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The Cauchy-Dirichlet Problem for Parabolic Equations with VMO Coefficients

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Abstract—In this note, we prove the well posedness for a Cauchy-Dirichlet problem in the class $W_p^{1,2}$, for every p between 1 and ∞.

The main part of the linear parabolic operator is symmetric, uniformly elliptic and belongs to the Sarason's class VMO.

Consequently, we prove a boundary estimate of the solution u, where the related constant depends on coefficients only through the ellipticity constant and the VMO moduli.

Keywords—Parabolic equations, Cauchy-Dirichlet problem.

1. INTRODUCTION

Let $\mathcal{L}$ be a linear parabolic operator of the form,

$$\mathcal{L}u \equiv u_t - \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x'_i x'_j} + \sum_{i=1}^{n} b_i(x)\, u_{x'_i} + cu, \tag{1.1}$$

where $x = (x', t) = (x'_1, x'_2, \dots, x'_n, t) \in \mathbb{R}^{n+1}$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain and Q_T, the cylinder $\Omega \times (0, T)$.

Aim of this note is to study the Cauchy-Dirichlet problem,

$$\begin{aligned} \mathcal{L}u &= f, && \text{a.e. } x \in Q_T, \\ u &= 0, && \text{on } \partial\Omega \times (0,T), \\ u(x', 0) &= 0, && \text{in } \Omega, \end{aligned} \tag{1.2}$$

where $f \in L^p(Q_T)$, $1 < p < \infty$. The coefficients $a_{ij}(x)$ are discontinuous; precisely, they are in the space VMO defined by Sarason [1], symmetric and also uniformly elliptic, i.e.,

$$\exists\, \tau \geq 1 : \tau^{-1} |\xi|^2 \leq a_{ij}(x)\, \xi_i \xi_j \leq \tau |\xi|^2, \qquad \forall \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in Q_T. \tag{1.3}$$

Let us assume that the coefficients b_i and c belong to suitable Lebesgue spaces.

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The literature about Cauchy-Dirichlet problem is very wide, we recall for instance the study by Campanato [2], Guglielmino [3], the papers by Gagliardo [4,5], the book by Ladyženskaya and Ural'ceva [6].

With different methods than that one used by Ladyženskaya and Ural'ceva, Gagliardo proved the existence and the uniqueness of the solution of Cauchy Dirichlet problem in [4]. In this paper, Gagliardo begins to study the regularity problem of the solutions, and interesting developments have been achieved in the study of this problem in the note [5].

A few years later, Arena with different hypotheses for the coefficients than those by Guglielmino in [7,8] studied the well posedness of the Cauchy-Dirichlet problem proving theorems of existence and uniqueness for solutions $u \in H^{2,1}(Q_T)$.

In the present note, the author considers a Cauchy-Dirichlet problem related to nondivergence form parabolic equation with discontinuous coefficients and proves the existence, uniqueness, and regularity of the solution. This problem is inspired by the study made by Bramanti and Cerutti [9]. We point out that our fundamental tools are some properties related to the products of the terms b_i and c with u and its derivatives. The technique used to obtain these results is based on dividing the cylinder Q_T in sections and obtaining the requested estimates in each part of the subdivision.

2. SOME DEFINITIONS

Let us introduce in $\mathbb{R}^{n+1}$ the following *parabolic metric* $d(x,y) = \rho(x-y)$, where

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + 4t^2}}{2}}.$$

DEFINITION 2.1. *Let us assume parabolic cubes of center* $x = (x', t) \in \mathbb{R}^{n+1}$ *and radius* r

$$I = I_r(x) = \left\{ y = (y', s) \in \mathbb{R}^{n+1} : |x' - y'| < r,\ |t - s| < r^2 \right\}.$$

DEFINITION 2.2. *(See [10].) Let* $f \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$. *We say that* f *belongs to the parabolic BMO space (BMO*$(\mathbb{R}^{n+1})$*) if the seminorm,*

$$\|f\|_* = \sup_I \frac{1}{|I|} \int_I |f(x) - f_I| \, dx,$$

is finite, where I *ranges in the class of the parabolic cubes in* $\mathbb{R}^{n+1}$ *and* f_I *is the average* $1/|I| \int_I f(x)\, dx$.

Let $f \in$ BMO and

$$\eta_f(r) = \sup_{\rho \le r} \frac{1}{|I_\rho|} \int_{I_\rho} \left| f(x) - f_{I_\rho} \right| dx$$

where I_ρ ranges over the class of parabolic cubes in $\mathbb{R}^{n+1}$ of radius ρ.

