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Lexicographic preferences representable by real-branching trees with countable height: A dichotomy result

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Abstract

The linear ordering $\mathbb{R}^{<\omega}_{\text{lex}}$ is the lexicographic linearization of the tree of $\mathbb{R}$-valued functions defined on a finite initial segment of ω and ordered by extension. We identify suitable notions of smallness and largeness for linear orderings that embed into $\mathbb{R}^{<\omega}_{\text{lex}}$ by using tree representations of chains. Specifically, small linear orderings are representable by inversely well-founded trees, and large linear orderings are representable by fully uncountably branching trees. We prove the rather surprising result that all linear orderings embeddable into $\mathbb{R}^{<\omega}_{\text{lex}}$ are either small or large. This fact sheds some light on the complicated structure of the linear ordering $\mathbb{R}^{<\omega}_{\text{lex}}$, and can be useful in applications to utility theory and preference modeling.

Keywords: Lexicographic ordering; Lexicographic preferences; Tree representation; Small chain; Large chain; Utility representation

1. Introduction

Embeddings of linear orderings into real lexicographic products have been an object of careful study in the literature. This is due to their theoretical interest as well as their utility in many applied sciences. From a theoretical point of view, this topic has been extensively analyzed within

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the theory of linear orderings [4,8,10,11,14,15,17,20]. For what concerns applications to other fields of research, embeddings into lexicographic products are of major interest in the branch of mathematical economics called *utility theory* [2,3,5,6,9,12,16,23]. In this field, researchers have focused their attention on finding suitable codomains of *utility functions* (i.e., order-preserving embeddings of chains), which are "larger" than the usual set of real numbers, thus allowing a representation of many preference relations which naturally arise in economics (e.g., the lexicographic plane, the symmetric long line, large spaces of functions, etc.). Further applications to economics concern the cases in which preference relations are modeled by using any type of lexicographic ordering (see [7] for an excellent, despite not updated, survey on the topic), e.g., lexicographic tradeoff structures, where the lexicographic importance of criteria varies according to suitable thresholds [19].

In this paper we provide some insight into the structure of the family of all linear orderings that embed into $\mathbb{R}^{<\omega}_{\text{lex}}$. The chain $\mathbb{R}^{<\omega}_{\text{lex}} = (\mathbb{R}^{<\omega}, \sqsubseteq_{\text{lex}})$ is the lexicographic linearization of the rooted tree $(\mathbb{R}^{<\omega}, \sqsubseteq)$ of all $\mathbb{R}$-valued functions defined on a finite initial segment of ω and ordered by extension. Note that several types of dynamic decisions in economics can be naturally modeled as $\mathbb{R}$-branching trees whose ascending paths are of finite length: see [22] for the original notion of *extensive form*, [18] for a classical generalization, and [1] for a recent extension of this approach. Therefore, apart from its theoretical interest, our analysis of the structure of the linear orderings embeddable into $\mathbb{R}^{<\omega}_{\text{lex}}$ may be useful in applications, allowing one to determine the evolution of decisions which are taken over a time-dependent variable (e.g., states of nature).

We determine suitable notions of smallness and largeness for linear orderings that are embeddable into $\mathbb{R}^{<\omega}_{\text{lex}}$. Specifically, we define the following two classes of chains within the family of linear orderings that embed into $\mathbb{R}^{<\omega}_{\text{lex}}$: (i) *small* chains, which can be represented as suitable subsets of an inversely well-founded tree; (ii) *large* chains, which contain a copy of a fully uncountably branching tree. Despite the fact that their definition appears to be rather restrictive, the following dichotomy result holds:

Theorem. *Each linear ordering embeddable into* $\mathbb{R}^{<\omega}_{\text{lex}}$ *is either small or large.*

The paper is organized as follows. Section 2 collects some preliminary terminology and notions. In Section 3 we use tree representations of chains to define the two classes of small and large chains within the family of linear orderings that are embeddable into $\mathbb{R}^{<\omega}_{\text{lex}}$. Section 4 is devoted to the analysis of a special type of order-theoretic properties of the nodes of a tree. In Section 5 we use the results obtained in the previous sections to prove the dichotomy theorem.

2. Preliminary notions

Here we recall some basic notions on linear orderings and trees [20,21]; the reader familiar with these well known notions may decide to skip them. In the last part of this section, we introduce the notions of $(\mathbb{R}, \omega)$-tree, $(\mathbb{R}, \omega)$-chain and tree representation, which will be used throughout the paper.

A *chain* is a linearly ordered set $(L, \prec)$. A *subchain* of $(L, \prec)$ is a set $M \subseteq L$, endowed with the induced order. For any two subchains A and B of $(L, \prec)$, the notation $A \prec B$ means that $a \prec b$ for each $a \in A$ and $b \in B$; in particular, if $A = \{a\}$, then we simply write $a \prec B$. Intervals and rays in $(L, \prec)$ are denoted as usual, namely, $(a, b)_L = \{x \in L : a \prec x \prec b\}$, $(a, \rightarrow)_L = \{x \in L : a \prec x\}$, $(\leftarrow, b]_L = \{x \in L : x \preceq b\}$, etc.. (The notation $x \preceq b$ stands for $x \prec b$ or $x = b$.) Whenever there is no risk of confusion, we drop the subscript L and simply write (a, b), $(a, \rightarrow)$, $(\leftarrow, b]$, etc.. The interval notation is also used for posets.

A *homomorphism* is an order-preserving function $f:(L,\prec_L)\to(M,\prec_M)$ between two chains, i.e., for each $a,b\in L$, if $a\preceq_L b$ then $f(a)\preceq_M f(b)$. An *embedding* is an injective homomorphism between chains, generically denoted by $L\hookrightarrow M$; a chain that embeds into the reals is called $\mathbb{R}$-*embeddable*. An *isomorphism* is a bijective homomorphism between chains, generically denoted by $L\cong M$. As usual, *ordinals* are the canonical representatives for the isomorphism classes of well-ordered sets, and they are identified with the set of all ordinals below it; in particular, ω and ω_1 denote, respectively, the first infinite ordinal and the first uncountable ordinal.

Let $\{(L_i,\prec_i):i\in I\}$ be a family of chains indexed by a chain $(I,<)$. The *sum* of $\{(L_i,\prec_i):i\in I\}$ is the chain $\sum_{i\in I}L_i$, whose underlying set is $\bigcup_{i\in I}\{i\}\times L_i$, and whose linear order $\prec$ is defined by $(j,a_j)\prec(k,a_k)$ if either $j<k$, or $j=k$ and $a_j\prec_j a_k$ in L_j. For a finite sum, we use the notation $L_1+L_2+\cdots+L_n$. If $(I,<)$ is the first infinite ordinal ω, then we define the *lexicographic product* of $\{(L_n,\prec_n):n<\omega\}$ as the chain $\prod^{\mathrm{lex}}_{n<\omega}L_n$, whose order relation $\prec_{\mathrm{lex}}$ is defined as follows: for any $\mathbf{a}=(a_n)_{n<\omega},\mathbf{b}=(b_n)_{n<\omega}\in\prod_{n<\omega}L_n$, let $\mathbf{a}\prec_{\mathrm{lex}}\mathbf{b}$ if there exists a natural number n_0 such that $a_{n_0}\prec_{n_0}b_{n_0}$ and $a_n=b_n$ for all $n<n_0$. For the lexicographic product of n chains, we use the notations $L_1\times_{\mathrm{lex}}\cdots\times_{\mathrm{lex}}L_n$ and L^n_{lex}, where in the second case the n factors are all equal to L. By definition, we set $L^1_{\mathrm{lex}}:=L$, and $L^0_{\mathrm{lex}}:=\mathbf{1}$ (the chain with one element). Finally, observe that the lexicographic product $L\times_{\mathrm{lex}}M$ is equal to the sum $\sum_{x\in L}M_x$, where $M_x=M$ for each $x\in L$.

