,

$$\hat{\varphi}_+(u_0) < 0 = \hat{\varphi}_+(0),$$

which clearly means  $u_0 \neq 0$ . Now, through (3.23) we get  $\hat{\varphi}'_+(u_0) = 0$ , namely

$$\int_{\Omega} \nabla u_0 \cdot \nabla v \, dx + \int_{\Omega} (a + \hat{a})u_0 v \, dx + \int_{\partial\Omega} \beta u_0 v \, d\sigma = \int_{\Omega} [f(x, u_0^+) + \hat{a}u_0^+]v \, dx, \quad v \in X. \tag{3.25}$$

Using Lemma 2.1 and (3.25) written for  $v := -u_0^-$  produces

$$\hat{b}\|u_0^-\|^2 \leq \mathcal{E}(u_0^-) + \hat{a}\|u_0^-\|^2 = - \int_{\partial\Omega} \beta(x)u_0^-(x)^2 \, d\sigma \leq 0,$$

whence  $u_0 \geq 0$ . Therefore,  $u_0$  is a nontrivial nonnegative solution to (1.1), because (3.25) becomes

$$\int_{\Omega} \nabla u_0 \cdot \nabla v \, dx + \int_{\Omega} a u_0 v \, dx + \int_{\partial\Omega} \beta u_0 v \, d\sigma = \int_{\Omega} f(x, u_0)v \, dx \quad \forall v \in X.$$

By Lemma 3.2 one has  $u_0 \in \text{int}(C_+)$  while (3.16) forces  $\varphi|_{C_+} = \hat{\varphi}_+|_{C_+}$ . Thus, [21, Proposition 3] ensures that  $u_0$  turns out to be a local minimizer for  $\varphi$ . We may evidently assume  $u_0$  isolated in  $K(\varphi)$ , otherwise infinitely many solutions there would exist. The same reasoning made in the proof of [1, Proposition 29] provides here  $\rho > 0$  fulfilling

$$\varphi(u_0) < c_\rho := \inf_{u \in \partial B_\rho(u_0)} \varphi(u). \tag{3.26}$$

From (3.2) it easily follows

$$\lim_{\tau \rightarrow -\infty} \varphi(\tau \hat{u}_1) = -\infty.$$

Thanks to Proposition 3.1, Condition (C) holds true for  $\varphi$ . Consequently, the Mountain Pass Theorem gives  $u_1 \in X \setminus \{u_0\}$  such that  $\varphi'(u_1) = 0$  and

$$c_\rho \leq \varphi(u_1) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)), \tag{3.27}$$

where

$$\Gamma := \{\gamma \in C^0([0, 1], X) : \gamma(0) = -\tau \hat{u}_1, \gamma(1) = u_0\}$$

with sufficiently large  $\tau > 0$ . Obviously,  $u_1$  solves (1.1). Through the regularity arguments exploited in the proof of Lemma 3.2 we achieve  $u_1 \in C^1(\bar{\Omega})$ . Thus, the only thing to check is that  $u_1 \neq 0$ . This will be a consequence of the inequality

$$\varphi(u_1) < 0, \tag{3.28}$$

which, due to (3.27), derives from the claim below.

$$\text{There exists a path } \tilde{\gamma} \in \Gamma \text{ such that } \varphi(\tilde{\gamma}(t)) < 0 \text{ for all } t \in [0, 1]. \tag{3.29}$$

Pick  $\varepsilon > 0$  and choose  $\delta > 0$  as in (3.24). Combining (p<sub>8</sub>) with [14, Lemma 2.1] entails

$$\max_{t \in [-1,1]} \mathcal{E}(\gamma_\varepsilon(t)) < \hat{\lambda}_2 + \varepsilon \tag{3.30}$$

for appropriate  $\gamma_\varepsilon \in C^0([-1, 1], C^1(\bar{\Omega})) \cap \Gamma_1$ . Since  $\gamma_\varepsilon([-1, 1])$  is compact in  $C^1(\bar{\Omega})$  while  $\tau\hat{u}_1, u_0 \in \text{int}(C_+)$  we can find  $\eta > 0$  so small that

$$-\tau\hat{u}_1(x) \leq \eta\gamma_\varepsilon(t)(x) \leq u_0(x), \quad |\eta\gamma_\varepsilon(t)(x)| \leq \delta$$

whenever  $x \in \Omega, t \in [-1, 1]$ . On account of (3.24), (3.30), and the equality  $\|\gamma_\varepsilon(t)\|_2 = 1$  one has

$$\begin{aligned} \varphi(\eta\gamma_\varepsilon(t)) &= \frac{\eta^2}{2} \mathcal{E}(\gamma_\varepsilon(t)) - \int_{\Omega} F(x, \eta\gamma_\varepsilon(t)(x)) \, dx \\ &< \frac{\eta^2}{2} \left( \hat{\lambda}_2 + \varepsilon + \int_{\Omega} (\varepsilon - a_1) |\gamma_\varepsilon(t)|^2 \, dx \right) < \frac{\eta^2}{2} \left( \int_{\Omega} (\hat{\lambda}_2 - a_1) |\gamma_\varepsilon(t)|^2 \, dx + 2\varepsilon \right) < 0 \end{aligned}$$

