TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 3, pp. 1031-1037, June 2013
DOI: 10.11650/tjm.17.2013.2565
This paper is available online at http://journal.taiwanmathsoc.org.tw

A NOTE ON A RESULT OF MIRONESCU AND RADULESCU

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Abstract. In the present note we deal with a bifurcation problem involving an asymptotically linear function. By employing a recent result of Ricceri we make much more precise the conclusion of a theorem of Mironescu and Radulescu.

1. RESULTS

Let us consider the problem

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 2$), $f : [0, +\infty[\to \mathbb{R}$ is non-negative, convex and of class C^1 in $[0, +\infty[$, such that $f(0) > 0$ and $f'(0) > 0$, λ is a positive parameter. By a positive solution of (P_λ) we mean a function $u \in C^2(\Omega) \bigcap C(\bar{\Omega})$ which is positive in Ω, such that the equation $-\Delta u(x) = \lambda f(u(x))$ holds for every $x \in \Omega$ and $u(x) = 0$ for every $x \in \partial\Omega$ (i.e. a classical solution). It is well known (see the work of Brezis et al. [2]) that if f satisfies the above assumptions and is also asymptotically linear, i.e.

$$\lim_{t \to +\infty} \frac{f(t)}{t} = a \in (0, +\infty) \tag{1.1}$$

then,

(a_1) there exists $\lambda^\star > 0$ such that (P_λ) has a positive solution for $\lambda \in (0, \lambda^\star)$, no solution for $\lambda \in (\lambda^\star, +\infty)$;

(a_2) for any $\lambda \in (0, \lambda^\star)$, (P_λ) has a minimal positive solution u_λ (i.e. for every u solution of (P_λ), $u_\lambda(x) \leq u(x)$ for every $x \in \Omega$);

Received October 24, 2012, accepted December 9, 2012.
Communicated by Biagio Ricceri.
2010 *Mathematics Subject Classification*: 35J61, 35B32.
Key words and phrases: Semilinear elliptic equation, Bifurcation problem, Variational methods.

(a_3) the function $\lambda \to u_\lambda$ is a C^1, convex, increasing function;

(a_4) u_λ is the only solution u of (P_λ) such that the operator $-\Delta - \lambda f'(u)$ is coercive.

The problem of the uniqueness of the solution, as well as of the properties of u_λ for λ close to $\lambda^\star$, the existence or non-existence of solutions of (P_λ) for $\lambda = \lambda^\star$ were investigated by Mironescu and Radulescu (see [5] and [6]) who proved the following:

Theorem 1.1. ([5], Theorem A and [6], Theorem 1). *Assume that $f : [0, +\infty[\to \mathbb{R}$ is non-negative, convex and of class C^1 in $[0, +\infty[$, such that $f(0) > 0$, $f'(0) > 0$ and*

$$\lim_{t \to +\infty} (f(t) - at) = l \geq 0 \tag{1.2}$$

for some positive a. Then,

(b_1) *$\lambda^\star = \frac{\lambda_1}{a}$ (λ_1 being the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$);*

(b_2) *$\lim_{\lambda \to \lambda^\star} u_\lambda = +\infty$ uniformly on compact subsets of Ω;*

(b_3) *u_λ is the only solution of (P_λ) when $\lambda \in (0, \lambda^\star)$;*

(b_4) *$(P_{\lambda^\star})$ has no solution.*

See also [4] in connection with conclusion (b_4) and [1] and [3] for extension of the above theorem to the biharmonic equation and to p-Laplacian problems.

Remark 1.1. Notice that hypothesis (1.2) implies (1.1).

The purpose of this note is simply to point out that, under the same assumptions as those of Theorem 1.1, many further additional assertions, besides $(b_1) - (b_4)$, can be deduced by combining the conclusions of the above theorem with some recent results by Ricceri. More precisely, in our result, we will exhibit some extra properties of the function u_λ in connection with well posedness problems on spheres and some continuity properties of suitable mappings involving the energy functional related to problem (P_λ).

