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MULTIPLICITY RESULTS FOR SOME ELLIPTIC PROBLEMS OF n-LAPLACE TYPE

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Abstract. By using a recent result of Ricceri, we prove the existence of multiple solutions for perturbed n-Laplacian equations with Dirichlet boundary conditions. The Trudinger Moser inequality allows us to deal with perturbations with exponential growth.

1. Introduction

In the present paper we deal with an n-Laplace problem of the following type:

$$(P) \qquad \begin{cases} -\Delta_n u = f(x,u) & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary ($n \geq 2$), $\Delta_n u = div\left(|\nabla u|^{n-2}\,\nabla u\right)$ is the n-Laplacian, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

According to the definition given by Adimurthi in [1] and [2] (see also [3]), we say that f has a *subcritical growth* if, for every $\delta > 0$,

$$\lim_{|t|\to\infty} \sup_{x\in\Omega} \frac{|f(x,t)|}{e^{\delta|t|^{\frac{n}{n-1}}}} = 0. \tag{1.1}$$

Such a notion of criticality is motivated by the Trudinger Moser inequality after the celebrated papers [9] and [6] (see Section 2).

In the present paper we will assume that the nonlinearity f belongs to the class $\mathcal{A}$ of the Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ having a subcritical growth and satisfying the following condition:

$$\text{for every } \; M > 0, \quad \sup_{|t|\leq M} |f(x,t)| \in L^\infty(\Omega). \tag{1.2}$$

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Problems of the above type arise for instance in conformal geometry. When $n = 2$, the *prescribed Gaussian curvature equation* is a subcritical equation on a two dimensional manifold M with metric tensor g of the type

$$-\Delta u = k(x)e^{2u} + h(x)$$

where Δ is the Laplace-Beltrami operator associated to the metric g, k is the Gaussian curvature and h a given Hölder function.

The solvability of nonlinear boundary value problems in the presence of an exponential nonlinearity has been considered by several authors with the purpose to generalize to a wider class of nonlinearities, classical results from the critical point theory.

In [3], de Figueiredo, Myiagaki and Ruf obtained a non trivial solution for problem (P) in the subcritical case both when the nonlinearity has a local minimum at zero and when the nonlinearity has a saddle point at zero. Similar results have been proved by do O' in [4] for a more general class of nonlinearities and operators of elliptic type by the means of a min-max approach. In these results the main difficulty consists in verifying the conditions which allow to employ the classical critical points theory, such as the Palais Smale condition. Existence results of positive radial solutions have been obtained also by Adimurthi in [1] for a semilinear Dirichlet problem with subcritical growth in the unit disc of $\mathbb{R}^2$.

By using topological and variational argument, in [11], Calanchi, Ruf and Zhang, achieved the existence of two solutions (one negative and the other one sign changing) for a semilinear elliptic problem in bounded domains of $\mathbb{R}^2$ with a nonlinearity having an exponential growth at $+\infty$ and a linear growth at $-\infty$. A more precise conclusion for the same problem has been reached in [7] by Mugnai, who proved that a semilinear Dirichlet problem in the presence of subcritical exponential nonlinearities has, near resonance, four solutions (one negative and other three which change sign).

In the present paper we will prove the existence of three weak solutions for the perturbed problem

$$(P_{\lambda,\mu}) \qquad \begin{cases} -\Delta_n u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f and g belong to the class $\mathcal{A}$, λ and μ are non negative parameters. As usual, a weak solution of problem $(P_{\lambda,\mu})$ is any $u \in W_0^{1,n}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{n-2}\nabla u(x)\nabla v(x)\,dx = \int_\Omega (\lambda f(x,u(x)) + \mu g(x,u(x)))v(x)\,dx$$

for every $v \in W_0^{1,n}(\Omega)$.

Our main result is the following

Theorem 1. *Let $f \in \mathcal{A}$ be such that*

$$\max\left\{\limsup_{t\to 0}\frac{\sup_{x\in\Omega}F(x,t)}{|t|^n},\limsup_{|t|\to+\infty}\frac{\sup_{x\in\Omega}F(x,t)}{|t|^n}\right\}\le 0 \tag{1.3}$$

and

$$\sup_{u\in W_0^{1,n}(\Omega)}\int_\Omega F(x,u(x))\,dx>0 \tag{1.4}$$

where $F(x,t)=\int_0^t f(x,s)\,ds$.
Then, if we set

$$\theta=\frac{1}{n}\inf\left\{\frac{\int_\Omega|\nabla u(x)|^n\,dx}{\int_\Omega F(x,u(x))\,dx}: u\in W_0^{1,n}(\Omega),\int_\Omega F(x,u(x))\,dx>0\right\}, \tag{1.5}$$

for each compact interval $[a,b]\subset\,]\theta,+\infty[$, there exists $r>0$ with the following property: for every $\lambda\in[a,b]$, and every function $g\in\mathcal{A}$, there exists $\sigma>0$ such that for each $\mu\in[0,\sigma]$, the problem

$$(P_{\lambda,\mu})\qquad\begin{cases}-\Delta_n u=\lambda f(x,u)+\mu g(x,u) & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

has at least three weak solutions in $W_0^{1,n}(\Omega)$ whose norms are less than r.