provided  $\varepsilon < 2^{-1} \int_{\Omega} (a_1 - \hat{\lambda}_2) |\gamma_\varepsilon(t)|^2 \, dx$ . Consequently,

$$\varphi|_{\eta\gamma_\varepsilon([-1,1])} < 0. \tag{3.31}$$

Next, write  $a := \hat{\varphi}_+(u_0)$ . It is evident that  $a < 0$ . Further, we may suppose

$$K(\hat{\varphi}_+) = \{0, u_0\},$$

otherwise the conclusion would be straightforward. Hence, no critical value of  $\hat{\varphi}_+$  lies in  $(a, 0)$  while

$$K_a(\hat{\varphi}_+) = \{u_0\}.$$

Due to the Second Deformation Lemma [8, Theorem 5.1.33], there exists a continuous function  $h : [0, 1] \times (\hat{\varphi}_+^0 \setminus \{0\}) \rightarrow \hat{\varphi}_+^0$  satisfying

$$h(0, u) = u, \quad h(1, u) = u_0, \quad \text{and} \quad \hat{\varphi}_+(h(t, u)) \leq \hat{\varphi}_+(u)$$

for all  $(t, u) \in [0, 1] \times (\hat{\varphi}_+^0 \setminus \{0\})$ . Let  $\gamma_+(t) := h(t, \eta\hat{u}_1)^+, t \in [0, 1]$ . Then  $\gamma_+(0) = \eta\hat{u}_1, \gamma_+(1) = u_0$ , as well as

$$\varphi(\gamma_+(t)) = \hat{\varphi}_+(\gamma_+(t)) \leq \hat{\varphi}_+(h(t, \eta\hat{u}_1)) \leq \hat{\varphi}_+(\eta\hat{u}_1) = \varphi(\eta\gamma_\varepsilon(1)) < 0; \tag{3.32}$$

cf. (3.31) besides (3.16) and Lemma 2.1. Finally, define

$$\gamma_-(t) := -[t\eta + (1 - t)\tau]\hat{u}_1 \quad \forall t \in [0, 1].$$

Since through (3.22) we obtain

$$\varphi(\gamma_-(t)) \leq \frac{1}{2} (\mathcal{E}(\gamma_-(t)) - a_3 \|\gamma_-(t)\|_2^2) = \frac{1}{2} [t\eta + (1 - t)\tau]^2 (\hat{\lambda}_1 - a_3) < 0, \tag{3.33}$$

concatenating  $\gamma_-, \eta\gamma_\varepsilon,$  and  $\gamma_+$  produces a path  $\tilde{\gamma} \in \Gamma$  which, in view of (3.31)–(3.33), fulfills (3.29).  $\square$

**Remark 3.1.** The information  $u_0 > 0$  basically derives from (3.23). Instead,  $u_1$  turns out to be a mountain pass type critical point for  $\varphi$ . This merely gives  $u_1 \neq 0$ .

A variant of [Theorem 3.1](#) that does not change overall problem's geometry is the one below, where

(f<sub>5</sub>) *There exist  $m \geq \max\{n_0, 2\}$  as well as  $a_1, a_2 \in L^\infty(\Omega) \setminus \{\hat{\lambda}_m, \hat{\lambda}_{m+1}\}$  such that  $\hat{\lambda}_m \leq a_1 \leq a_2 \leq \hat{\lambda}_{m+1}$  and*

$$a_1(x) \leq \liminf_{t \rightarrow 0} \frac{f(x, t)}{t} \leq \limsup_{t \rightarrow 0} \frac{f(x, t)}{t} \leq a_2(x)$$

*uniformly in  $x \in \Omega$ .*

(f<sub>6</sub>)  *$f(x, \cdot)$  is differentiable at zero and  $f'_t(x, 0) = \lim_{t \rightarrow 0} \frac{f(x, t)}{t}$  uniformly with respect to  $x \in \Omega$ . Moreover, for appropriate  $m \geq \max\{n_0, 2\}$  and  $a_2 \in L^\infty(\Omega) \setminus \{\hat{\lambda}_{m+1}\}$  one has*

$$\hat{\lambda}_m \leq f'_t(\cdot, 0) \leq a_2 \leq \hat{\lambda}_{m+1}.$$

**Lemma 3.3.** *If (f<sub>1</sub>), (f<sub>2</sub>), (f<sub>3</sub>), and either (f<sub>5</sub>) or (f<sub>6</sub>) hold true then  $C_q(\varphi, 0) = \delta_{q, d_m} \mathbb{Z}$  for all  $q \in \mathbb{N}_0$ , where  $d_m := \dim(\bar{H}_m)$ .*

**Proof.** 1) *Under Condition (f<sub>5</sub>).*

Pick any  $\theta \in L^\infty(\Omega)$  satisfying  $a_1 \leq \theta \leq a_2$  and set

$$\psi(u) := \frac{1}{2} \left( \mathcal{E}(u) - \int_{\Omega} \theta(x) u(x)^2 dx \right), \quad u \in X.$$