DEFINITION 2.3. *(See e.g., [11].) We say that the function* $f \in$ *BMO is in the Sarason class VMO*$(\mathbb{R}^{n+1})$ *if*

$$\lim_{r \to 0^+} \eta(r) = 0.$$

We will refer to η as the VMO modulus of f.

DEFINITION 2.4. *Let us denote $L^{p,q}(Q_T)$, $1 \le p,\ q \le \infty$ the space of measurable functions $h(x,t)$ such that*

$$\|h\|_{L^{p,q}(Q_T)} \equiv \left(\int_0^T \left(\int_\Omega |h(x,t)|^p \, dx \right)^{q/p} dt \right)^{1/q} < \infty,$$

with obvious modifications if p or q or both the exponents are infinite. If $p = q$, we simply write $L^p(Q_T)$ instead of $L^{p,p}(Q_T)$.

Moreover, if $h \in L^{p,q}(Q_T)$, we set

$$\omega(\sigma) = \sup_{|D| \le \sigma} \|h\|_{L^{p,q}(D)},$$

where $D \subset \mathbb{R}^{n+1}$ is Lebesgue measurable and $|D|$ is its Lebesgue measure.

The function $\omega(\sigma)$ is decreasing in $]0, |Q_T|[$ and is such that

$$\lim_{\sigma \to 0} \omega(\sigma) = 0.$$

We will refer to $\omega(\sigma)$ as the AC modulus of $|h|$.

We intend consider $W_p^{2,1}(Q_T)$ as the space of functions $u(x,t)$ with p-summability in Q_T such that the derivative with respect to t belongs to $L^p(Q_T)$ and as functions of t with values in $L^p(\Omega)$, belong to $L^p(0,T,W_p^2(\Omega))$.

Let us assume in $W^{2,1}(Q_T)$ as a norm the quantity,

$$\|u\|_{W_p^{2,1}(Q_T)} = \left(\int_{Q_T} \left(u_t^2 + \sum_{\alpha=1}^{2} |D^\alpha u|^2 \right)^{p/2} dx\, dt \right)^{1/p}.$$

Let us define $W_{0,p}(Q_T)$ to be the closure in the $W_p^{2,1}(Q_T)$ norm of the space

$$\mathcal{C} = \left\{ \phi \in C^\infty \left(\bar{Q}_T \right) : \phi = 0, \text{ for } t = 0, \text{ or } x \in \partial\Omega \right\}.$$

We are interested in the following Cauchy-Dirichlet problem,

$$\mathcal{L}u = f, \qquad \text{a.e. } x \in Q_T,$$
$$u \in W_{0,p}(Q_T),$$

associated with a nondivergence form parabolic equation.

3. PRELIMINARY LEMMA AND MAIN RESULTS

In the sequel, we need the next result and we refer the readers to [12] for the proof of the lemma.

LEMMA 3.1. *For every function $u \in W_p^{2,1}(Q_T)$, we have*

$$\text{ess}_{Q_T}\sup \left| D^j u \right| \le C \|u\|_{W_p^{2,1}(Q_T)}, \tag{3.1}$$

if $0 \le |j| \le 1$, $2p - 2 - |j|p > n$;

$$\left\| D^j u \right\|_{L^{\rho,\infty}(Q_T)} + \left\| D^j u \right\|_{L^{\infty,\rho}(Q_T)} \le C \|u\|_{W_p^{2,1}(Q_T)}, \tag{3.2}$$

if $0 \le |j| \le 1$, $2p - 2 - |j|p = n$; $1 \le \rho < \infty$;

$$\left\| D^j u \right\|_{L^{\rho,(p/1-\delta)}(Q_T)} \le C \|u\|_{W_p^{2,1}(Q_T)}, \tag{3.3}$$

if $0 \le |j| \le 1$, $2p - 2 - |j|p < n$, $\rho = np/(n - 2p + 2\theta + |j|p)$, *and moreover,*

$$\max\left(0, \frac{2p - |j|\,p - n}{2}\right) \le \theta \le \min\left(1, \frac{p\,(2 - |j|)}{2}\right). \tag{3.4}$$

If $2p - |j|p - n = 0$ *in (3.4), we have not to read the equal sign on the left. If* $2p - 2 - |j|p - n \le 0$ *and* $2 < p < \infty$ *in (3.4) we have not to read the equal sign on the right.*

The constant C *does not depend on* Ω, p, T *and on* ρ *in the case of (3.2), or on* θ *in the case of (3.3). If* u *has a trace equal to zero on* $\Omega \times \{T = 0\}$, *the constant* C *is not decreasing when* T *is growing.*

Let us assume the following hypotheses.