A *tree* is a poset $(T,\preceq)$ whose initial segments $(\leftarrow,t):=\{x\in T:x\prec t\}$ are all well-ordered by $\preceq$ for each $t\in T$. The elements of a tree are called *nodes*. For each two distinct nodes $s,t\in T$, the notation $s\perp t$ stands for *incomparability* between s and t, i.e., $s\npreceq t$ and $t\npreceq s$. A tree T is *rooted* if it has a minimum element, called the *root* of T. (All trees considered in this paper are rooted.) A maximal node of a tree is also called a *terminal node*. An *immediate successor* of a node $t\in T$ is a node $u\in T$ such that $t\prec u$ and the open interval $(t,u)\subseteq T$ is empty. The set of immediate successors of $t\in T$ is denoted by $\mathrm{Succ}(t,T)$ (or simply by $\mathrm{Succ}(t)$ in unambiguous cases). A *subtree* of T is a subposet $T'\subseteq T$ such that for each $t,t'\in T$, if $t'\in T'$ and $t\preceq t'$, then $t\in T'$. A *path* of T is a subtree P of T, which is linearly ordered by the induced order. An *antichain* of T is a set of nodes $A\subseteq T$ such that any two distinct elements in it are incomparable. Let $t\in T$ be a node of a tree $(T,\preceq)$. The *open cone above* t is the set $(t,\rightarrow):=\{x\in T:t\prec x\}$; similarly, the *closed cone above* t is the set $[t,\rightarrow):=\{x\in T:t\preceq x\}$. The *height of* t in T is the order-type of the initial segment $(\leftarrow,t)$ and is denoted by $\mathrm{ht}(t,T)$ (or simply by $\mathrm{ht}(t)$ if there is no risk of ambiguity). The α-*th level* of T is the set $\mathrm{Lev}_\alpha(T):=\{t\in T:\mathrm{ht}(t)=\alpha\}$, and the *height of* T is the ordinal $\mathrm{ht}(T):=\min\{\alpha:\mathrm{Lev}_\alpha(T)=\emptyset\}$. (Thus, the level zero of a rooted tree is formed by its root only.)

In this paper we work on the tree $(\mathbb{R}^{<\omega},\sqsubseteq)$ of all $\mathbb{R}$-valued functions defined on a finite initial segment of ω and ordered by extension. Thus, its underlying set is $\mathbb{R}^{<\omega}:=\bigcup_{n<\omega}\mathbb{R}^n$, and its partial order $\sqsubseteq$ is defined by $x\sqsubseteq y$ if x is a restriction of y; in particular, for each $n<\omega$, we have $\mathrm{Lev}_n(\mathbb{R}^{<\omega},\sqsubseteq)=\mathbb{R}^n$. Observe that $(\mathbb{R}^{<\omega},\sqsubseteq)$ has height ω, infinitely many branches of length ω and no terminal nodes. The main object of our analysis is the *lexicographic linearization* of the tree $(\mathbb{R}^{<\omega},\sqsubseteq)$, i.e., the chain $\mathbb{R}^{<\omega}_{\mathrm{lex}}=(\mathbb{R}^{<\omega},\sqsubseteq_{\mathrm{lex}})$, whose linear order $\sqsubseteq_{\mathrm{lex}}$ is the completion of the partial order $\sqsubseteq$ obtained as follows: whenever two nodes s and t are incomparable in $(\mathbb{R}^{<\omega},\sqsubseteq)$, we complete the order by looking at the first elements where the paths leading to s and t differ. More precisely, for each $s,t\in\mathbb{R}^{<\omega}$, let $s\sqsubseteq_{\mathrm{lex}}t$ if either $s\sqsubseteq t$, or $s\perp t$ and $s^*<t^*$, where $s^*\in(\leftarrow,s]$ and $t^*\in(\leftarrow,t]$ are the first elements in the respective paths which do not belong to $(\leftarrow,s)\cap(\leftarrow,t)$. (Note that this definition is well given because $(\leftarrow,s)\cap(\leftarrow,t)$ has a last element u, and the set of immediate successors of u is a subset of the reals.) More generally,

we will denote by $T_{\text{lex}} = (T, \preceq_{\text{lex}})$ the lexicographic linearization of a tree $(T, \preceq)$ of height at most ω (see [21] for the general procedure to obtain the lexicographic linearization of a tree).

Definition 2.1. An $(\mathbb{R}, \omega)$-*tree* is a rooted tree of height at most ω with the property that the set of the immediate successors of each node embeds into $\mathbb{R}$. An $(\mathbb{R}, \omega)$-*chain* is the lexicographic linearization of an $(\mathbb{R}, \omega)$-tree.

Note that $(\mathbb{R}^{<\omega}, \sqsubseteq)$ and $(\mathbb{R}^{<\omega}, \sqsubseteq_{\text{lex}})$ are universal for the classes of $(\mathbb{R}, \omega)$-trees and $(\mathbb{R}, \omega)$-chains, respectively: in fact, any $(\mathbb{R}, \omega)$-tree embeds into $(\mathbb{R}^{<\omega}, \sqsubseteq)$, and any $(\mathbb{R}, \omega)$-chain embeds into $(\mathbb{R}^{<\omega}, \sqsubseteq_{\text{lex}})$. Observe also that an $(\mathbb{R}, \omega)$-chain has a minimum element, namely, the one corresponding to the root of the associated $(\mathbb{R}, \omega)$-tree. Furthermore, a subtree of an $(\mathbb{R}, \omega)$-tree is an $(\mathbb{R}, \omega)$-tree, whereas a subchain of an $(\mathbb{R}, \omega)$-chain is not necessarily an $(\mathbb{R}, \omega)$-chain (for example, it might lack a minimum element).

We conclude this section by associating to each isomorphism class of $(\mathbb{R}, \omega)$-chains a set of (not necessarily isomorphic) $(\mathbb{R}, \omega)$-trees.

Definition 2.2. Define an equivalence relation $\sim_{\text{lex}}$ on the class of $(\mathbb{R}, \omega)$-trees as follows. For each couple of $(\mathbb{R}, \omega)$-trees S and T, let $S \sim_{\text{lex}} T$ if S_{lex} and T_{lex} are isomorphic chains. Thus each isomorphism class of $(\mathbb{R}, \omega)$-chains has a class of $(\mathbb{R}, \omega)$-trees associated to it. If L is an $(\mathbb{R}, \omega)$-chain, each $(\mathbb{R}, \omega)$-tree T whose lexicographic linearization is isomorphic to L is said to be a *tree representation* of L. If S and T are two tree representations of the same linear ordering L, then the notation $S \overset{\text{lex}}{=} T$ means the following: (i) S and T have the same underlying set of nodes; (ii) the isomorphism between S_{lex} and T_{lex} is the identity map. More generally, if S and T are two $(\mathbb{R}, \omega)$-trees having the same set of nodes, then we say that S is a *refinement* of T if the partial order $\preceq_S$ contains the partial order $\preceq_T$.

3. Small and large chains

In this section we define suitable notions of smallness and largeness for chains that embed into $\mathbb{R}^{<\omega}_{\text{lex}}$. To this aim, we introduce two special classes of linear orderings embeddable into $\mathbb{R}^{<\omega}_{\text{lex}}$, namely, pseudo-small chains and pseudo-large chains. Then, a chain is defined to be small if is *contained* in a pseudo-small chain, and, dually, it is large if it *contains* a pseudo-large chain. We obtain sufficient conditions for a chain to be small (Theorem 3.7) and large (Theorem 3.11). These two facts, along with the main result of Section 4 (Theorem 4.6), will be used to prove the dichotomy theorem (Theorem 5.4).

Definition 3.1. A *pseudo-small tree* is an $(\mathbb{R}, \omega)$-tree with no infinite branches. A *pseudo-small chain* is an $(\mathbb{R}, \omega)$-chain with a pseudo-small tree representation. A chain is *small* if it embeds into a pseudo-small chain.

Within the family of pseudo-small trees associated to a small chain L, we can find a tree T such that L is displayed as a set of terminal nodes of T.