Thanks to (f<sub>5</sub>), [Lemma 2.2](#) can be applied. So,  $u = 0$  is a non-degenerate critical point of  $\psi$  with Morse index  $d_m$ , which forces

$$C_q(\psi, 0) = \delta_{q, d_m} \mathbb{Z} \quad \forall q \in \mathbb{N}_0; \tag{3.34}$$

see [\(2.2\)](#). Now, let  $h : [0, 1] \times X \rightarrow \mathbb{R}$  given by

$$h(t, u) := (1 - t)\varphi(u) + t\psi(u), \quad (t, u) \in [0, 1] \times X.$$

We shall prove that there exists  $r > 0$  such that  $0 \notin h([0, 1] \times \partial B_r)$ . In fact, if not, one might construct two sequences  $\{t_n\} \subseteq [0, 1]$  and  $\{u_n\} \subseteq X \setminus \{0\}$  with the properties

$$t_n \rightarrow t \in [0, 1], \quad u_n \rightarrow 0 \quad \text{in } X, \quad h'_u(t_n, u_n) = 0 \quad \forall n \in \mathbb{N}.$$

Consequently, letting  $v_n := \|u_n\|^{-1}u_n$  we have

$$\int_{\Omega} (\nabla v_n \cdot \nabla w + a v_n w) dx + \int_{\partial\Omega} \beta v_n w d\sigma = \int_{\Omega} \left[ (1 - t_n) \frac{f(x, u_n)}{\|u_n\|} + t_n \theta v_n \right] w dx \tag{3.35}$$

whatever  $w \in X$  as well as [\(3.6\)](#). Because of (f<sub>1</sub>) the sequence  $\{\|u_n\|^{-1}N_f(u_n)\} \subseteq L^2(\Omega)$  is bounded. The same reasoning made in [\[17, pp. 317–318\]](#) produces a function  $\hat{\theta} \in L^\infty(\Omega)$  such that  $a_1 \leq \hat{\theta} \leq a_2$  and

$$\frac{1}{\|u_n\|} N_f(u_n) \rightharpoonup \hat{\theta} v \quad \text{in } L^2(\Omega). \tag{3.36}$$

Thanks to [\(3.35\)](#) besides [\(3.6\)](#) we get, as  $n \rightarrow +\infty$ ,



$$\int_{\Omega} (\nabla v \cdot \nabla w + avw) dx + \int_{\partial\Omega} \beta vw d\sigma = \int_{\Omega} [(1-t)\hat{\theta} + t\theta]vw dx, \quad w \in X,$$

namely  $v$  is a weak solution of the problem

$$-\Delta u + a(x)u = \theta_t(x)u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} + \beta(x)u = 0 \quad \text{on } \partial\Omega,$$

where  $\theta_t(x) := (1-t)\hat{\theta}(x) + t\theta(x)$ . From  $a_1 \leq \theta_t \leq a_2$ , assumption (f<sub>5</sub>), and (p<sub>9</sub>) it follows

$$\hat{\lambda}_m(\theta_t) < \hat{\lambda}_m(\hat{\lambda}_m) = 1 = \hat{\lambda}_{m+1}(\hat{\lambda}_{m+1}) < \hat{\lambda}_{m+1}(\theta_t),$$

whence  $v = 0$ . Through (3.6), (3.36), and (3.35) written for  $w := v_n - v$  we easily infer that  $v_n \rightarrow 0$  in  $X$ , but this is impossible, because  $\|v_n\| \equiv 1$ . Finally, combining the homotopy invariance property of critical groups with (3.34) completes the proof.

2) Under Condition (f<sub>6</sub>).

Define, like before,

$$\psi(u) := \frac{1}{2} \left( \mathcal{E}(u) - \int_{\Omega} f'_t(x, 0)u(x)^2 dx \right), \quad u \in X.$$

Thanks to (f<sub>6</sub>) and (p<sub>4</sub>) one has  $\psi(u) \leq 0$  for all  $u \in \bar{H}_m$ . If  $\hat{c} > 0$  is furnished by Lemma 2.2 for  $n := m + 1$  then

$$\psi(u) \geq \frac{1}{2} \left( \mathcal{E}(u) - \int_{\Omega} a_2(x)u(x)^2 dx \right) \geq \frac{\hat{c}}{2} \|u\|^2 > 0 \quad \forall u \in \hat{H}_{m+1} \setminus \{0\},$$

since  $f'_t(x, 0)t^2 \leq a_2(x)t^2$  in  $\Omega \times \mathbb{R}$ . Now, Proposition 2.3 of [25] ensures that (3.34) holds. Due to (f<sub>6</sub>) again, given any  $\varepsilon > 0$  we can find  $\delta > 0$  fulfilling

$$|f(x, t) - f'_t(x, 0)t| \leq \varepsilon|t|, \quad (x, t) \in \Omega \times [-\delta, \delta].$$

This entails

$$|\varphi(u) - \psi(u)| \leq \int_{\Omega} \left( \int_0^{|u(x)|} |f(x, t) - f'_t(x, 0)t| dt \right) dx \leq \frac{\varepsilon}{2} \|u\|_2^2$$

as well as

$$|\langle \varphi'(u) - \psi'(u), v \rangle| \leq \int_{\Omega} |f(x, u) - f'_t(x, u)u| |v| dx \leq \varepsilon \|u\|_2 \|v\|_2 \quad \forall v \in X$$

provided  $|u(x)| \leq \delta$ . Consequently, to every  $\varepsilon > 0$  there corresponds  $\rho > 0$  such that

$$\|\varphi - \psi\|_{C^1(D_\rho)} \leq \varepsilon,$$

where  $D_\rho := \{u \in C^1(\bar{\Omega}) : \|u\|_{C^1(\bar{\Omega})} \leq \rho\}$ . Corollary 5.1.25 of [4] thus yields

$$C_q(\varphi|_{C^1(\bar{\Omega})}, 0) = C_q(\psi|_{C^1(\bar{\Omega})}, 0), \quad q \in \mathbb{N}_0,$$

which actually means  $C_q(\varphi, 0) = C_q(\psi, 0)$ , because  $C^1(\bar{\Omega})$  is dense in  $X$ ; see, e.g., [18]. Now the conclusion directly follows from (3.34).  $\square$