The above result is a new application of a recent three critical points theorem by Ricceri ([8], see Section below). In [8], an application to a p-Laplacian problem of $(P_{\lambda,\mu})$ type is given for any p. In the case when $p=n$, the nonlinearities f and g are assumed to have the classical polynomial growth, that is

$$|f(x,s)|,|g(x,s)|\le c(1+|s|^q),\quad q>0.$$

In our result a wider class of nonlinearities is allowed.

2. Variational Setting

Our main tool is a recent three critical points theorem by Ricceri. Let us first introduce the following notation. If X is a real Banach space, we denote by $\mathcal{W}_X$ the class of functionals $\Phi: X\to\mathbb{R}$, having the following property: if $\{u_n\}$ is a sequence in X converging weakly to $u\in X$ and verifying $\liminf_{n\to\infty}\Phi(u_n)\le\Phi(u)$, then $\{u_n\}$ has a subsequence strongly converging to u.

Theorem 2. ([8], Theorem 2). *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$, a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X, and whose derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ a C^1 functional with compact derivative. Assume that Φ has a strict local minimum u_0 with $\Phi(u_0) = J(u_0) = 0$. Finally, setting*

$$\alpha = \max\left\{0, \limsup_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\},$$
$$\beta = \sup_{u\in\Phi^{-1}(]0,+\infty[)} \frac{J(u)}{\Phi(u)},$$

assume that $\alpha < \beta$.
Then, for each compact interval $[a,b] \subset \left]\frac{1}{\beta}, \frac{1}{\alpha}\right[$ (with the conventions $\frac{1}{0} = +\infty$, $\frac{1}{\infty} = 0$) there exists $r > 0$ with the following property: for every $\lambda \in [a,b]$, and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\sigma > 0$ such that for each $\mu \in [0,\sigma]$, the equation

$$\Phi'(u) = \lambda J'(u) + \mu\Psi'(u)$$

has at least three solutions in X whose norms are less than r.

We denote by X the space $W_0^{1,n}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^n \, dx\right)^{\frac{1}{n}}.$$

From the Rellich Kondrachov Theorem, X is continuously embedded in $L^s(\Omega)$ for every $s \geq 1$, that is, there exists a constant C_s' such that, if $\|u\|_s = \left(\int_\Omega |u(x)|^s \, dx\right)^{\frac{1}{s}}$,

$$\|u\|_s \leq C_s' \|u\|.$$

It is also well known (see the papers by Trudinger and Moser [6, 9]) that if $L_\phi(\Omega)$ is the Orlicz-Lebesgue space generated by the function $\phi(t) = \exp\left(|t|^{\frac{n}{n-1}}\right) - 1$, i.e.

$$L_\phi(\Omega) = \{u : \Omega \to \mathbb{R}, \text{ measurable } : \int_\Omega \phi(u(x))\, dx < \infty\},$$

and $L_{\phi^*}(\Omega)$ is the linear hull of $L_\phi(\Omega)$ equipped with the norm

$$\|u\|_{L_{\phi^*}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \phi\Big(\frac{u(x)}{\lambda}\Big) dx \leq 1\right\},$$

X is continuously embedded in $L_{\phi^*}(\Omega)$. Throughout the sequel we will use the following facts:

(TM_1) for every $u \in X$, for every $\delta > 0$,

$$\exp\left(\delta |u(\cdot)|^{\frac{n}{n-1}}\right) \in L^1(\Omega);$$

(TM_2) there exists a constant C_n depending on n and on the measure of Ω such that, for every $0 < \delta \leq \alpha_n$, where $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$, being ω_{n-1} the measure of the $(n-1)$ dimensional surface of the unit sphere in $\mathbb{R}^n$,

$$\sup_{\|u\|\leq 1} \int_\Omega \exp\left(\delta |u(x)|^{\frac{n}{n-1}}\right) dx \leq C_n.$$

Due to the above result, problem $(P_{\lambda,\mu})$ can be treated variationally in X, i.e. solutions of $(P_{\lambda,\mu})$ will be obtained as critical points of a suitable energy functional defined on X.