For every $i = 1, \dots, n$, $b_i \in L^{t,\bar{t}}(Q_T)$, where

$$t = \bar{t} = p, \qquad \text{if } p > n + 2,$$
$$t = p, \quad \bar{t} = \frac{p\rho}{\rho - p} \quad \text{or} \quad t = \frac{p\rho}{\rho - p}, \quad \bar{t} = p, \quad \forall\, \rho > p, \qquad \text{if } p = n + 2,$$
$$t = \frac{p\rho}{\rho - p}, \quad \bar{t} = \frac{p}{\theta}, \qquad \text{if } 1 < p < n + 2.$$

If $p < n + 2$, we set $\rho = np/(n - p + 2\theta)$ and $\max(0, (p - n)/2) \le \theta \le \min(1, p/2)$. Let us suppose $c \le 0$ a.e. in Q_T and also $c \in L^{s,\bar{s}}(Q_T)$, where

$$s = \bar{s} = p, \qquad \text{if } p > \frac{n+2}{2},$$
$$s = p, \quad \bar{s} = \frac{p\rho}{\rho - p} \quad \text{or} \quad s = \frac{p\rho}{\rho - p}, \quad \bar{s} = p \quad \forall\, \rho > p, \qquad \text{if } p = \frac{n+2}{2},$$
$$s = \frac{p\rho}{\rho - p}, \quad \bar{s} = \frac{p}{\theta}, \qquad \text{if } 1 < p < \frac{n+2}{2}.$$

If $p < n + 2/2$, we assume $\rho = np/(n - 2p + 2\theta)$, $\max(0, (2p - n)/2) \le \theta \le \min(1, p)$.

THEOREM 3.2. *Let us assume the coefficients* b_i *and* c *as above,* $\forall\, i = 1, \dots, n$.

Then, for every function $u \in W^{2,1}_p(Q_T)$, *we have*

$$b_i Du, cu \in L^p(Q_T), \qquad \forall\, 1 < p < \infty,$$

and it results

$$\|b_i Du\|_{L^p(Q_T)} + \|cu\|_{L^p(Q_T)} \le C\,\|u\|_{W^{2,1}_p(Q_T)},$$

where C *depends on* $T, p, n, \Omega, \|c\|_{L^{s,\bar{s}}(Q_T)}, \|b_i\|_{L^{t,\bar{t}}(Q_T)}$ *and the Sobolev constant.*

PROOF. To prove some of the following inequalities it is useful to consider Lemma 3.1.

Let $\mathbf{p} > \mathbf{n} + \mathbf{2}$.

Then,

$$\left(\int_0^T \left(\int_\Omega |b_i Du|^p\,dx\right) dt\right)^{1/p} \le \left(\int_0^T \left(\int_\Omega |b_i|^p \left(\sup_\Omega |Du|\right)^p dx\right) dt\right)^{1/p}$$
$$= \|Du\|_{L^\infty(Q_T)}\,\|b_i\|_{L^p(Q_T)}.$$

Let $\mathbf{p} > (\mathbf{n} + \mathbf{2})/\mathbf{2}$.

$$\left(\int_0^T \left(\int_\Omega |cu|^p\,dx\right) dt\right)^{1/p} \le \left(\int_0^T \left(\int_\Omega \left(\sup_\Omega |u|\right)^p |c|^p\,dx\right) dt\right)^{1/p}$$
$$= \|u\|_{L^\infty(Q_T)}\,\|c\|_{L^p(Q_T)}.$$

Let $\mathbf{p} = \mathbf{n} + \mathbf{2}$.

$$\left(\int_\Omega |b_i Du|^p \, dx\right) \le \left[\left(\int_\Omega |Du|^\rho \, dx\right)^{1/\rho}\right]^p \cdot \left[\left(\int_\Omega |b_i|^{py} \, dx\right)^{1/py}\right]^p,$$

where $1/(\rho/p) + 1/y = 1$ or equivalently $y = \rho/(\rho - p)$, $\forall\, \rho \ge 1$, such that $\rho > p$, then

$$\int_0^T \left(\int_\Omega |b_i Du|^p \, dx\right) dt \le \int_0^T \left(\int_\Omega |Du|^\rho \, dx\right)^{p/\rho} \cdot \left(\left(\int_\Omega |b_i|^{py} \, dx\right)^{1/py}\right)^p dt$$

$$\left(\int_0^T \int_\Omega |b_i Du|^p \, dx\, dt\right)^{1/p} \le \|Du\|_{L^{\rho,\infty}(Q_T)} \|b_i\|_{L^{p\rho/(\rho-p),p}(Q_T)}.$$