**Theorem 3.2.** *Let (2.3), (f<sub>1</sub>)–(f<sub>3</sub>), and either (f<sub>5</sub>) or (f<sub>6</sub>) be satisfied. Then the same conclusion of Theorem 3.1 holds.*

**Proof.** An argument analogous to that employed in showing Theorem 3.1 provides here two solutions,  $u_0 \in \text{int}(C_+)$  and  $u_1 \in C^1(\bar{\Omega})$ . So, it remains to see whether  $u_1 \neq 0$ . By [17, Proposition 6.100] we have  $C_1(\varphi, u_1) \neq 0$  while Lemma 3.3 entails  $C_1(\varphi, 0) = \delta_{1,d_m}\mathbb{Z}$ . Since  $d_m \geq 2$ , the function  $u_1$  cannot be trivial.  $\square$

**Remark 3.2.** Although (f<sub>5</sub>) and (f<sub>6</sub>) look less general than (f<sub>4</sub>), inequality (3.22) is not taken on.

The next variant of Theorem 3.1 exhibits a different geometry at zero. Indeed, instead of (f<sub>4</sub>), (f<sub>5</sub>), or (f<sub>6</sub>), we shall suppose

(f<sub>7</sub>) *There exist  $a_4 > 0$ ,  $q \in (0, 2)$ , and  $\delta > 0$  such that*

$$a_4|t|^q \leq f(x, t)t \leq qF(x, t) \quad \forall (x, t) \in \Omega \times [-\delta, \delta].$$

Condition (f<sub>7</sub>) allows to get further information on the critical groups of  $\varphi$  at zero. This has previously been pointed out in [16] concerning a different problem; cf. also [12].

**Lemma 3.4.** *Under (2.3), (f<sub>1</sub>), (f<sub>7</sub>), and the assumption that zero is an isolated critical point of  $\varphi$ , one has  $C_q(\varphi, 0) = 0$  for all  $q \in \mathbb{N}_0$ .*

**Proof.** Let  $\psi : X \rightarrow \mathbb{R}$  be defined by

$$\psi(u) := \frac{1}{2} (\mathcal{E}(u) + \hat{a}\|u\|_2^2) - \int_{\Omega} F(x, u(x))dx, \quad u \in X.$$

Obviously, zero turns out to be an isolated critical point of  $\psi$ , because

$$\|\psi'(u) - \varphi'(u)\|_{X^*} \leq \hat{a}\|u\|_2.$$

Reasoning as in the proof of Lemma 3.3 we get  $C_q(\varphi, 0) = C_q(\psi, 0)$ . Thus, the conclusion is achieved once

$$C_q(\psi, 0) = 0, \quad q \in \mathbb{N}_0.$$

Thanks to (f<sub>1</sub>) and (f<sub>7</sub>), given any  $r \in (2, 2^*)$ , there exists  $c_4 > 0$  fulfilling

$$F(x, t) \geq \frac{a_4}{q}|t|^q - c_4|t|^r \quad \text{and} \quad qF(x, t) - f(x, t)t \geq -c_4|t|^r \quad \text{in } \Omega \times \mathbb{R}. \tag{3.37}$$

If  $u \in (X \setminus \{0\}) \cap \psi_0$  then

$$\begin{aligned} \frac{d}{d\tau} \psi(\tau u)|_{\tau=1} &= \langle \psi'(u), u \rangle \geq \langle \psi'(u), u \rangle - q\psi(u) \\ &= \left(1 - \frac{q}{2}\right) (\mathcal{E}(u) + \hat{a}\|u\|_2^2) + \int_{\Omega} [qF(x, u) - f(x, u)u] dx \\ &\geq \left(1 - \frac{q}{2}\right) \hat{b}\|u\|^2 - c_4\|u\|^r \end{aligned}$$

by (3.37) besides Lemma 2.1. Consequently

$$\frac{d}{d\tau} \psi(\tau u)|_{\tau=1} > 0$$

whenever  $\|u\|$  is sufficiently small, say  $u \in (\bar{B}_{2\rho} \setminus \{0\}) \cap \psi_0$  for some  $\rho > 0$ . Hence, in particular,  $\tau_0 > 0$  and  $\tau_0 u \in (\bar{B}_{2\rho} \setminus \{0\}) \cap \psi_0$  imply

$$\frac{d}{d\tau} \psi(\tau u)|_{\tau=\tau_0} = \frac{1}{\tau_0} \frac{d}{d\tau} \psi(\tau \tau_0 u)|_{\tau=1} > 0.$$