We present now some examples of functions belonging to $\mathcal{A}$. We drop, for simplicity, the dependence from x.

Examples

1. Let $j : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{|t|\to\infty} j(t) = 0$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = \exp(j(t)|t|^{\frac{n}{n-1}})$. Then, f belongs to $\mathcal{A}$. As a particular case, choose $q \in]0, \frac{n}{n-1}[$ and let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) = \exp(|t|^q)$.
2. Let $q > 0$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = |t|^q$. Then, f belongs to $\mathcal{A}$.

Now we state and prove the following lemma which will be used throughout the proof of Theorem 1.

Lemma 3. *If $h \in \mathcal{A}$, then the functional $\mathcal{H} : X \to \mathbb{R}$ defined by $\mathcal{H}(u) = \int_\Omega H(x, u(x))\, dx$, where $H(x,t) = \int_0^t h(x,s)\, ds$, is continuously differentiable in X with compact derivative and for all $u, v \in X$,*

$$\mathcal{H}'(u)(v) = \int_\Omega h(x, u(x)) v(x)\, dx.$$

Proof. From condition (1.1), for fixed $\delta > 0$, there exists $c > 0$ such that for every $x \in \Omega$ and $t \in \mathbb{R}$,

$$|h(x,t)| \leq c \exp(\delta |t|^{\frac{n}{n-1}}).$$

Then, for every $u \in X$, and almost every $x \in \Omega$,

$$|H(x, u(x))| \leq c|u(x)| \exp(\delta |u(x)|^{\frac{n}{n-1}})$$

and, by (TM_1) and Hölder inequality, we obtain that $\mathcal{H}$ is well defined on X.

Let us prove that $\mathcal{H}$ is Gâteaux differentiable with derivative given by

$$\mathcal{H}'(u)(v) = \int_\Omega h(x, u(x)) v(x)\, dx \tag{2.1}$$

for all $u, v \in X$.

Let $u, v \in X$. For $t \in]0, 1[$, we have

$$\frac{\mathcal{H}(u+tv)-\mathcal{H}(u)}{t}=\int_\Omega h(x,u(x)+t\tau(x)v(x))v(x)\,dx$$

where τ is a measurable function with values in $[0,1]$. Lebesgue Theorem implies that

$$\lim_{t\to 0^+}\frac{\mathcal{H}(u+tv)-\mathcal{H}(u)}{t}=\int_\Omega h(x,u(x))v(x)\,dx$$

which is claim (2.1).

Notice now that, if $\{u_k\}$ is a bounded sequence in X, then, for every $q>0$, one has

$$\sup_k\int_\Omega |h(x,u_k(x))|^q dx<\infty. \tag{2.2}$$

Indeed, let $M>0$ such that $\|u_k\|\le M$ for every $k\in\mathbb{N}$ and fix $q>0$. Choose then $\delta\in\left]0,\frac{\alpha_n}{qM^{\frac{n}{n-1}}}\right[$. So, there exists $c>0$ such that, for almost every $x\in\Omega$,

$$|h(x,u_k(x))|\le c\exp(\delta|u_k(x)|^{\frac{n}{n-1}})$$

and

$$\begin{aligned}\int_\Omega |h(x,u_k(x))|^q\,dx &\le c^q\int_\Omega \exp(q\delta|u_k(x)|^{\frac{n}{n-1}})\,dx\\ &\le c^q\int_\Omega \exp\left(q\delta M^{\frac{n}{n-1}}\left(\frac{|u_k(x)|}{\|u_k\|}\right)^{\frac{n}{n-1}}\right)dx\le c^qC_n,\end{aligned}$$

where C_n is as in (TM_2). Claim (2.2) is achieved.

Notice also that if $\{u_k\}$ is a bounded sequence in X, there exists $u\in X$ such that, up to a subsequence, $u_k\to u$ a.e. in Ω. Let us show that, for every $q>0$, one has

$$h(\cdot,u_k(\cdot))\to h(\cdot,u(\cdot))\ \text{ in } L^q(\Omega). \tag{2.3}$$

Indeed, $h(\cdot,u_k(\cdot))\to h(\cdot,u(\cdot))$ a.e. in Ω. Moreover, due to (2.2), for fixed $p>1$, there exists some positive constant c_1 such that, for all $k\in\mathbb{N}$,

$$\int_\Omega |h(x,u_k(x))|^{qp}\,dx\le c_1.$$

Let $\varepsilon>0$, and E be a measurable subset of Ω. By Hölder inequality, for every $k\in\mathbb{N}$, one has