Also, we have

$$\left(\int_\Omega |b_i Du|^p \, dx\right) \le \left(\sup_\Omega |Du|\right)^p \cdot \left(\int_\Omega |b_i|^p \, dx\right),$$

integrating between zero and T, we obtain

$$\int_0^T \left(\int_\Omega |b_i Du|^p \, dx\right) dt \le \left(\int_0^T \left(\sup_\Omega |Du|\right)^\rho dt\right)^{p/\rho} \cdot \left[\left(\int_0^T \left(\int_\Omega |b_i|^p \, dx\right)^{y/p} dt\right)^{1/y}\right]^p,$$

with $1/(\rho/p) + 1/(y/p) = 1$, or equivalently, $y = p\rho/(\rho - p)$.

Then,

$$\left[\int_0^T \left(\int_\Omega |b_i Du|^p \, dx\right) dt\right]^{1/p} \le \|Du\|_{L^{\infty,\rho}(Q_T)} \|b_i\|_{L^{p,p\rho/(\rho-p)}(Q_T)}, \qquad \rho > p.$$

Let $\mathbf{p} = (n+2)/2$.

We obtain

$$\left(\int_\Omega |cu|^p \, dx\right) \le \left[\left(\int_\Omega |u|^\rho \, dx\right)^{1/\rho}\right]^p \cdot \left[\left(\int_\Omega |c|^{py} \, dx\right)^{1/py}\right]^p,$$

with $y = \rho/(\rho - p)$; then

$$\int_0^T \left(\int_\Omega |cu|^p \, dx\right) dt \le \int_0^T \left[\int_\Omega |u|^\rho \, dx\right]^{p/\rho} \cdot \left[\left(\int_\Omega |c|^{py} \, dx\right)^{1/py}\right]^p dt.$$

It follows

$$\left[\int_0^T \left(\int_\Omega |cu|^p \, dx\right) dt\right]^{1/p} \le \|u\|_{L^{\rho,\infty}(Q_T)} \|c\|_{L^{p\rho/(\rho-p),p}(Q_T)}.$$

Moreover, in this case, we have

$$\left(\int_\Omega |cu|^p \, dx\right) \le \left(\sup_\Omega |u|\right)^p \left(\int_\Omega |c|^p \, dx\right) = \|u\|^p_{L^\infty(Q_T)} \left(\int_\Omega |c|^p \, dx\right)$$

integrating between zero and T we obtain

$$\int_0^T \left(\int_\Omega |cu|^p \, dx\right) dt \le \left(\int_0^T \|u\|^\rho_{L^\infty(Q_T)} \, dt\right)^{p/\rho} \left(\int_0^T \left(\int_\Omega |c|^p \, dx\right)^{y/p} dt\right)^{(1/y)p},$$

with $\rho > p$ and $y = p\rho/(\rho - p)$.

It implies

$$\left(\int_0^T \left(\int_\Omega |cu|^p\, dx\right)^p dt\right)^{1/p} \le \left(\int_0^T \|u\|^\rho_{L^\infty(Q_T)}\, dt\right)^{1/\rho} \|c\|_{L^{p,p\rho/(\rho-p)}(Q_T)}$$
$$= \|u\|_{L^{\infty,\rho}(Q_T)} \|c\|_{L^{p,p\rho/(\rho-p)}(Q_T)}\,.$$

Let $\mathbf{p} < n+2$, $\rho = np/(n-p+2\theta)$, $\max(0;(p-n)/2) \le \theta \le \min(1;p/2)$.
Then,

$$\int_\Omega |b_i Du|^p\, dx \le \left(\left(\int_\Omega |b_i|^{py}\, dx\right)^{1/py}\right)^p \left(\left(\int_\Omega |Du|^\rho\, dx\right)^{1/\rho}\right)^p,$$

where $y = \rho/(\rho-p)$, so that

$$\int_0^T \left(\int_\Omega |b_i Du|^p\, dx\right) dt \le \int_0^T \left(\|b_i\|^p_{L^{p\rho/(\rho-p)}(Q_T)} \left(\int_\Omega |Du|^\rho\, dx\right)^{p/\rho}\right) dt$$
$$\le \left(\int_0^T \|b\|^{p/\theta}_{L^{p\rho/(\rho-p)}(Q_T)}\, dt\right)^\theta \left(\int_0^T \|Du\|^{p/(1-\theta)}_{L^\rho(Q_T)}\, dt\right)^{1-\theta}$$
$$= \|b\|^p_{L^{p\rho/(\rho-p),(p/\theta)}(Q_T)} \|Du\|^p_{L^{\rho,p/(1-\theta)}(Q_T)}\,.$$