This means that the  $C^1$ -function  $\tau \mapsto \psi(\tau u)$ ,  $\tau \in (0, +\infty)$ , turns out to be increasing at the point  $\tau$  provided  $\tau u \in (\bar{B}_{2\rho} \setminus \{0\}) \cap \psi_0$ . So, it vanishes at most once in the open interval  $(0, 2\|u\|^{-1}\rho)$ . On the other hand, (3.37) yields

$$\psi(\tau u) \leq \frac{\tau^2}{2} (\mathcal{E}(u) + \hat{a}\|u\|_2^2) - a_4 \frac{\tau^q}{q} \|u\|_q^q + c_4 \tau^r \|u\|_r^r,$$

whence  $\psi(\tau u) < 0$  for all  $\tau > 0$  small enough, since  $q < 2 < r$ . Summing up, given any  $u \in \bar{B}_{2\rho} \setminus \{0\}$ , either  $\psi(\tau u) < 0$  as soon as  $\tau u \in \bar{B}_{2\rho}$  or

$$\exists \text{ a unique } \bar{\tau}(u) > 0 \text{ such that } \bar{\tau}(u)u \in \bar{B}_{2\rho} \setminus \{0\}, \psi(\bar{\tau}(u)u) = 0. \tag{3.38}$$

Moreover, if  $u \in (\bar{B}_{2\rho} \setminus \{0\}) \cap \psi_0$  then  $0 < \bar{\tau}(u) \leq 1$  and

$$\psi(\tau u) < 0 \quad \forall \tau \in (0, \bar{\tau}(u)), \quad \psi(\tau u) > 0 \quad \forall \tau > \bar{\tau}(u) \quad \text{with} \quad \tau u \in \bar{B}_{2\rho}.$$

Let  $\tau : \bar{B}_\rho \setminus \{0\} \rightarrow (0, +\infty)$  be defined by

$$\tau(u) := \begin{cases} 1 & \text{when } u \in (\bar{B}_\rho \setminus \{0\}) \cap \psi^0, \\ \bar{\tau}(u) & \text{when } u \in (\bar{B}_\rho \setminus \{0\}) \cap \psi_0. \end{cases}$$

We claim that the function  $\tau(u)$  is continuous. This immediately follows once one knows that  $\bar{\tau}(u)$  turns out to be continuous on  $(\bar{B}_\rho \setminus \{0\}) \cap \psi_0$ , because, by uniqueness,  $u \in \bar{B}_\rho \setminus \{0\}$  and  $\psi(u) = 0$  evidently force  $\bar{\tau}(u) = 1$ ; cf. (3.38). Pick  $\hat{u} \in (\bar{B}_\rho \setminus \{0\}) \cap \psi_0$ . The function  $\phi(t, u) := \psi(tu)$  belongs to  $C^1(\mathbb{R} \times X)$  and, on account of (3.38), we have

$$\phi(\bar{\tau}(\hat{u}), \hat{u}) = 0, \quad \frac{\partial \phi}{\partial u}(\bar{\tau}(\hat{u}), \hat{u}) = \bar{\tau}(\hat{u})\psi'(\bar{\tau}(\hat{u})\hat{u}).$$

Since zero turns out to be an isolated critical point for  $\psi$ , there is no loss of generality in assuming  $K_\varphi \cap \bar{B}_\rho = \{0\}$ . So, the Implicit Function Theorem furnishes  $\varepsilon > 0$ ,  $\sigma \in C^1(B_\varepsilon(\hat{u}))$  such that

$$\phi(\sigma(u), u) = 0 \quad \forall u \in B_\varepsilon(\hat{u}), \quad \sigma(\hat{u}) = \bar{\tau}(\hat{u}).$$

Through  $0 < \bar{\tau}(\hat{u}) \leq 1$  we thus get  $0 < \sigma(u) < 2$  for all  $u \in U$ , where  $U \subseteq B_\varepsilon(\hat{u})$  denotes a convenient neighborhood of  $\hat{u}$ . Consequently,

$$\sigma(u)u \in \bar{B}_{2\rho} \setminus \{0\} \quad \text{and} \quad \psi(\sigma(u)u) = 0 \quad \text{provided} \quad u \in (\bar{B}_\rho \setminus \{0\}) \cap \psi_0 \cap U.$$

By (3.38) this results in  $\sigma(u) = \bar{\tau}(u)$ , from which the continuity of  $\bar{\tau}(u)$  at  $\hat{u}$  follows. As  $\hat{u}$  was arbitrary, the function  $\bar{\tau}(u)$  turns out to be continuous on  $(\bar{B}_\rho \setminus \{0\}) \cap \psi_0$ .