$$\begin{aligned}\int_E |h(x,u_k(x))|^q\,dx &\le (\text{meas}(E))^{\frac{1}{p'}}\left(\int_\Omega |h(x,u_k(x))|^{qp}\,dx\right)^{\frac{1}{p}}\\ &\le c_1^{\frac{1}{p}}\,(\text{meas}(E))^{\frac{1}{p'}},\end{aligned}$$

where p' is the conjugate of p, that is $\frac{1}{p} + \frac{1}{p'} = 1$. Hence, if meas$(E)$ is sufficiently small, for every $k \in \mathbb{N}$

$$\int_E |h(x, u_k(x))|^q \, dx \leq \varepsilon,$$

and Vitali's convergence Theorem applies ensuring at once claim (2.3).

We are ready to prove now that $\mathcal{H}'$ is a continuous, compact map from X to its dual X^*.

Let $\{u_k\}$ be a sequence in X converging to u. Then, $\{u_k\}$ is bounded and $u_k \to u$ a.e. in Ω. We have, for $v \in X$ with $\|v\| \leq 1$,

$$\begin{aligned}
&|(\mathcal{H}'(u_k) - \mathcal{H}'(u))(v)| \\
&\leq \left(\int_\Omega |h(x, u_k(x)) - h(x, u(x))|^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \left(\int_\Omega |v(x)|^n \, dx \right)^{\frac{1}{n}} \\
&\leq C'_n \left(\int_\Omega |h(x, u_k(x)) - h(x, u(x))|^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}}
\end{aligned}$$

(here we have used the continuous embedding of X in $L^n(\Omega)$). Due to (2.3), the last hand side goes to zero as k tends to ∞ and it follows that

$$\mathcal{H}'(u_k) \to \mathcal{H}'(u) \qquad \text{in } X^*.$$

In a similar way it is possible to prove that $\mathcal{H}'$ is compact. Indeed, let $\{u_k\}$ be a bounded sequence in X. Then, there exists u in X such that $u_k \to u$ a.e. in Ω and (2.3) holds. As above, we get that $\mathcal{H}'(u_k) \to \mathcal{H}'(u)$ in X^*. ∎

3. Proof of the Main Result

In the present Section we apply Theorem 2 to the functionals Φ and J defined on X as follows

$$\Phi(u) = \frac{1}{n}\|u\|^n,$$

$$J(u) = \int_\Omega F(x, u(x)) \, dx$$

for every $u \in X$. It is well known that Φ is coercive, weakly lower semicontinuous, bounded on bounded subsets of X and it belongs to $\mathcal{W}_X$. Moreover Φ is continuously Gâteaux differentiable in X with derivative

$$\Phi'(u)(v) = \int_\Omega |\nabla u(x)|^{n-2} \nabla u(x) \nabla v(x) \, dx$$

for all $u, v \in X$ and Φ' admits a continuous inverse on X^* (see [10], Theorem 26. A).

In view of Lemma 3, J is well defined and continuously Gâteaux differentiable with compact derivative $J^{'}$ given by

$$J^{'}(u)(v) = \int_{\Omega} f(x, u(x))v(x)\, dx$$

for all $u, v \in X$.

Let us prove now that

$$\limsup_{u\to 0} \frac{J(u)}{\Phi(u)} \leq 0. \tag{3.1}$$

By the assumption (1.3),

$$\limsup_{t\to 0} \frac{\sup_{x\in\Omega} F(x,t)}{|t|^n} \leq 0,$$

and so for every $\varepsilon > 0$ there exists some positive ρ such that, for every $x \in \Omega$ and $|t| < \rho$,

$$F(x,t) < \varepsilon|t|^n.$$

As f belongs to $\mathcal{A}$, for fixed $\delta > 0$ and $q > n$ there exists $c > 0$ such that, for every $x \in \Omega$ and $|t| \geq \rho$

$$F(x,t) \leq c|t|^q \exp(\delta|t|^{\frac{n}{n-1}}).$$

Then, for every $x \in \Omega$ and $t \in \mathbb{R}$, one has

$$F(x,t) \leq \varepsilon|t|^n + c\exp(\delta|t|^{\frac{n}{n-1}})|t|^q.$$

After choosing $p > 1$, we apply Hölder's inequality to get

$$\int_{\Omega} \exp(\delta|u(x)|^{\frac{n}{n-1}})|u(x)|^q\, dx$$
$$\leq \left[\int_{\Omega} \exp\left(p\delta\|u\|^{\frac{n}{n-1}}\left(\frac{|u(x)|}{\|u\|}\right)^{\frac{n}{n-1}}\right) dx\right]^{\frac{1}{p}} \left(\int_{\Omega} |u(x)|^{p'q}\, dx\right)^{\frac{1}{p'}},$$

where p' is the conjugate of p.