Lemma 3.2. *For each small chain L, there exist a pseudo-small tree T and an embedding $f : L \hookrightarrow T_{\text{lex}}$ whose image $f[L]$ is a set of terminal nodes of T.*

Proof. Let L be a small chain. Therefore, there exists a pseudo-small tree T and an embedding $f : L \hookrightarrow T_{\text{lex}}$. We construct a pseudo-small tree T^* and an embedding $f^* : L \hookrightarrow T^*{}_{\text{lex}}$ such that $f^*[L]$ is contained in the set of terminal nodes of T^*. To build T^*, start with T and for each non-terminal node $t \in T$, add a new node t^* in a way that $\text{Succ}(t, T^*) := \{t^*\} \cup \text{Succ}(t, T)$,

$t^* \prec_{\text{lex}} \text{Succ}(t, T)$, and $\text{Succ}(t^*, T^*) = \emptyset$. It is immediate to check that T^* is a pseudo-small tree. Next, define a map $f^* \colon L \to T^*_{\text{lex}}$ as follows for each $x \in L$: let $f^*(x) := f(x)$ if $f(x)$ is a terminal node in T, and $f^*(x) := (f(x))^*$ if $f(x)$ is non-terminal in T. Then f^* is an embedding such that $f^*[L]$ is a set of terminal nodes in T^*. □

Example 3.3. For each $n < \omega$, the lexicographic power $\mathbb{R}^n_{\text{lex}}$ is a small chain, which can be identified with the set of terminal nodes of the pseudo-small tree $(\mathbb{R}^{\leq n}, \sqsubseteq)$ via the canonical inclusion $\iota_n \colon \mathbb{R}^n_{\text{lex}} \hookrightarrow \mathbb{R}^{\leq n}_{\text{lex}}$. More generally, in [13] we define a $\mathbf{c}^+$-sequence of linear orderings $\{\uparrow_\gamma \mathbb{R} : \gamma < \mathbf{c}^+\}$, whose initial ω-segment is the sequence of finite lexicographic powers $\{\mathbb{R}^n_{\text{lex}} : n < \omega\}$. We show (Corollary 6.4) that the transfinite sequence $\{\uparrow_\gamma \mathbb{R} : \gamma < \mathbf{c}^+\}$ is a *hierarchy*, i.e., each linear ordering embeds into all chains following it and into none of the chains preceding it. Furthermore, each linear ordering in this hierarchy is a small chain, which is displayed as the set of all terminal nodes of a suitable pseudo-small tree (Lemma 8.6).

The notion of smallness is an invariant of finite lexicographic operations.

Lemma 3.4. *A lexicographic product of finitely many small chains is a small chain. A sum over a small chain of small chains is a small chain.*

Proof. For the first statement, it suffices to show that the claim holds for two small chains. Since a lexicographic product of two chains is a particular case of a sum of chains over another chain, it suffices to show that the second statement holds. Let $\{M_x : x \in X\}$ be a family of small chains indexed over a small chain X. By Lemma 3.2, we can assume that all small chains under consideration are displayed as sets of terminal nodes of pseudo-small trees. Therefore, there exist: (i) a pseudo-small tree U and an embedding $g \colon X \hookrightarrow U_{\text{lex}}$ such that $g[X]$ is a set of terminal nodes of U; (ii) a family of pseudo-small trees $\{U_x : x \in X\}$ and a family of embeddings $\{g_x : x \in X\}$, where each map $g_x \colon M_x \hookrightarrow (U_x)_{\text{lex}}$ is such that the image $g_x[M_x]$ is a set of terminal nodes in U_x. In the sequel we define a pseudo-small tree T and an embedding $f \colon \sum_{x \in X} M_x \hookrightarrow T_{\text{lex}}$ such that the image $f\left[\sum_{x \in X} M_x\right]$ is a set of terminal nodes of T. This will prove that $\sum_{x \in X} M_x$ is a small chain.

Let T be the $(\mathbb{R}, \omega)$-tree obtained substituting the terminal node $g(x) \in U$ by the tree U_x for each $x \in X$. Further, let $f \colon \sum_{x \in X} M_x \to T_{\text{lex}}$ be the map defined by $f(x, m) := (g(x), g_x(m))$ for each $(x, m) \in \sum_{x \in X} M_x$, where $(g(x), g_x(m)) \in T$ denotes the node $g_x(m) \in U_x$ once that the terminal node $g(x) \in U$ has been substituted by the tree U_x. It is immediate to check that T is a pseudo-small tree and f is an embedding such that $f\left[\sum_{x \in X} M_x\right]$ is a set of terminal nodes of T. Thus the claim holds. □

A key property of small chains is that in the process of checking embeddability into a pseudo-small chain, any countable subset can be disregarded (Theorem 3.7). To prove this fact, we introduce the notion of the Dedekind duplicate of a chain.

Definition 3.5. Let $(L, \prec_L)$ be a chain. The set of *Dedekind cuts* of L is

$$\text{Cut}(L) := \{(A, B) : A, B \subseteq L \ \wedge \ A \prec_L B \ \wedge \ A \cup B = L\}.$$

(Recall that $A \prec_L B$ means that $a \prec_L b$ for each $a \in A$ and $b \in B$.) Endow $\text{Cut}(L)$ with a linear order $\prec_{\text{Cut}(L)}$ defined by $(A, B) \prec_{\text{Cut}(L)} (C, D)$ if $A \subsetneqq C$ for each $(A, B), (C, D) \in \text{Cut}(L)$. The *Dedekind duplicate* of L is the chain $\left(\text{Ded}(L), \prec_{\text{Ded}(L)}\right)$, whose underlying set is $\text{Ded}(L) := L \cup \text{Cut}(L)$, and whose linear order extends the orders on L and $\text{Cut}(L)$ as follows:

- if $x, y \in L$, then let $x \prec_{\text{Ded}(L)} y$ if and only if $x \prec_L y$;

- if $x, y \in \mathrm{Cut}(L)$, then let $x \prec_{\mathrm{Ded}(L)} y$ if and only if $x \prec_{\mathrm{Cut}(L)} y$;
- if $x \in L$ and $y = (A, B) \in \mathrm{Cut}(L)$, then let $x \prec_{\mathrm{Ded}(L)} y$ if and only if $x \prec_L B$.

Note that $\big(\mathrm{Ded}(L), \prec_{\mathrm{Ded}(L)}\big)$ has a minimum, namely, $(\emptyset, L)$, as well as a maximum, namely, either $\max L$ or $(L, \emptyset)$ in case L has no maximum. Furthermore, if L is empty, then $\mathrm{Ded}(L) = \mathrm{Cut}(L) = \{(\emptyset, \emptyset)\}$. Let M be a subchain of L. For each $x, y \in L$ such that $x \prec_L y$, the notation $(x, y)_M$ stands for $(x, y)_L \cap M$; furthermore, we denote by $\overline{M}$ the closure of M in L.

Lemma 3.6. *If L and M are chains such that $M \subsetneq L$, then L embeds into* $\mathrm{Ded}(\overline{M}) \times_{\mathrm{lex}} (L \setminus M)$.

Proof. The result holds trivially if $M = \emptyset$ or $\overline{M} = L$. Thus, assume that $\emptyset \subsetneq \overline{M} \subsetneq L$. For each $x \in L \setminus \overline{M}$, let x^+ be the element of $\mathrm{Cut}(\overline{M}) \subseteq \mathrm{Ded}(\overline{M})$ defined as follows:

$$x^+ := \big((\leftarrow, x)_{\overline{M}}, [x, \rightarrow)_{\overline{M}}\big).$$

Definition 3.5 yields that for each $x, y \in L \setminus \overline{M}$ such that $x \prec_L y$, we have $x^+ \preceq_{\mathrm{Ded}(\overline{M})} y^+$. Define a function $f : L \to \mathrm{Ded}(\overline{M}) \times_{\mathrm{lex}} (L \setminus M)$ as follows for each $x \in L$:

$$f(x) := \begin{cases} (x, z_0) & \text{if } x \in \overline{M} \\ (x^+, x) & \text{if } x \in L \setminus \overline{M} \end{cases}$$

where z_0 is a fixed element of $L \setminus M$. To complete the proof, we show that f is an embedding. Let $x, y \in L$ be such that $x \prec_L y$. We prove that $f(x) \prec_{\mathrm{lex}} f(y)$ in each of the four possible cases. If $x, y \in \overline{M}$, then $f(x) = (x, z_0) \prec_{\mathrm{lex}} (y, z_0) = f(y)$ holds. Dually, if $x, y \in L \setminus \overline{M}$, then we get $f(x) = (x^+, x) \prec_{\mathrm{lex}} (y^+, y) = f(y)$ because of the second components. Further, if $x \in \overline{M}$ and $y \in L \setminus \overline{M}$, then $f(x) = (x, z_0) \prec_{\mathrm{lex}} (y^+, y) = f(y)$ by comparing the first components. Finally, assume that $x \in L \setminus \overline{M}$ and $y \in \overline{M}$. By definition of closure, there exists an open interval $(a, b) \subseteq L$ such that $x \in (a, b)$ and $(a, b) \cap M = \emptyset$. It follows that $x^+ = \big((\leftarrow, a]_{\overline{M}}, [b, \rightarrow)_{\overline{M}}\big)$ and $b \preceq_L y$. Now Definition 3.5 yields that $x^+ \prec_{\mathrm{Ded}(\overline{M})} y$, and so we have $f(x) = (x^+, x) \prec_{\mathrm{lex}} (y, z_0) = f(y)$ also in this case. □

Finally, we obtain the following sufficient condition for the smallness of chains.