Next, observe that  $\tau u \in \bar{B}_\rho \cap \psi^0$  for all  $\tau \in [0, 1]$ ,  $u \in \bar{B}_\rho \cap \psi^0$ . Hence, if

$$h(t, u) := (1 - t)u, \quad (t, u) \in [0, 1] \times (\bar{B}_\rho \cap \psi^0)$$

then  $h([0, 1] \times (\bar{B}_\rho \cap \psi^0)) \subseteq \bar{B}_\rho \cap \psi^0$ , namely  $\bar{B}_\rho \cap \psi^0$  is contractible in itself. Moreover, the function

$$g(u) := \tau(u)u \quad \forall u \in \bar{B}_\rho \setminus \{0\}$$

is continuous and one has  $g(\bar{B}_\rho \setminus \{0\}) \subseteq (\bar{B}_\rho \cap \psi^0) \setminus \{0\}$ . Since

$$g|_{(\bar{B}_\rho \cap \psi^0) \setminus \{0\}} = \text{id}|_{(\bar{B}_\rho \cap \psi^0) \setminus \{0\}},$$

the set  $(\bar{B}_\rho \cap \psi^0) \setminus \{0\}$  turns out to be a retract of  $\bar{B}_\rho \setminus \{0\}$ . Being  $\bar{B}_\rho \setminus \{0\}$  contractible in itself because  $X$  is infinite dimensional, we get (see, e.g., [9, p. 389])

$$C_q(\psi, 0) := H_q(\bar{B}_\rho \cap \psi^0, (\bar{B}_\rho \cap \psi^0) \setminus \{0\}) = 0, \quad q \in \mathbb{N}_0,$$

as desired.  $\square$

**Remark 3.3.** This proof is patterned after that of [12, Theorem 3.1].

**Theorem 3.3.** Let (2.3), (f<sub>1</sub>)–(f<sub>3</sub>), and (f<sub>7</sub>) be satisfied. Then the same conclusion of Theorem 3.1 holds.

**Proof.** Reasoning exactly as in the proof of the above-mentioned result yields (3.23) for some  $u_0 \in X$ . Furthermore, with (f<sub>4</sub>) replaced by (f<sub>7</sub>), one achieves both  $u_0 \neq 0$  and  $u_0 \in \text{int}(C_+)$ ; cf. Lemma 3.2. So,  $u_0$  turns out to be a local minimizer for  $\varphi$ , which entails  $u_0 \in K(\varphi)$ . Proposition 3.1 guarantees that  $\varphi$  fulfills Condition (C). Thus, the arguments exploited in the proof of Theorem 3.1 provide a second solution  $u_1 \in C^1(\bar{\Omega})$ . Thanks to [17, Proposition 6.100] we have  $C_1(\varphi, u_1) \neq 0$ . Since  $C_1(\varphi, 0) = 0$  by Lemma 3.4, the function  $u_1$  cannot be zero.  $\square$

### 3.2. Existence of at least three nontrivial solutions

From now on, we shall suppose that  $f(\cdot, 0) = 0$ ,  $f(x, \cdot) \in C^1(\mathbb{R})$  for every  $x \in \Omega$ , and

(f'<sub>1</sub>) There exists  $a_0 \in L^\infty(\Omega)$  satisfying  $|f'_t(x, t)| \leq a_0(x)$  in  $\Omega \times \mathbb{R}$ .

**Lemma 3.5.** Under (2.3), (f'<sub>1</sub>), (f<sub>2</sub>), and (f<sub>3</sub>), one has  $C_q(\varphi, \infty) = 0$  for all  $q \in \mathbb{N}_0$ .

**Proof.** Pick  $\mu \in (\hat{\lambda}_k, \hat{\lambda}_{k+1})$ . Define, provided  $(t, u) \in [0, 1] \times X$ ,

$$h(t, u) := \frac{1}{2}\mathcal{E}(u) - t \int_{\Omega} F(x, u) dx + \frac{1-t}{2} (\hat{a}\|u^+\|_2^2 - \mu\|u^-\|_2^2).$$

Clearly,  $h$  maps bounded sets into bounded sets. On account of Proposition 3.1, both  $h(0, \cdot)$  and  $h(1, \cdot)$  satisfy Condition (C). Due to (f'<sub>1</sub>), the functionals  $t \mapsto h'_t(t, u)$  and  $u \mapsto h'_u(t, u)$  are locally Lipschitz continuous. Let us next verify that

$$\exists a \in \mathbb{R}, \delta > 0 \quad \text{fulfilling} \quad h(t, u) \leq a \implies (1 + \|u\|)\|h'_u(t, u)\|_{X^*} \geq \delta\|u\|^2. \quad (3.39)$$

If the assertion were false, then we might find two sequences  $\{t_n\} \subseteq [0, 1]$ ,  $\{u_n\} \subseteq X$  with the properties below:

$$\begin{aligned} t_n \rightarrow t, \quad \|u_n\| \rightarrow +\infty, \quad h(t_n, u_n) \rightarrow -\infty, \\ (1 + \|u_n\|)\|h'_u(t_n, u_n)\|_{X^*} < \frac{1}{n}\|u_n\|^2 \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.40}$$

Put  $v_n := \|u_n\|^{-1}u_n$ . Reasoning as in the proof of Lemma 3.3 produces  $v \in X$  such that  $v_n \rightarrow v$  and

$$\int_{\Omega} [\nabla v \cdot \nabla w + avw + (1 - t)\hat{a}v^+w] dx + \int_{\partial\Omega} \beta v w d\sigma = \int_{\Omega} [t\theta v^+ - \eta_t v^-] w dx \tag{3.41}$$

for all  $w \in X$ , where  $\eta_t(x) := t\eta(x) + (1 - t)\mu$  while  $\theta, \eta \in L^\infty(\Omega)$  comply with  $\theta \leq \hat{\lambda}_1$ ,  $\hat{\lambda}_k \leq \eta \leq \hat{\lambda}_{k+1}$ . Hence,  $v$  is a weak solution to the problem

$$-\Delta u + a(x)u + (1 - t)\hat{a}u^+ = t\theta(x)u^+ - \eta_t(x)u^- \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} + \beta(x)u = 0 \quad \text{on } \partial\Omega.$$