By combining the previous inequality with (TM_2) and bearing in mind that X is continuously embedded in $L^s(\Omega)$ for every $s \geq 1$, for $\|u\| \leq \left(\frac{\alpha_n}{p\delta}\right)^{\frac{n-1}{n}}$, one has

$$J(u) \leq \varepsilon(C_n')^n\|u\|^n + c(C_{p'q}')^q C_n^{\frac{1}{p}}\|u\|^q.$$

Therefore,

$$\frac{J(u)}{\Phi(u)} \leq n(C_n')^n\varepsilon + cn(C_{p'q}')^q C_n^{\frac{1}{p}}\|u\|^{q-n}.$$

Since $q > n$, claim (3.1) immediately follows.

Let us prove now that

$$\limsup_{\|u\|\to\infty} \frac{J(u)}{\Phi(u)} \leq 0. \tag{3.2}$$

By the assumption (1.3),

$$\limsup_{|t|\to\infty} \sup_{x\in\Omega} \frac{F(x,t)}{|t|^n} \leq 0,$$

and so for every $\varepsilon > 0$, there exists some positive ρ such that, for every $x \in \Omega$ and $|t| > \rho$,

$$F(x,t) \leq \varepsilon |t|^n.$$

From condition (1.2), there exists some constant $c > 0$ such that, for every $x \in \Omega$,

$$\sup_{|t|\leq\rho} |f(x,t)| \leq c.$$

Then, for every $x \in \Omega$ and $t \in \mathbb{R}$,

$$F(x,t) \leq c\rho + \varepsilon |t|^n$$

and so

$$J(u) \leq c\rho \text{meas}(\Omega) + \varepsilon \int_\Omega |u(x)|^n dx.$$

Since X is continuously embedded into $L^n(\Omega)$, we get

$$\frac{J(u)}{\Phi(u)} \leq n \frac{c\rho \text{meas}(\Omega)}{\|u\|^n} + \varepsilon n C_n',$$

and claim (3.2) follows at once.

In view of (3.1) and (3.2), we get

$$\max\left\{ \limsup_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to 0} \frac{J(u)}{\Phi(u)} \right\} \leq 0.$$

and all the assumptions of Theorem 2 are satisfied with $\alpha = 0$ and $\beta = \frac{1}{\theta}$, where θ is as in (1.5). Choose $[a,b] \subseteq]\theta, +\infty[$ and $g \in \mathcal{A}$.

Notice that, if $G : \Omega \times \mathbb{R} \to \mathbb{R}$ is the function $G(\xi,t) = \int_0^t g(\xi,s)\,ds$, by Lemma 3, the functional $\Psi(u) = \int_\Omega G(x,u(x))\,dx$ is continuously Gâteaux differentiable in X, it has compact derivative, and for every $u, v \in X$

$$\Psi^{'}(u)(v) = \int_\Omega g(x,u(x))v(x)\,dx.$$

Then, Theorem 2 applies and there exists $r > 0$ such that for every $\lambda \in [a,b]$, it is possible to find $\sigma > 0$ verifying the following condition: for each $\mu \in [0,\sigma]$, the functional $\Phi - \lambda J - \mu\Psi$ has at least three critical points whose norms are less than r. It is clear that critical points of $\Phi - \lambda J - \mu\Psi$ are precisely weak solutions of problem $(P_{\lambda,\mu})$. The proof is concluded. ■

Remark 4. In order to guarantee assumption 1.4 in Theorem 1, one can require, for instance, the following condition:

there exist $\xi_0 \in \mathbb{R}$ and an open, non-empty set $\Omega_0 \subseteq \Omega$ such that $F(x, \xi_0) > 0$ for all $x \in \Omega_0$.

We give now an example of application of Theorem 1.

Example 5. Let $p \in]0, n[$. Then, there exists a positive number θ such that for each compact interval $[a, b] \subset]\theta, +\infty[$, there exists $r > 0$ with the following property: for every $\lambda \in [a, b]$, and every $q \in]0, \frac{n}{n-1}[$, there exists $\sigma > 0$ such that for each $\mu \in [0, \sigma]$, the problem

$$(P_{\lambda,\mu}) \qquad \begin{cases} -\Delta_n u = \lambda \dfrac{|u|^{p+n-2}u(p|u|^n + p + n)}{(1 + |u|^n)^2} + \mu \exp(|u|^q) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least three weak solutions in $W_0^{1,n}(\Omega)$ whose norms are less than r.

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