Theorem 3.7. *Let L be a chain and M a countable subchain of L. If $L \setminus M$ is small, then so is L.*

Proof. The result is obvious in the case that $M = L$, so assume that $M \subsetneq L$. If $L \setminus M$ is a small chain, then Lemma 3.6 implies that L embeds into $\mathrm{Ded}(\overline{M}) \times_{\mathrm{lex}} (L \setminus M)$. Countability of M yields the embeddability $\mathrm{Ded}(\overline{M}) \hookrightarrow \mathbb{R}^2_{\mathrm{lex}}$, which implies that $\mathrm{Ded}(\overline{M})$ is a small chain. Therefore, the lexicographic product $\mathrm{Ded}(\overline{M}) \times_{\mathrm{lex}} (L \setminus M)$ is a small chain by Lemma 3.4. The claim follows. □

Now we introduce a notion of largeness for chains embeddable into $\mathbb{R}^{<\omega}_{\mathrm{lex}}$.

Definition 3.8. A *pseudo-large tree* is an $(\mathbb{R}, \omega)$-tree with no terminal nodes and such that the set of immediate successors of each node is an uncountable chain with no minimum. A *pseudo-large chain* is an $(\mathbb{R}, \omega)$-chain that has a pseudo-large tree representation. A chain embeddable into $\mathbb{R}^{<\omega}_{\mathrm{lex}}$ is *large* if it contains a copy of a pseudo-large chain.

We shall prove a sufficient condition for a chain to be large (Theorem 3.11). To this aim, we introduce the definition of a branching generator in an $(\mathbb{R}, \omega)$-tree.

Definition 3.9. Let U be an $(\mathbb{R}, \omega)$-tree and $\mathcal{A}(U)$ the family of all antichains in U. A *branching generator* in U is a pair (S, σ), where S is a set of nodes containing the root of U, and $\sigma : S \to \mathcal{A}(U)$ is a map such that for each $s \in S$, the antichain $\sigma(s)$ satisfies the following properties:

- $\sigma(s)$ is uncountable;
- $\sigma(s)$ is contained in $S \cap (s, \to)_U$;
- $(\sigma(s))_{\text{lex}}$ embeds into $\mathbb{R}$.

The next result shows that the existence of a branching generator is preserved by refinements of $(\mathbb{R}, \omega)$-trees (see the last part of Definition 2.2 for the notion of refinement).

Lemma 3.10. *Let T and U be $(\mathbb{R}, \omega)$-trees having the same underlying set of nodes. If U has a branching generator and T refines U, then T has a branching generator as well.*

Proof. Assume that the partial order $\preceq_U$ is contained into the partial order $\preceq_T$, and let (S, σ) be a branching generator for U. We construct a branching generator (R, ρ) for T, with $R \subseteq S$ and $\rho : R \to \mathcal{A}(T)$ such that $\rho(r) = \sigma(r) \cap R$ for each $r \in R$. In what follows we build a sequence of pairs $\{(R_n, \rho_n) : n < \omega\}$ such that R_0 has the root r_T of T as its only element, $\rho_0(r_T)$ is an uncountable antichain of T whose lexicographic linearization embeds into the reals, and for each $n < \omega$, we have:

- $R_{n+1} = R_n \cup \rho_n[R_n] \subseteq S$;
- $\rho_{n+1} : R_{n+1} \to \mathcal{A}(T)$ is such that $\rho_{n+1} \restriction R_n = \rho_n$, and for each $r \in R_{n+1} \setminus R_n$, the set $\rho_{n+1}(r)$ is an uncountable antichain, which is contained in the open cone $(r, \to)_T$ and whose lexicographic linearization embeds into the reals.

Then the pair (R, ρ), where $R := \bigcup_{n<\omega} R_n$ and $\rho := \bigcup_{n<\omega} \rho_n$, is a branching generator for T.

Let r_U be the root of U. By hypothesis, r_U belongs to S and $\sigma(r_U)$ is an uncountable antichain of U contained in $S \cap (r_U, \to)_U$ and whose lexicographic linearization embeds into $\mathbb{R}$. Note that since $\preceq_U$ is included into $\preceq_T$, it follows that T and U have the same root $r_T = r_U$, and the inclusion $\mathcal{A}(T) \subseteq \mathcal{A}(U)$ holds. Let $R_0 := \{r_T\}$. Countability of $\text{ht}(T)$ implies that we can select an uncountable set $A_{r_T} \subseteq \sigma(r_T) \subseteq (r_T, \to)_T$ with the property that all nodes in A_{r_T} belong to $\text{Lev}_k(T)$ for some $k \geq 1$; in particular, A_{r_T} is an antichain of T whose lexicographic linearization embeds into $\mathbb{R}$. Let $\rho_0 : R_0 \to \mathcal{A}(T)$ be the map defined by $r_T \mapsto A_{r_T}$. For the successor step, define $R_{n+1} := R_n \cup \rho_n[R_n]$ and $\rho_{n+1} \restriction R_n := \rho_n$. To complete the definition of ρ_{n+1}, let $r \in R_{n+1} \setminus R_n$. Since $\preceq_U$ is contained in $\preceq_T$, it follows that $(r, \to)_U \subseteq (r, \to)_T$. Thus, $\sigma(r)$ is an uncountable antichain of U, which is contained in the open cone $(r, \to)_T$, and whose lexicographic linearization embeds into the reals. Select an uncountable subset $A_r \subseteq \sigma(r)$ such that A_r is contained in $\text{Lev}_k(T)$ for some $k \geq n + 1$, and set $\rho_{n+1}(r) := A_r$. This completes the definition of the sequence $\{(R_n, \rho_n) : n < \omega\}$. $\square$

Theorem 3.11. *Let T be an $(\mathbb{R}, \omega)$-tree and $X \subseteq T$ a set of nodes. If T has a branching generator and X is upward dense in T, then X_{lex} is a large chain.*

Proof. Let (S, σ) be a branching generator for T. We aim at obtaining a branching generator (R, ρ) for T such that R_{lex} contains a pseudo-large chain and $R_{\text{lex}} \hookrightarrow X_{\text{lex}}$. We create (R, ρ) from (S, σ) by selecting a set $R \subseteq S$ as follows.

Let $r_T \in R$. Then there exists a node $r_T^- \in \sigma(r_T)$ such that in the linear ordering $\sigma(r_T)_{\text{lex}}$ the final segment $(r_T^-, \to)_{\text{lex}}$ is isomorphic to an uncountable subset of the reals. Set $\rho(r_T) :=$

$(r_T^-, \to)$. Repeat the construction for each $r \in R$ by choosing a node $r^- \in \sigma(r)$ such that the final segment $(r^-, \to)_{\text{lex}} \subseteq \sigma(r)_{\text{lex}}$ is isomorphic to an uncountable subset of the reals. Set $\rho(r) := (r^-, \to) \subseteq \sigma(r)$. Then (R, ρ) is a branching generator for T.