If  $t = 1$  then (3.41) written for  $w := v^+$  entails

$$\mathcal{E}(v^+) = \int_{\Omega} \theta(x)v^+(x)^2 dx.$$

Two situations may now occur:

1)  $\theta \not\equiv \hat{\lambda}_1$ . Lemma 2.2 immediately forces  $v^+ = 0$ . Consequently,

$$-\Delta v + a(x)v = \eta(x)v \quad \text{in } \Omega, \quad \frac{\partial v}{\partial n} + \beta(x)v = 0 \quad \text{on } \partial\Omega.$$

Since  $v \neq 0$ , because  $\|v\| = 1$ , and  $\hat{\lambda}_k \leq \eta \leq \hat{\lambda}_{k+1}$ , through (p<sub>9</sub>) we see that  $v$  must change sign, which is absurd.

2)  $\theta = \hat{\lambda}_1$ . Likewise the proof of Proposition 3.2, (3.40) give rise to a contradiction.

Therefore,  $t < 1$ . Letting  $w := v^+$  in (3.41) yields

$$(1 - t)\mathcal{E}(v^+) + (1 - t)\hat{a}\|v^+\|_2^2 = t \left[ \int_{\Omega} \theta(v^+)^2 dx - \mathcal{E}(v^+) \right].$$

From Lemmas 2.1–2.2 it thus follows  $(1 - t)\hat{b}\|v^+\|^2 \leq 0$ , whence  $v = -v^-$ . Now, (3.41) becomes

$$-\Delta v + a(x)v = \eta_t(x)v \quad \text{in } \Omega, \quad \frac{\partial v}{\partial n} + \beta(x)v = 0 \quad \text{on } \partial\Omega,$$

and, as before,  $v$  has to be nodal, since  $\hat{\lambda}_k \leq \eta_t \leq \hat{\lambda}_{k+1}$  by the choice of  $\mu$ . However, this is impossible. Thus, (3.39) holds true. Via Proposition 2.1 we obtain

$$C_q(\varphi, \infty) = C_q(h(1, \cdot), \infty) = C_q(h(0, \cdot), \infty) \quad \forall q \in \mathbb{N}_0. \tag{3.42}$$

Observe next that

$$C_q(h(0, \cdot), \infty) = C_q(h(0, \cdot), 0). \quad (3.43)$$

In fact, if  $u \in K(h(0, \cdot))$  then

$$\int_{\Omega} (\nabla u \cdot \nabla v + auv + \hat{a}u^+v) dx + \int_{\partial\Omega} \beta uv d\sigma = -\mu \int_{\Omega} u^- v dx, \quad v \in X. \quad (3.44)$$

Choosing  $v := u^+$  furnishes  $\mathcal{E}(u^+) + \hat{a}\|u^+\|_2^2 = 0$ , namely  $u^+ = 0$ ; cf. [Lemma 2.1](#). So, (3.44) actually means

$$-\Delta u + a(x)u = \mu u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} + \beta(x)u = 0 \quad \text{on } \partial\Omega,$$

and, a fortiori,  $u = 0$ , because  $\hat{\lambda}_k < \mu < \hat{\lambda}_{k+1}$ . This shows that  $K(h(0, \cdot)) = \{0\}$ , from which (3.43) follows at once.

Let us finally compute  $C_q(h(0, \cdot), 0)$ . Consider the homotopy

$$\hat{h}(t, u) := h(0, u) + t \int_{\Omega} u(x) dx \quad \forall (t, u) \in [0, 1] \times X.$$

We claim that

$$\hat{h}'_u(t, u) \neq 0, \quad (t, u) \in [0, 1] \times (X \setminus \{0\}). \quad (3.45)$$

By contradiction, suppose there exists  $(t, u) \in (0, 1] \times (X \setminus \{0\})$  fulfilling  $h'_u(t, u) = 0$ . The same arguments exploited above produce here  $u \leq 0$  and

$$-\Delta u + a(x)u = \mu u - t \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} + \beta(x)u = 0 \quad \text{on } \partial\Omega.$$

Hence, likewise the proof of [Lemma 3.2](#),  $u \in -\text{int}(C_+)$ . Define, for every  $v \in \text{int}(C_+)$ ,

$$R(v, -u) := |\nabla v|^2 - \nabla(-u) \cdot \nabla \left( \frac{v^2}{-u} \right).$$

Using Picone's identity [[17, Proposition 9.61](#)] yields

$$\begin{aligned} 0 &\leq \int_{\Omega} R(v, -u)(x) dx = \|\nabla v\|_2^2 - \int_{\Omega} (-\Delta u) \frac{v^2}{u} dx - \int_{\partial\Omega} \beta u \frac{v^2}{-u} d\sigma \\ &= \|\nabla v\|_2^2 + \int_{\Omega} av^2 dx + \int_{\partial\Omega} \beta v^2 d\sigma - \mu \int_{\Omega} v^2 dx + t \int_{\Omega} \frac{v^2}{u} dx \\ &< \|\nabla v\|_2^2 + \int_{\Omega} av^2 dx + \int_{\partial\Omega} \beta v^2 d\sigma - \mu \int_{\Omega} v^2 dx. \end{aligned}$$