Let $\mathbf{p} < (\mathbf{n}+\mathbf{2})/\mathbf{2}$, $\rho = np/(n-p+2\theta)$, $\max(0;(2p-n)/2) \le \theta \le \min(1;p)$, we have

$$\left(\int_\Omega |cu|^p\, dx\right) \le \left(\int_\Omega |c|^{py}\, dx\right)^{1/(py)p} \left(\left(\int_\Omega |u|^{p(\rho/p)}\, dx\right)^{1/\rho}\right)^p,$$

where y is such that $1/(\rho/p) + 1/y = 1$, and then

$$\int_0^T \left(\int_\Omega |cu|^p\, dx\right) dt \le \int_0^T \left(\|c\|^p_{L^{p\rho/(\rho-p)}(Q_T)} \left(\int_\Omega |u|^\rho\, dx\right)^{p/\rho}\right) dt$$
$$= \left(\int_0^T \|c\|^{p/\theta}_{L^{p\rho/(\rho-p)}(Q_T)}\, dt\right)^\theta \left(\int_0^T \|u\|^{p/(1-\theta)}_{L^\rho(Q_T)}\, dt\right)^{1-\theta}$$
$$= \|c\|^p_{L^{p\rho/(\rho-p),(p/\theta)}(Q_T)} \|u\|^p_{L^{\rho,p/(1-\theta)}(Q_T)}\,.$$

THEOREM 3.3. *Let $f \in L^p(Q_T)$, $1 < p < \infty$, the coefficients b_i, c satisfying the above assumptions and $a_{ij} \in VMO \cap L^\infty(Q_T)$ are symmetric and uniformly elliptic.*

Then, there exists a constant k such that for any u solution of the Cauchy-Dirichlet problem (2.1), we have

$$\|u\|_{W^{2,1}_p(Q_T)} \le k\left(\|\mathcal{L}u\|_{L^p(Q_T)} + \|u\|_{L^p(Q_T)}\right),$$

where the constant k depends on $n; p; \tau; \eta; |\Omega|; \partial\Omega; T;$ $\|b_i\|_{L^{t,\bar t}}$, $\forall\, i = 1,\dots,n$; $\|c\|_{L^{s,\bar s}}$ and the AC moduli of b_i and c.

PROOF. We start by observing that for the above Cauchy-Dirichlet problem Bramanti and Cerutti [9, Theorem 4.3] have proved the estimate,

$$\|u\|_{W^{2,1}_p(Q_T)} \le C \left\|\mathcal{L}u - \sum_{i=1}^n b_i D^i u - cu\right\|_{L^p(Q_T)}.$$

Therefore, we obtain

$$\|u\|_{W^{2,1}_p(Q_T)} \le C\left\{\|\mathcal{L}u\|_{L^p(Q_T)} + \left\|\sum_{i=1}^n b_i D^i u\right\|_{L^p(Q_T)} + \|cu\|_{L^p(Q_T)}\right\}$$
$$\le C\left\{\|\mathcal{L}u\|_{L^p(Q_T)} + \sum_{i=1}^n \left\|b_i D^i u\right\|_{L^p(Q_T)} + \|cu\|_{L^p(Q_T)}\right\}.$$

Let us define

$$Q(L)=\{(x,t)\in Q_T:|b_i(x,t)|\geq L\},$$

then

$$\left(\int_{Q_T}|b_i|^p|D_iu|^p\,dx\,dt\right)^{1/p}\leq L\left(\int_{Q_T\setminus Q(L)}|D_iu|^p\,dx\,dt\right)^{1/p}+\left(\int_{Q(L)}|b_i|^p|D_iu|^p\,dx\,dt\right)^{1/p}.$$

Using Lemma 3.2 and Sobolev inequality, we obtain

$$\left(\int_{Q_T}|b_i|^p|D_iu|^p\,dx\,dt\right)^{1/p}\leq L\left(\int_{Q_T\setminus Q(L)}|D_iu|^p\,dx\,dt\right)^{1/p}+c\,\|b_i\|_{L^{t,\bar t}(Q(L))}\cdot\|u\|_{W_p^{2,1}(Q_T)},$$

where the constant c depends on the Sobolev constant.
It is also known that

$$\|D_iu\|_{L^p(Q_T)}\leq\epsilon\left\|D_i^2u\right\|_{L^p(Q_T)}+c(\epsilon)\|u\|_{L^p(Q_T)}.$$