By removing the maximal well-ordered (hence countable) initial segment from each subchain $\rho(r)_{\text{lex}}$, we obtain an $(\mathbb{R}, \omega)$-chain P_{lex} such that the set of successors of each element in P (considered as a tree on its own) is isomorphic to an uncountable subset of the reals with no minimum element. Therefore P_{lex} is a pseudo-large chain. By the upward density of X in T, for each $r^- \in \sigma(r)$, there exists $x_r \in X$ such that $r^- \preceq_T x_r$. Since for each $r_1, r_2 \in R$, the chain of implications

$$r_1 \neq r_2 \;\Rightarrow\; r_1^- \neq r_2^- \;\Rightarrow\; x_{r_1} \neq x_{r_2}$$

holds, it follows that

$$P_{\text{lex}} \;\subseteq\; R_{\text{lex}} \;\cong\; \{x_r : r \in R\}_{\text{lex}} \;\subseteq\; X_{\text{lex}}.$$

Therefore X_{lex} is a large chain, as claimed. □

4. Lexicographically intrinsic properties

In this section we introduce the notion of a lexicographically intrinsic property of the nodes of $(\mathbb{R}, \omega)$-trees, which is defined by the invariance of the lexicographic linearization of the cone above a node. This order-theoretic property has some peculiar features, which allow one to select a suitable representative within each equivalence class of $(\mathbb{R}, \omega)$-trees. The content of this section is rather technical, however its main result (Theorem 4.6) plays a fundamental role in the proof of the dichotomy theorem. The reader who is only interested in a compact proof of the dichotomy theorem might want to (temporarily) skip all technical details (Lemma 4.3 and onwards).

Definition 4.1. Let $\mathfrak{P}$ be a property of the nodes of $(\mathbb{R}, \omega)$-trees. For each $(\mathbb{R}, \omega)$-tree T, denote by $\mathfrak{P}(T)$ the set of nodes in T satisfying property $\mathfrak{P}$. We say that $\mathfrak{P}$ is *lexicographically intrinsic* if for each pair of $(\mathbb{R}, \omega)$-trees T and T', for each pair of nodes $t \in T$ and $t' \in T'$, whenever $[t, \to) \sim_{\text{lex}} [t', \to)$ as trees (in the sense of Definition 2.2), we have that $t \in \mathfrak{P}(T)$ if and only if $t' \in \mathfrak{P}(T')$.

Thus, a lexicographically intrinsic property is an invariant of the lexicographic linearization of the cone above a node. An example of a lexicographically intrinsic property is the following.

Example 4.2. Let T be an $(\mathbb{R}, \omega)$-tree and t a node of T. We say that t is a *pseudo-small node* if the lexicographic linearization $[t, \to)_{\text{lex}}$ of the closed cone $[t, \to)$ is a pseudo-small chain. It is easy to check that the property of being (or not being) a pseudo-small node is lexicographically intrinsic.

We aim at showing that for each $(\mathbb{R}, \omega)$-tree T, if $\mathfrak{P}(T) \subseteq T$ is the set of all nodes of T satisfying a lexicographically intrinsic property $\mathfrak{P}$, then there exists an $(\mathbb{R}, \omega)$-tree U with the same lexicographic linearization as T, and whose own set of nodes $\mathfrak{P}(U)$ is displayed at its lowest levels. The proof of this fact is based on a recursive construction. We need three preliminary lemmas: two are related to a "pulling down" operation of the cones above selected nodes, and one is related to a "fusion" of the levels of suitably compatible trees.

Lemma 4.3 (*Pulling Down #1*)**.** *Let $\mathfrak{P}$ be a lexicographically intrinsic property of $(\mathbb{R}, \omega)$-trees. For each $(\mathbb{R}, \omega)$-tree T such that $\mathfrak{P}(T)$ is uncountable, there exists an $(\mathbb{R}, \omega)$-tree T' satisfying* *the following properties:*

(1) $T' \stackrel{\text{lex}}{=} T$;
(2) *for each* $t \in T$, $\mathrm{ht}(t, T') \leq \mathrm{ht}(t, T)$;
(3) *for each* $t \in T$, *either* $[t, \rightarrow)_{T'} \stackrel{\text{lex}}{=} [t, \rightarrow)_T$ *or* t *is a terminal node of* T';
(4) T *is a refinement of* T';
(5) $\downarrow \mathfrak{P}(T') \cap \mathrm{Lev}_1(T')$ *is uncountable.*

Proof. Toward a contradiction, let T be a counterexample to the claim of the theorem. Specifically, assume that $\mathfrak{P}(T)$ is uncountable and there is no $(\mathbb{R}, \omega)$-tree T' satisfying properties (1)–(5). Further, since there exists $k \geq 2$ such that the set of nodes $A :=\downarrow \mathfrak{P}(T) \cap \mathrm{Lev}_k(T)$ is uncountable, we assume that this natural number k is minimal among all such counterexamples. Note that if T is a counterexample and T' is another $(\mathbb{R}, \omega)$-tree that satisfies all properties but (5), then T' is also a counterexample. In the sequel we construct from T an $(\mathbb{R}, \omega)$-tree T' satisfying properties (1)–(4) and such that $\downarrow \mathfrak{P}(T') \cap \mathrm{Lev}_{k-1}(T')$ is uncountable.

Minimality of k implies that there is a node $u \in \mathrm{Lev}_{k-1}(T)$ such that the set $B := A \cap \mathrm{Succ}(u, T)$ is uncountable. Let T' be the tree obtained from T by pulling the open cone $(u, \rightarrow)_T$ down one level as follows. In the original tree T, let u' be the immediate predecessor of u. Further, let

$$\mathrm{Succ}(u', T) = X \cup \{u\} \cup Y \subseteq \mathrm{Lev}_{k-1}(T)$$

be the $\mathbb{R}$-embeddable chain of immediate successors of u', where $X < u < Y$ in the order of the real numbers. In the new tree T', both the underlying set of nodes and the partial order are the same as in T, with only two exceptions: (a) the set of immediate successors of u' is the $\mathbb{R}$-embeddable chain

$$\mathrm{Succ}(u', T') = X \cup \{u\} \cup \mathrm{Succ}(u, T) \cup Y$$

where $X < u < \mathrm{Succ}(u, T) < Y$ in the order of the real numbers; and (b) the node u is terminal in T'. It is straightforward to check that T' is an $(\mathbb{R}, \omega)$-tree satisfying properties (1)–(4). Note that the described operation affects only the node u with respect to $\mathfrak{P}$, because $\mathfrak{P}$ is a lexicographically intrinsic property. Therefore all nodes in B belong to $\downarrow \mathfrak{P}(T')$, and so the set $\downarrow \mathfrak{P}(T') \cap \mathrm{Lev}_{k-1}(T')$ is uncountable.

If $k = 2$, it follows that T' is an $(\mathbb{R}, \omega)$-tree that satisfies properties (1)–(5), which contradicts the fact that T is a counterexample. On the other hand, if $k > 2$, then T' is a counterexample such that $\downarrow \mathfrak{P}(T') \cap \mathrm{Lev}_{k-1}(T')$ is uncountable, contradicting the minimality of k. $\square$

Lemma 4.4 (*Pulling Down #2*)**.** *Let* $\mathfrak{P}$ *be a lexicographically intrinsic property of* $(\mathbb{R}, \omega)$*-trees. For each* $(\mathbb{R}, \omega)$*-tree* T *such that* $\downarrow \mathfrak{P}(T) \cap \mathrm{Lev}_1(T)$ *is uncountable, there exists an* $(\mathbb{R}, \omega)$*-tree* T' *satisfying the following properties:*

(1) $T' \stackrel{\text{lex}}{=} T$;
(2) *for each* $t \in T$, $\mathrm{ht}(t, T') \leq \mathrm{ht}(t, T)$;
(3) *for each* $t \in T$, *either* $[t, \rightarrow)_{T'} \stackrel{\text{lex}}{=} [t, \rightarrow)_T$ *or* t *is a terminal node of* T';
(4) T *is a refinement of* T';
(5) $\mathfrak{P}(T') \cap \big(\mathrm{Lev}_1(T') \cup \mathrm{Lev}_2(T')\big)$ *is uncountable.*

Proof. Toward a contradiction, let T be a counterexample. By hypothesis the set $\downarrow \mathfrak{P}(T)$ contains an uncountable set of distinct nodes $\{t_\alpha : \alpha < \omega_1\} \subseteq \mathrm{Lev}_1(T)$. Thus, for each $\alpha < \omega_1$ there exists a node $z_\alpha \in \mathfrak{P}(T)$ such that $t_\alpha \preceq_T z_\alpha$. Since $\mathrm{ht}(T) \leq \omega$ and there is no $(\mathbb{R}, \omega)$-tree T' satisfying (1)–(5), we can assume without loss of generality that there exists a natural number $k \geq 3$ such that for each $\alpha < \omega_1$, we have $z_\alpha \in \mathrm{Lev}_k(T)$. Suppose that T is chosen with minimal such k.