On account of (p<sub>6</sub>) this entails, for  $v := \hat{u}_1$ ,

$$0 < \hat{\lambda}_1 - \mu < 0,$$

which is clearly absurd. Thanks to (3.45) and Theorem 5.1.2 in [[4](#)] we have, for  $\rho > 0$  small enough,

$$H_q(\hat{h}(1, \cdot)^0 \cap B_\rho, \hat{h}(1, \cdot)^0 \cap B_\rho \setminus \{0\}) = 0 \tag{3.46}$$

while the homotopy invariance of singular homology forces

$$H_q(\hat{h}(0, \cdot)^0 \cap B_\rho, \hat{h}(0, \cdot)^0 \cap B_\rho \setminus \{0\}) = H_q(\hat{h}(1, \cdot)^0 \cap B_\rho, \hat{h}(1, \cdot)^0 \cap B_\rho \setminus \{0\}). \tag{3.47}$$

Since  $\hat{h}(0, \cdot) = h(0, \cdot)$ , (3.46)–(3.47) provide

$$C_q(h(0, \cdot), 0) = 0 \quad \forall q \in \mathbb{N}_0. \tag{3.48}$$

Gathering (3.42), (3.43), and (3.48) together yields the conclusion.  $\square$

**Theorem 3.4.** *Let (2.3),  $(f'_1)$ ,  $(f_2)$ ,  $(f_3)$ , and  $(f_6)$  be satisfied. Then Problem (1.1) admits at least three nontrivial solutions  $u_0 \in \text{int}(C_+)$ ,  $u_1, u_2 \in C^1(\bar{\Omega})$ .*

**Proof.** The same arguments adopted in the proofs of Theorems 3.1–3.2 give  $u_0$  and  $u_1$ . Moreover,

$$C_q(\varphi, u_0) = \delta_{q,0}\mathbb{Z} \quad \forall q \in \mathbb{N}_0, \tag{3.49}$$

because  $u_0$  is a local minimizer for  $\varphi$ , while  $u_1$  turns out to be a mountain pass type critical point of  $\varphi$ . Observe next that  $\varphi \in C^2(X)$  and one has

$$\langle \varphi''(u_1)(v), w \rangle = \int_{\Omega} [\nabla v \cdot \nabla w + avw - f'_t(x, u_1)vw] dx + \int_{\partial\Omega} \beta v w d\sigma, \quad v, w \in X. \tag{3.50}$$

If the Morse index of  $u_1$  is zero then, by (3.50),

$$\|\nabla v\|_2^2 + \int_{\partial\Omega} \beta v^2 d\sigma \geq \int_{\Omega} [f'_t(x, u_1) - a] v^2 dx \quad \text{in } X. \tag{3.51}$$

Two situations may now occur.

1)  $(f'_t(\cdot, u_1) - a)^+ = 0$ . Given  $u \in \ker(\varphi''(u_1))$ , from (3.50) we immediately infer

$$\|\nabla u\|_2^2 + \int_{\partial\Omega} \beta u^2 d\sigma \leq 0,$$

whence, on account of (2.3), the function  $u$  must be constant.

2)  $(f'_t(\cdot, u_1) - a)^+ \not\equiv 0$ . Inequality (3.51) entails  $\hat{\lambda}_1(\alpha) \geq 1$ , where  $\alpha := (f'_t(\cdot, u_1) - a)$ . So, due to (3.50),  $\hat{\lambda}_1(\alpha) = 1$  as soon as  $u \in \ker(\varphi''(u_1)) \setminus \{0\}$ .

Consequently, in either case,  $\dim(\ker(\varphi''(u_1))) \leq 1$ , and Corollary 6.102 of [17] yields

$$C_q(\varphi, u_1) = \delta_{q,1}\mathbb{Z} \quad \forall q \in \mathbb{N}_0. \tag{3.52}$$

Finally, if  $K(\varphi) = \{0, u_0, u_1\}$  then the Morse relation written for  $t = -1$ , Lemma 3.3, (3.49), (3.52), besides Lemma 3.5 would imply

$$(-1)^{d_m} + (-1)^0 + (-1)^1 = 0,$$

which is impossible. Thus, there exists  $u_1 \in K(\varphi) \setminus \{0, u_0, u_1\}$ , i.e., a third nontrivial solution to (1.1). Standard regularity arguments (see the proof of Lemma 3.2) ensure that  $u_1 \in C^1(\bar{\Omega})$ .  $\square$

**Example 3.1.** Let  $k > \max\{n_0, 2\}$  and let  $b \in [\hat{\lambda}_k, \hat{\lambda}_{k+1}]$ . The function  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  defined by, for every  $(x, t) \in \Omega \times \mathbb{R}$ ,

$$f(x, t) := \begin{cases} bt & \text{if } t \leq 1, \\ \hat{\lambda}_1 t - \sqrt{t} + c \log t + d & \text{otherwise,} \end{cases}$$

where  $c := b - \hat{\lambda}_1 + 2^{-1}$  and  $d := b - \hat{\lambda}_1 + 1$  satisfies all the assumptions of Theorem 3.4.

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