Hence, we obtain

$$\int_{Q_T}|b_i|^p|D_iu|^p\,dx\,dt\leq L\left(\epsilon\|u\|_{W_p^{2,1}(Q_T)}+c(\epsilon)\|u\|_{L^p(Q_T)}\right)+\|b_i\|_{L^{t,\bar t}(Q(L))}\cdot\|u\|_{W_p^{2,1}(Q_T)}.$$

In the following, we will have to estimate the last term of the above inequality.
We have that

$$\int_0^T\int_{\{(x,t):|b_i(x,t)|>L\}}|b_i(x,t)|\,dx\,dt\geq L\int_0^T dt\int_{\{(x,t):|b_i(x,t)|>L\}}dx=L\cdot|Q(L)|.$$

Moreover, using Hölder inequality

$$\begin{aligned}
&\int_0^T\int_{\{(x,t):|b_i(x,t)|>L\}}|b_i(x,t)|\,dx\,dt\\
&\leq\int_0^T dt\left(\int_\Omega|b_i(x,t)|\,dx\right)\\
&\leq\int_0^T dt\left(\int_\Omega|b_i(x,t)|^t\,dx\right)^{1/t}\cdot\left(\int_\Omega dx\right)^{1-(1/t)}\\
&\leq\int_0^T(|\Omega|)^{1-(1/t)}\cdot\left(\int_\Omega|b_i(x,t)|^t\,dx\right)^{1/t}dt\\
&\leq(|\Omega|)^{1-(1/t)}\cdot\left(\int_0^T\left(\int_\Omega|b_i(x,t)|^t\,dx\right)^{(1/t)\cdot\bar t}dt\right)^{(1/\bar t)}\cdot\left(\int_0^T dt\right)^{1-(1/\bar t)}\\
&=(|\Omega|)^{1-(1/t)}\cdot\|b_i\|_{L^{t,\bar t}(Q_T)}T^{1-(1/\bar t)}.
\end{aligned}$$

From inequality (3.6) and the last one, we obtain

$$|Q(L)|\leq\frac{1}{L}(|\Omega|)^{1-(1/t)}\cdot\|b_i\|_{L^{t,\bar t}(Q_T)}T^{1-(1/\bar t)}.$$

Let L be large enough and let us assume

$$\sigma=\frac{1}{L}(|\Omega|)^{1-(1/t)}\cdot\|b_i\|_{L^{t,\bar t}(Q)}T^{1-(1/\bar t)}.$$

We know (see [13]) that if $f \in L^{p,p_1}(Q)$ $\forall \epsilon > 0$ $\exists \sigma > 0 : \forall E$ subset of $Q : |E| < \sigma$

$$\left(\int_0^T \left(\int_{E(t)} |f(x,t)|^p \, dx \right)^{p_1/p} dt \right)^{1/p_1} < \epsilon,$$

where $E(t)$ is a section of E at level t.

Then, if we choose $f(x,t) = b_i(x,t)$, we have

$$\|b_i\|_{L^{t,\bar{t}}(Q(L))} < \epsilon$$

and the requested inequality is obtained.

Let us now consider $\|cu\|_{L^p(Q_T)}$.

We have that

$$\|cu\|_{L^p(Q_T)} \leq L \cdot \|u\|_{L^p(Q_T)} + \|u\|_{L^{s,\bar{s}}(Q_T)} \cdot \|u\|_{W_p^{2,1}(Q_T)},$$

then from (3.6) on, similar arguments allows us to get to the conclusion.

THEOREM 3.4. *Let $f \in L^p(Q_T)$, $1 < p < \infty$, the coefficients a_{ij}, b_i, c satisfying the above assumptions.*

Then, the Cauchy-Dirichlet problem (2.1) has a unique solution $u \in W_p^{2,1}(Q_T)$ and there exists a constant C_0 such that

$$\|u\|_{W_p^{2,1}(Q_T)} \leq C_0 \|f\|_{L^p(Q_T)}.$$

The constant C_0 depends on n; p; τ; η; $|\Omega|$; $\partial\Omega$; T; $\|b_i\|_{L^{t,\bar{t}}}$, $\forall i = 1, \dots, n$; $\|c\|_{L^{s,\bar{s}}}$ and the AC moduli of b_i and c.