Fix $\alpha < \omega_1$. Let y_α be the immediate predecessor of z_α, and x_α the immediate predecessor of y_α. Note that $t_\alpha \preceq_T x_\alpha \prec_T y_\alpha \prec_T z_\alpha$ because $k \geq 3$. Let

$$\mathrm{Succ}(y_\alpha, T) = Z_\alpha^- \cup \{z_\alpha\} \cup Z_\alpha^+ \subseteq \mathrm{Lev}_k(T)$$

be the $\mathbb{R}$-embeddable chain of immediate successors of y_α, where $Z_\alpha^- < z_\alpha < Z_\alpha^+$ in the order of the real numbers. Further, let

$$\mathrm{Succ}(x_\alpha, T) = Y_\alpha^- \cup \{y_\alpha\} \cup Y_\alpha^+ \subseteq \mathrm{Lev}_{k-1}(T)$$

be the $\mathbb{R}$-embeddable chain of immediate successors of x_α, where $Y_\alpha^- < y_\alpha < Y_\alpha^+$ in the order of the real numbers.

We define a new tree T_α by "pulling down" one level both the closed cone $[z_\alpha, \rightarrow)_T$ and the union of closed cones $\uparrow Z_\alpha^+ \subseteq T$. The underlying set of nodes and partial order of T_α are the same as in T, with only two exceptions: (a) y_α is a terminal node in T_α; and (b) the set of immediate successors of x_α in T_α is the $\mathbb{R}$-embeddable chain

$$\mathrm{Succ}(x_\alpha, T_\alpha) = Y_\alpha^- \cup \{y_\alpha\} \cup Z_\alpha^- \cup \{z_\alpha\} \cup Z_\alpha^+ \cup Y_\alpha^+$$

where $Y_\alpha^- < y_\alpha < Z_\alpha^- < z_\alpha < Z_\alpha^+ < Y_\alpha^+$ in the order of real numbers. Note that T_α is an $(\mathbb{R}, \omega)$-tree such that $T_\alpha \overset{\mathrm{lex}}{=} T$ and $\mathrm{ht}(t, T_\alpha) \leq \mathrm{ht}(t, T)$ for each $t \in T$. Observe also that the operation described above modifies only the lexicographic linearization of the closed cone above y_α, whereas the lexicographic linearization of all the other closed cones remains unchanged. Finally, it is straightforward to check that the partial order on T_α is contained in the partial order of T, since we have eliminated some comparisons and added none. Therefore, properties (1)–(4) hold for T_α and, since $\mathfrak{P}$ is lexicographically intrinsic, the node z_α belongs to $\mathfrak{P}(T_\alpha) \cap \mathrm{Lev}_{k-1}(T_\alpha)$.

Next, we create a tree T' by simultaneously performing the "pulling down" operation on each node in the set $\{z_\alpha : \alpha < \omega_1\}$. The tree T' has the same underlying set of nodes and the same lexicographic linearization as T, but the set of nodes $\{z_\alpha : \alpha < \omega_1\}$ lies at level $k-1$ in the new tree T'. Observe that the construction never increases the height of any node, and so the tree T' satisfies (1) and (2). Furthermore, the set $\mathrm{Succ}(t', T')$ is an $\mathbb{R}$-embeddable chain for each $t' \in T'$: indeed, the only sets of immediate successors that have been altered in T' are $\mathrm{Succ}(y_\alpha, T')$ and $\mathrm{Succ}(x_\alpha, T')$ for each $\alpha < \omega_1$, and all of them are embeddable into $\mathbb{R}$. It follows that T' is $(\mathbb{R}, \omega)$-tree. Finally, note that the only nodes in T' whose closed cones have been modified are the nodes in the set $\{y_\alpha : \alpha < \omega_1\}$, which are terminal in T'. Therefore, properties (3) and (4) hold for T' as well. Since $\mathfrak{P}$ is lexicographically intrinsic, the uncountable set of nodes $\{z_\alpha : \alpha < \omega_1\}$ lies in $\mathfrak{P}(T') \cap \mathrm{Lev}_{k-1}(T')$.

If $k = 3$, then T' is an $(\mathbb{R}, \omega)$-tree satisfying properties (1)–(5). This contradicts the fact that T is a counterexample. On the other hand, if $k > 3$, then T' is another counterexample to the statement of the theorem, contradicting the minimality of k. This completes the proof. $\square$

For each $(\mathbb{R}, \omega)$-tree T and for each $n < \omega$, the notation $T \upharpoonright (n+1)$ stands for the subtree formed by its first n levels, i.e., $\bigcup_{i \leq n} \mathrm{Lev}_i(T)$.

Lemma 4.5 (*Fusion Lemma*). *Let T be an $(\mathbb{R}, \omega)$-tree and $\{U^{(n)} : n < \omega\}$ a sequence of $(\mathbb{R}, \omega)$-trees satisfying the following properties for each $n < \omega$:*

(a) $U^{(n)} \upharpoonright (m+1) = U^{(m)} \upharpoonright (m+1)$ *for each $m \leq n$;*

(b) $U^{(n)} \overset{\text{lex}}{=} T$;
(c) $\text{ht}(t, U^{(n)}) \leq \text{ht}(t, U^{(m)}) \leq \text{ht}(t, T)$ *for each* $t \in T$ *and* $m \leq n$.

Then the tree U*, whose* n*-th level is defined by* $\text{Lev}_n(U) := \text{Lev}_n\left(U^{(n)}\right)$ *for each* $n < \omega$*, is a well-defined* $(\mathbb{R}, \omega)$*-tree satisfying following properties:*

(i) $U \restriction (n+1) = U^{(n)} \restriction (n+1)$;
(ii) $U \overset{\text{lex}}{=} T$.

Proof. Let U be the tree defined by levels as in the statement of the theorem. We prove that U is an $(\mathbb{R}, \omega)$-tree satisfying properties (i)–(ii). First of all, observe that U is well-defined. In fact, for each $u \in U$ there exists a natural number $k := \text{ht}(u, T) + 1$ such that for all $n \geq k$, we have $(\leftarrow, u]_U = (\leftarrow, u]_{U(n)}$ by properties (a) and (c) in the hypothesis. Therefore, the initial segment having u as endpoint stabilizes into a (well-defined) well-ordered set.

To prove (i), fix $n < \omega$. For each $k \leq n$, hypothesis (a) and the definition of U yield the chain of equalities $\text{Lev}_k(U^{(k)}) = \text{Lev}_k(U^{(n)}) = \text{Lev}_k(U)$. Property (i) follows.

Next, we show that U is an $(\mathbb{R}, \omega)$-tree. Hypothesis (c) implies that $\text{ht}(U^{(n)}) \leq \text{ht}(T) \leq \omega$ for each $n < \omega$, hence $h(U) \leq \omega$. We claim that the set of immediate successors in U of each node $u \in U$ is an $\mathbb{R}$-embeddable chain; this will show that U is an $(\mathbb{R}, \omega)$-tree. Let u be an arbitrary node in U. Then there exists $n < \omega$ such that $u \in \text{Lev}_n(U)$. Property (i) yields the equality $U \restriction (n+2) = U^{(n+1)} \restriction (n+2)$. In particular, $\text{Succ}(u, U) = \text{Succ}(u, U^{(n+1)})$, hence the chain $\text{Succ}(u, U)$ embeds into $\mathbb{R}$, because $U^{(n+1)}$ is an $(\mathbb{R}, \omega)$-tree by hypothesis.

To prove (ii), let $u_1, u_2 \in U$ such that $u_1 \prec_{T_{\text{lex}}} u_2$. Then there exist $n_1, n_2 < \omega$ such that $u_1 \in \text{Lev}_{n_1}(U)$ and $u_2 \in \text{Lev}_{n_2}(U)$. Let $n := \max\{n_1, n_2\}$. Since $U^{(n)} \overset{\text{lex}}{=} T$ by hypothesis (b), we have $u_1 \prec_{{U^{(n)}}_{\text{lex}}} u_2$. Thus property (i) and the definition of lexicographic linearization of a tree imply that $u_1 \prec_{U_{\text{lex}}} u_2$, as claimed. $\square$

We are ready to prove the main result of this section. Its content can be roughly summarized as follows: for each lexicographic intrinsic property $\mathfrak{P}$ of an $(\mathbb{R}, \omega)$-tree, we can always find another $(\mathbb{R}, \omega)$-tree "similar" to the original one, and such that for each node, the locally uncountable character of $\mathfrak{P}$ is always displayed at the first two levels above the node. For the sake of synthesis, for any tree T and node $t \in T$, we use the following notation:

$$\begin{aligned} \text{Succ}^{(2)}(t, T) &:= \bigcup_{u \in \text{Succ}(t,T)} \text{Succ}(u, T), \\ \text{Succ}^{(1,2)}(t, T) &:= \text{Succ}(t, T) \cup \text{Succ}^{(2)}(t, T). \end{aligned} \tag{1}$$

Theorem 4.6. *Let* $\mathfrak{P}$ *be a lexicographically intrinsic property of* $(\mathbb{R}, \omega)$*-trees, and* T *an* $(\mathbb{R}, \omega)$*-tree. There exists an* $(\mathbb{R}, \omega)$*-tree* U *satisfying the following properties:*

(i) $U \overset{\text{lex}}{=} T$;
(ii) T *is a refinement of* U;
(iii) *for each* $u \in U$*, either* $\mathfrak{P}(U) \cap \text{Succ}^{(1,2)}(u, U)$ *is uncountable or* $\mathfrak{P}(U) \cap [u, \rightarrow)_U$ *is countable.*

Proof. If $\mathfrak{P}(T)$ is countable, then the result holds trivially for $U := T$. Therefore, assume that $\mathfrak{P}(T)$ is uncountable. Denote by $\mathfrak{Q}(T)$ be the set of the nodes of T that do not satisfy (iii), i.e.,

$$\mathfrak{Q}(T) := \{t \in T : |\mathfrak{P}(T) \cap \text{Succ}^{(1,2)}(t, T)| \leq \omega \ \wedge \ |\mathfrak{P}(T) \cap (t, \rightarrow)_T| \geq \omega_1\}.$$

We construct by recursion a sequence $\{(U^{(n)}, \preceq_{U^{(n)}}) : n < \omega\}$ of $(\mathbb{R}, \omega)$-trees satisfying the following properties for each $n < \omega$ (for consistency, let $U^{(-1)} := T$ by definition):

(a) $U^{(n)} \restriction (m+1) = U^{(m)} \restriction (m+1)$ for each $m \le n$;
(b) $[x, \to)_{U^{(n)}} \overset{\text{lex}}{=} [x, \to)_{U^{(n-1)}}$ for each $x \in U^{(n)} \restriction (n+1)$;
(c) $U^{(n)} \overset{\text{lex}}{=} T$;
(d) $\mathrm{ht}(t, U^{(n)}) \le \mathrm{ht}(t, U^{(m)}) \le \mathrm{ht}(t, T)$ for each $t \in T$ and $m \le n$;
(e) $\preceq_{U^{(n)}} \ \subseteq \ \preceq_{U^{(m)}} \ \subseteq \ \preceq_T$ for each $m \le n$;
(f) $\mathfrak{Q}(U^{(n)}) \cap U^{(n)} \restriction (n+1) = \emptyset$.

For the base step of the recursive construction, a joint application of the two "pulling down" Lemmas 4.3 and 4.4 yields an $(\mathbb{R}, \omega)$-tree T' such that (1) $T' \overset{\text{lex}}{=} T$, (2) $\mathrm{ht}(t, T') \le \mathrm{ht}(t, T)$ for each $t \in T$, (3) either $[t, \to)_{T'} \overset{\text{lex}}{=} [t, \to)_T$ or t is a terminal node of T' for each $t \in T$, (4) T is a refinement of T', and (5) $|\mathfrak{P}(T') \cap (\mathrm{Lev}_1(T') \cup \mathrm{Lev}_2(T'))| \ge \omega_1$. Set $U^{(0)} := T'$. It is easy to check that $U^{(0)}$ satisfies properties (a)–(f).

Next, assume that the $(\mathbb{R}, \omega)$-tree $U^{(n)}$ satisfying properties (a)–(f) has been defined. If the set $\mathfrak{Q}(U^{(n)}) \cap \mathrm{Lev}_{n+1}(U^{(n)})$ is empty, then set $U^{(n+1)} := U^{(n)}$. Otherwise, for each node $u \in \mathfrak{Q}(U^{(n)}) \cap \mathrm{Lev}_{n+1}(U^{(n)})$, we apply the two "pulling down" Lemmas 4.3 and 4.4 to the $(\mathbb{R}, \omega)$-tree $T_u := [u, \to)_{U^{(n)}}$ and obtain an $(\mathbb{R}, \omega)$-tree $T_u{}'$ such that:

(1) $T_u{}' \overset{\text{lex}}{=} T_u$;
(2) $\mathrm{ht}(t, T_u{}') \le \mathrm{ht}(t, T_u)$ for each $t \in T_u$;
(3) either $[t, \to)_{T_u{}'} \overset{\text{lex}}{=} [t, \to)_{T_u}$, or t is a terminal node of $T_u{}'$ for each $t \in T$;
(4) T_u is a refinement of $T_{u'}$;
(5) $|\mathfrak{P}(T_u{}') \cap (\mathrm{Lev}_1(T_u') \cup \mathrm{Lev}_2(T_u{}'))| \ge \omega_1$.

Since $\mathfrak{Q}(U^{(n+1)}) \cap \mathrm{Lev}_{n+1}(U^{(n+1)})$ is empty by construction, the inductive hypothesis implies that the tree $U^{(n+1)}$ satisfies properties (a)–(f). This completes the construction of the sequence $\{U^{(n)} : n < \omega\}$.

Let $(U, \preceq_U)$ be the tree whose n-th level is defined by $\mathrm{Lev}_n(U) := \mathrm{Lev}_n(U^{(n)})$ for each $n < \omega$. By the "fusion" Lemma 4.5 and by construction, U is an $(\mathbb{R}, \omega)$-tree such that the following properties hold for each $n < \omega$:

(A) $U \restriction (n+1) = U^{(n)} \restriction (n+1)$;
(B) $U \overset{\text{lex}}{=} T$;
(C) $\mathrm{ht}(u, U) \le \mathrm{ht}(u, U^{(n)}) \le h(u, T)$ for each $u \in U$;
(D) $[u, \to)_U \overset{\text{lex}}{=} [u, \to)_{U^{(n)}}$ for each $u \in U \restriction (n+1)$;
(E) $\preceq_U \ \subseteq \ \preceq_{U^{(n)}} \ \subseteq \ \preceq_T$;
(F) $\mathfrak{Q}(U^{(n)}) \cap U^{(n)} \restriction (n+1) = \emptyset$.

To complete the proof, we show that U satisfies property (iii) of the thesis, i.e., $\mathfrak{Q}(U) = \emptyset$. Let u be an arbitrary node of U. Then $u \in \mathrm{Lev}_k(T)$ for some $k < \omega$. Properties (C) and (F) imply that $u \notin \mathfrak{Q}(U^{(k)})$. Furthermore, the equality $[u, \to)_U \overset{\text{lex}}{=} [u, \to)_{U^{(k)}}$ holds by property (D). Since $\mathfrak{P}$ is lexicographically intrinsic, it follows that $u \notin \mathfrak{Q}(U)$, as claimed. $\square$

5. The dichotomy theorem

We apply Theorem 4.6 to a special lexicographically intrinsic property for the nodes of $(\mathbb{R}, \omega)$-trees: the property $\widetilde{\mathcal{S}}$ of *not* being a small node, defined as follows (cf. Example 4.2).