PROOF. Using inequality (3.5) let us prove the *uniqueness*.

Let u be a solution of the above Cauchy-Dirichlet problem with known term $f \equiv 0$.

Let us suppose $u \in \mathcal{C}$. Then, we set $u(x,t) = \int_0^t u_\zeta(x,\zeta) d\zeta$.

It follows $\|u\|_p \leq T \cdot \|u_t\|_p$, which, using (3.5), implies that there exists $\bar{T} > 0$, such that if $T \leq \bar{T}$

$$\|u\|_{W_p^{2,1}(Q_T)} \leq k \|\mathcal{L}u\|_{L^p(Q_T)}.$$

The above estimate is obtained, by density argument, for $u \in W_{0,p}(Q_T)$.

Let us also break up Q_T into a finite number of cylinders $Q_i = \Omega \times [T_i; T_{i+1}]$, such that $|T_{i+1} - T_i| \leq \bar{T}$. Applying the last inequality to each Q_i we have that $u \equiv 0$.

The *estimate* (3.7) is proved by contradiction.

Let us suppose that estimate (3.7) is not true, then there exists a sequence,

$$\mathcal{L}^{(k)} = \frac{\partial}{\partial t} - \sum_{i,j=1}^n a_{ij}^{(k)}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i^{(k)}(x) \frac{\partial}{\partial x_i} + c^{(k)},$$

such that the hypotheses for a_{ij}, b_i, c are true, such that the L^∞ norms of $a_{ij}^{(k)}$, $k \in N$, are uniformly bounded, the $L^{t,\bar{t}}(Q_T)$ and $L^{s,\bar{s}}(Q_T)$ norms respectively of $b_i^{(k)}, c^{(k)}$, $k \in N$, are uniformly bounded. Let us also suppose that there exists a sequence of functions $\{u^{(k)}\}, u^{(k)} \in W_0(Q_T)$,

$$\left\| u^{(k)} \right\|_{W_p^{2,1}(Q_T)} = 1, \qquad \lim_{k\to\infty} \left\| L^{(k)} u^{(k)} \right\|_{L^p(Q_T)} = 0.$$

As proved in [14, Theorem 4.4], it is possible to find a subsequence of $\{a_{ij}^{(k)}\}$ and we call it again $\{a_{ij}^{(k)}\}$, such that it converges a.e. in R^{n+1} to a function α_{ij} satisfying the above hypothesis of a_{ij}.

It is also possible to find two subsequences of $\{b_i^{(k)}\}$ $\{c^{(k)}\}$ weakly converging, respectively to $\beta_i \in L^{t,\bar{t}}(Q_T)$ and $\gamma \in L^{s,\bar{s}}(Q_T)$. Let us define

$$\mathcal{L}^{(\tau)} = \frac{\partial}{\partial t} - \sum_{i,j=1}^{n} \alpha_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,=1}^{n} \beta_i(x) \frac{\partial}{\partial x_i} + \gamma.$$

Moreover there exists a subsequence $\{u^{(k)}\}$ weakly convergent to a function $u^{(\tau)} \in W_{0,p}(Q_T)$. It follows that $\{\|u^{(k)}\|_{L^p(Q_T)}\}$ converges to $\|u^{(\tau)}\|_{L^p(Q_T)}$. Let us suppose $1 < p < n+2$, $z \in L^{p'}(Q_T)$, $p' = p/(p-1)$, then

$$\begin{aligned}
&\int_{Q_T} \left| \left(\mathcal{L}^{(k)} u^{(k)} - \mathcal{L}^{(\tau)} u^{(\tau)} \right) z \right| \, dx\, dt \\
&\leq \int_{Q_T} \left| \sum_{i,j=1}^{n} \left(u_{x_i x_j}^{(k)} - u_t^{(k)} - u_{x_i x_j}^{(\tau)} + u_t^{(\tau)} \right) \alpha_{ij} z \right| \, dx\, dt \\
&+ \sum_{i,j=1}^{n} \left\| u_{x_i x_j}^{(k)} - u_t^{(k)} \right\|_{L^p(Q_T)} \cdot \left\| \left(a_{ij}^{(k)} - \alpha_{ij} \right) z \right\|_{L^{p'}(Q_T)} \\
&+ \sum_{i=1}^{n} \left\| u_{x_i}^{(k)} - u_{x_i}^{(\tau)} \right\|_{L^q(Q_T)} \cdot \|\beta_i z\|_{L^{q'}(Q_T)} \\
&+ \sum_{i=1}^{n} \left\| b_i^{(k)} - \beta_i \right\|_{L^t(Q_T)} \cdot \|z\|_{L^{p'}(Q_T)} \cdot \left\| u_{x_i}^{(k)} - u_{x_i}^{(\tau)} \right\|_{L^q(Q_T)} \\
&+ \sum_{i=1}^{n} \int_{Q_T} \left| u_{x_i}^{(\tau)} z \left(b_i^{(k)} - \beta_i \right) \right| \, dx\, dt \\
&+ \left\| c^{(k)} - \gamma \right\|_{L^n(Q_T)} \cdot \|z\|_{L^{p'}(Q_T)} \cdot \left\| u^{(k)} - u^{(\tau)} \right\|_{L^t(Q_T)} \\
&+ \left\| u^{(k)} - u^{(\tau)} \right\|_{L^t(Q_T)} \cdot \|\gamma \cdot z\|_{L^{t'}(Q_T)} + \int_{Q_T} \left| \left(c^{(k)} - \gamma \right) u^{(\tau)} z \right| \, dx\, dt,
\end{aligned}$$