Definition 5.1. Let T be an $(\mathbb{R}, \omega)$-tree and t a node of T. We say that t is a *small node* if $[t, \rightarrow)_{\text{lex}}$ is a small chain. We use the following notation:

$$\mathcal{S}(T) := \{t \in T : t \text{ is a small node}\} \quad \text{and} \quad \widetilde{\mathcal{S}}(T) := T \setminus \mathcal{S}(T).$$

Let $A \subseteq T$ be a nonempty set in a tree $(T, \preceq)$. Recall that the *downward closure* of A in T is the set $\downarrow A := \bigcup_{a \in A} (\leftarrow, a]$, and A is *downward closed* in T if $A = \downarrow A$. (Note that each downward closed set is a subtree of T.) Dually, the *upward closure* of A in T is the set $\uparrow A := \bigcup_{a \in A} [a, \rightarrow)$, and A is *upward closed* in T if $A = \uparrow A$. Furthermore, we say that A is *upward dense* in T if for each $t \in T$ there exists $a \in A$ such that $t \preceq a$ (i.e., $\downarrow A = T$).

Lemma 5.2. *For each $(\mathbb{R}, \omega)$-tree T, the set $\mathcal{S}(T)$ is upward closed, and the set $\widetilde{\mathcal{S}}(T)$ is downward closed.*

Proof. Let T be an $(\mathbb{R}, \omega)$-tree. It suffices to show that $\mathcal{S}(T)$ is upward closed. Assume that $t, u \in T$ are such that $t \prec u$ and t is a small node. Since the linear ordering $[t, \rightarrow)_{\text{lex}}$ is a small chain, there exists a pseudo-small tree U such that $[t, \rightarrow)_{\text{lex}}$ embeds into U_{lex}. Thus we have $[u, \rightarrow)_{\text{lex}} \subseteq [t, \rightarrow)_{\text{lex}} \hookrightarrow U_{\text{lex}}$, hence u is a small node. $\square$

If an $(\mathbb{R}, \omega)$-tree has a non-small node, then it has uncountably many non-small nodes, as the next results shows.

Lemma 5.3. *For each $(\mathbb{R}, \omega)$-tree T, the following statements are equivalent:*

(i) *T_{lex} is a small chain;*
(ii) *the root of T is a small node;*
(iii) *all nodes of T are small;*
(iv) *all but at most countably many nodes of T are small.*

Therefore, the set $\widetilde{\mathcal{S}}(T)$ is either empty or uncountable.

Proof. Let T be an $(\mathbb{R}, \omega)$-tree and r_T the root of T. The equality $T = [r_T, \rightarrow)$ and Lemma 5.2 yield the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). To finish the proof, it suffices to show that (iv) implies (ii). Toward a contradiction, assume that the set $\widetilde{\mathcal{S}}(T)$ is countable and r_T fails to be a small node. We claim that T_{lex} is a small chain; this will contradict the equivalence (i) $\Leftrightarrow$ (ii).

By Theorem 3.7, to prove the claim it suffices to show that the linear ordering $T_{\text{lex}} \setminus \widetilde{\mathcal{S}}(T) = \mathcal{S}(T)_{\text{lex}}$ is a small chain. Let

$$B := \{b \in T : b \text{ is a minimal small node}\}$$

where "minimal" means that if $b \in B$, then there is no $b' \in B$ such that $b' \prec b$. Therefore, the set B is an antichain of T that does not contain r_T. For each $b \in B$, let t_b be the immediate predecessor of b. By the minimality of the nodes of B, the set

$$C := \{t_b : b \in B\}$$

contains only nodes that fail to be small. Note that Lemma 5.2 yields that $\uparrow B = \mathcal{S}(T)$ and $\downarrow C \subseteq \widetilde{\mathcal{S}}(T)$. By hypothesis, the set $\downarrow C$ is countable, hence so is the set C. Since T is an

$(\mathbb{R}, \omega)$-tree, we obtain

$$B_{\text{lex}} \hookrightarrow C_{\text{lex}} \times_{\text{lex}} \mathbb{R} \hookrightarrow \mathbb{R}^{\leq 2}_{\text{lex}}$$

hence B_{lex} is a small chain. Furthermore, we have

$$T_{\text{lex}} \setminus \widetilde{\mathcal{S}}(T) = \mathcal{S}(T)_{\text{lex}} = \uparrow B_{\text{lex}} = \sum_{b \in B_{\text{lex}}} [b, \to)_{\text{lex}}$$

where $[b, \to)_{\text{lex}}$ is a small chain by hypothesis. Therefore, Lemma 3.4 yields that $T_{\text{lex}} \setminus \widetilde{\mathcal{S}}(T)$ is a small chain and the proof is complete. □

Finally, we have:

Theorem 5.4. *Each chain embeddable into $\mathbb{R}^{<\omega}_{\text{lex}}$ is either small or large.*

Proof. Let L be a chain, and $f : L \hookrightarrow \mathbb{R}^{<\omega}_{\text{lex}}$ an embedding. Let $(T, \preceq_T)$ be the subtree of $(\mathbb{R}^{<\omega}, \sqsubseteq)$ obtained by taking the downward closure of the image $f[L]$. Then T is an $(\mathbb{R}, \omega)$-tree such that L embeds into the $(\mathbb{R}, \omega)$-chain T_{lex}.

We claim that there exists a tree representation $(U, \preceq_U)$ of T_{lex} with the following two properties: (i) T is a refinement of U; (ii) for each $u \in U$, either u is a small node or there are uncountably many nodes in $\text{Succ}^{(1,2)}(u, U)$ that fail to be small. Indeed, an application of Theorem 4.6 to the tree T and the intrinsic property $\widetilde{\mathcal{S}}$ yields an $(\mathbb{R}, \omega)$-tree representation U of T_{lex} such that for each $u \in U$, either $\widetilde{\mathcal{S}}(U) \cap \text{Succ}^{(1,2)}(u, U)$ is uncountable, or $\widetilde{\mathcal{S}}(U) \cap [u, \to)$ is countable. For each $u \in U$, if $\widetilde{\mathcal{S}}(U) \cap [u, \to)$ is countable, then Lemma 5.3 implies that the root u of the $(\mathbb{R}, \omega)$-tree $[u, \to)$ is a small node. The claim follows.

As a consequence of the claim, either (I) the root r_U of U is a small node, or (II) there are uncountably many nodes in $\text{Lev}_1(U) \cup \text{Lev}_2(U)$ that fail to be small. In case (I), Lemma 5.3 implies that U_{lex} is a small chain. Since L embeds into U_{lex}, it follows that L is a small chain as well.

Next, assume that (II) holds, hence L is not small. We claim that U has a branching generator (S, σ). Let $r_U \in S$. If there exists an uncountable set $A_1 \subseteq \text{Lev}_1(U)$ of non-small nodes, then set $\sigma(r_U) := A_1$. Otherwise, there is an uncountable set $A_2 \subseteq \text{Lev}_2(U)$ of non-small nodes. In this case, if there exists $u \in \text{Lev}_1(U)$ such that the set $A_2' := \text{Succ}(u, U) \cap A_2$ is uncountable, then set $\sigma(r_U) := A_2'$. On the other hand, if there is no such a node $u \in \text{Lev}_1(U)$, then select an uncountable set $A_2'' \subseteq A_2$ such that no elements in A_2'' have the same immediate predecessor, and set $\sigma(r_U) := A_2''$. It is immediate to check that the lexicographic linearization of the uncountable antichain $\sigma(r_U)$ is embeddable into the reals. Since each node $u \in \sigma(r_U)$ fails to be small, we can repeat the above construction for each such u and create recursively a branching generator for U. This proves the claim.

Since the partial order $\preceq_U$ is contained in the partial order $\preceq_T$, we can apply Lemma 3.10 to obtain a branching generator for T. By construction the chain L is upward dense in T, hence Theorem 3.11 yields that L is a large chain. □

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