with $1/q' + 1/q = 1$, $1 < q < ((n+2)p)/((n+2)-p)$, $1 < t < ((n+2)p)/((n+2)-2p)$, $1/t' + 1/t = 1$. Since $\{\mathcal{L}^{(k)} u^{(k)}\}$ is weakly convergent to $\mathcal{L}^{(\tau)} u^{(\tau)}$ in $L^p(Q_T)$ we have that $\mathcal{L}^{(\tau)} u^{(\tau)} = 0$ a.e. in Q_T and by the uniqueness it follows $u^{(\tau)} = 0$. Moreover $\|u^{(k)}\|_{L^p(Q_T)}$ converging to zero and (3.5) contradict $\|u^{(k)}\|_{W_p^{2,1}(Q_T)} = 1$. The cases $p = n+2$ and $p > n+2$ are similar.

The *existence* will follow by a standard approximation argument using smooth coefficients.

REFERENCES

1. D. Sarason, On functions of vanishing mean oscillation, *Trans. Amer. Math. Soc.* **207**, 391–405, (1975).
2. S. Campanato, Equazioni paraboliche del secondo ordine e spazi $\mathcal{L}^{2,\theta}(\Omega,\delta)$, *Annali di Matematica* **73** (4), 55–102, (1966).
3. F. Guglielmino, Sulle equazioni paraboliche del secondo ordine di tipo non variazionale, *Annali di Matematica* **4** (65), 127–150, (1964).
4. E. Gagliardo, Problema al contorno per equazioni differenziali lineari di tipo parabolico in n variabili, *Ricerche di Matematica* **5**, 169–205, (1956).
5. E. Gagliardo, Teoremi di esistenza e di unicita' per problemi al contorno relativi ad equazioni paraboliche lineari e quasilineari in n variabili, *Ricerche di Matematica* **5**, 239–257, (1956).
6. O.A. Ladyženskaya and N.N. Ural'ceva, A boundary value problem for linear and quasilinear parabolic equations, *Soviet Math.* **2**, 969–972, (1961).
7. O. Arena, Sul problema di Cauchy-Dirichlet per le equazioni paraboliche a coefficienti discontinui, *Le Matematiche* **XXII**, 160–181, (1967).
8. O. Arena, Sopra una classe di equazioni paraboliche, *Boll. U.M.I.* **4**, 127–151, (1969).
9. M. Bramanti and M.C. Cerutti, $W_p^{1,2}$ solvability for the Cauchy-Dirichlet problem for parabolic equations with VMO coefficients, *Comm. in Partial Differential Equations* **18** (9/10), 1735–1763, (1993).

10. F. John and L. Nirenberg, On functions of bounded mean oscillation, *Commun. Pure Appl. Math.* **14**, 415–426, (1961).
11. M.A. Ragusa, Regularity of solutions of divergence form elliptic equations, *Proc. Amer. Math. Soc.* **128**, 533–540, (1999).
12. O. Arena and A. Maugeri, Perturbazioni di operatori parabolici di ordine $2n$ con termini di ordine inferiore, *Boll. U.M.I.* **9** (4), 169–184, (1974).
13. F. Guglielmino, Nuovi contributi allo studio delle equazioni paraboliche del secondo ordine di tipo non variazionale, *Ricerche di Matematica* **XIV**, 124–144, (1965).
14. F. Chiarenza, M. Frasca and P. Longo, $W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients, *Trans. Amer. Math. Soc.* **336**, 841–853, (1993).