Let us introduce some notations and preliminary remarks. Put

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } s \geq 0 \\ f(0) & \text{if } s < 0 \end{cases},$$

and

$$F(t) = \int_0^t \tilde{f}(s) ds.$$

In the Sobolev space $H_0^1(\Omega)$, endowed with the norm $\|u\| = \left(\int_\Omega |\nabla u(x)|^2 dx\right)^{\frac{1}{2}}$, define the functional

$$J(u) = \int_\Omega F(u(x)) dx.$$

The equation appearing in (P_λ) is the Euler-Lagrange equation associated to the energy functional $\mathcal{E}_\lambda : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{E}_\lambda(u) = \frac{\|u\|^2}{2} - \lambda J(u)$$

which is continuously Gâteaux differentiable in $H_0^1(\Omega)$, with derivative at u given by

$$\mathcal{E}'_\lambda(u)v = \int_\Omega \nabla u(x)\nabla v(x)dx - \lambda \int_\Omega \tilde{f}(u(x))v(x)dx$$

for every $v \in H_0^1(\Omega)$. Since $\tilde{f}$ is non-negative, any critical point u of $\mathcal{E}_\lambda$ turns out to be non-negative, and so it is a *weak* solution of (P_λ). Actually, u is positive on Ω from the maximum principle and classical from bootstrap arguments.

Let

$$S_r = \{u \in H_0^1(\Omega) : \|u\|^2 = r\}.$$

Our main result is

Theorem 1.2. *Under the same assumptions as in Theorem 1.1, the following holds true:*

(c_1) *the function*

$$\lambda \to h(\lambda) := \|u_\lambda\|^2$$

is increasing in $(0, \lambda^\star)$ *and its range is* $(0, +\infty)$*;*

(c_2) *for each* $r > 0$*, the function* $u_{h^{-1}(r)}$ *is the unique global maximum of* $J_{|S_r}$ *and every maximizing sequence for* J *in* S_r *converges to it;*

(c_3) *the mapping* $r \to u_{h^{-1}(r)}$ *is continuous in* $(0, +\infty)$*;*

(c_4) *the mapping* $r \to J(u_{h^{-1}(r)})$ *is of class* C^1 *in* $(0, +\infty)$ *with derivative* $\frac{1}{2h^{-1}(r)}$*, in particular it is increasing and strictly concave in* $(0, +\infty)$.

The above result is an immediate consequence of an abstract theorem by Ricceri which gives an answer to a problem raised by Schechter and Tintarev in [13] about conditions providing uniqueness of maxima of integral functionals on spheres (see [12]). The quoted result of Mironescu and Radulescu turns out to be an essential tool in order to apply Ricceri's result.

Let X be an infinite dimensional real Hilbert space and $\tilde{J} : X \to \mathbb{R}$ a sequentially weakly continuous functional of class C^1 and such that $\tilde{J}(0) = 0$. Denote as above, by

$$S_r = \{u \in X : \|u\|^2 = r\},$$

$$\gamma(r) = \sup_{S_r} \tilde{J},$$

and

$$r^\star = \inf\{r > 0 : \gamma(r) > 0\}.$$

The following Theorem is a byproduct of previous results of Ricceri (see [7]-[11]):

Theorem 1.3. ([12], Theorem 1). *Set*

$$\rho = \limsup_{\|u\|\to+\infty} \frac{\tilde{J}(u)}{\|u\|^2}$$

and

$$\sigma = \sup_{u\in X\setminus\{0\}} \frac{\tilde{J}(u)}{\|u\|^2}.$$

Let ζ, η satisfy

$$\max\{0,\rho\} \leq \zeta < \eta \leq \sigma.$$

Assume that $\tilde{J}$ has no local maximum in $X \setminus \{0\}$ and that, for each $\mu \in (\zeta,\eta)$, the functional $u \to \mu\|u\|^2 - \tilde{J}(u)$ has a unique global minimum, say $\hat{y}_\mu$. Let M_ζ (resp. M_η if $\eta < +\infty$ or $M_\eta = \emptyset$ if $\eta = +\infty$) be the set of all global minima of the functional $u \to \zeta\|u\|^2 - \tilde{J}(u)$ (resp. $u \to \eta\|u\|^2 - \tilde{J}(u)$ if $\eta < +\infty$). Set

$$\alpha = \max\left\{0, \sup_{u\in M_\eta} \|u\|^2\right\}$$

and

$$\beta = \inf_{u\in M_\zeta} \|u\|^2.$$

Then, the following assertions hold:

(d_1) *one has $r^\star \leq \alpha < \beta$;*

(d_2) *the function $\mu \to g(\mu) := \|\hat{y}_\mu\|^2$ is decreasing in (ζ,η) and its range is (α,β);*

(d_3) *for each $r \in (\alpha,\beta)$, the point $\hat{x}_r := \hat{y}_{g^{-1}(r)}$ is the unique global maximum of $\tilde{J}|_{S_r}$ towards which every maximizing sequence in S_r converges;*

(d_4) *the function $r \to \hat{x}_r$ is continuous in (α,β);*

(d_5) *the function γ is of class C^1 with derivative $\gamma'(r) = g^{-1}(r)$ for all $r \in (\alpha,\beta)$, so in particular it is increasing and strictly concave in (α,β);*

(d_6) *one has $\tilde{J}'(\hat{x}_r) = 2g^{-1}(r)\hat{x}_r$ for all $r \in (\alpha,\beta)$.*

Proof of Theorem 1.2. We are going to apply the conclusions of Theorem 1.1 to Theorem 1.3 where we choose $X = H_0^1(\Omega)$ and $\tilde{J} = J$. From assumption (1.2), and the definition of $\tilde{f}$, J is sequentially weakly continuous, of class C^1 and satisfies $J(0) = \int_\Omega F(0)dx = 0$. We notice also that J has no local maxima in $H_0^1(\Omega)$. More precisely, J has no critical points in $H_0^1(\Omega)$. Indeed if u were a critical point of J, then, for every $v \in H_0^1(\Omega)$,

$$J'(u)v = \int_\Omega \tilde{f}(u(x))v(x)dx = 0,$$

which implies $\tilde{f}(u(x)) = 0$ a.e. in Ω, which, in view of the assumptions on f, can not be fulfilled by any $u \in H_0^1(\Omega)$.

From assumption (1.2), and from the definition of $\tilde{f}$ one has in particular that

$$\lim_{t\to+\infty} \frac{F(t)}{t^2} = \frac{a}{2}, \qquad \lim_{t\to-\infty} \frac{F(t)}{t^2} = 0$$

from which, it follows that

$$\limsup_{\|u\|\to\infty} \frac{J(u)}{\|u\|^2} \le \frac{a}{2\lambda_1}.$$

Hence, by using (b_1) of Theorem 1.1,

$$\limsup_{\|u\|\to\infty} \frac{J(u)}{\|u\|^2} \le \frac{1}{2\lambda^\star} \tag{1.3}$$

and so, we deduce also that the energy functional $\mathcal{E}_\lambda$ is coercive for $\lambda \in (0, \lambda^\star)$.

Moreover, since $f(0) > 0$, one has

$$\lim_{t\to 0^+} \frac{F(t)}{t^2} = +\infty$$

and so

$$\sup_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{J(u)}{\|u\|^2} = +\infty. \tag{1.4}$$

Hence, from (1.3) and (1.4),

$$\rho \le \frac{1}{2\lambda^\star} \quad \text{and} \quad \sigma = +\infty.$$

Choose then, in Theorem 1.3

$$\zeta = \frac{1}{2\lambda^\star} \quad \text{and} \quad \eta = +\infty.$$

We notice that the energy functional

$$\mathcal{F}_\mu(u) = \mu\|u\|^2 - J(u)$$

has a unique global minimum for every $\mu \in \left(\frac{1}{2\lambda^\star}, +\infty\right)$, say $\hat{y}_\mu$, if and only if the functional

$$\mathcal{E}_\lambda(u) = \frac{\|u\|^2}{2} - \lambda J(u)$$

has a unique global minimum for every $\lambda \in (0, \lambda^\star)$, that is the point

$$u_\lambda = \hat{y}_\mu \ \text{ with } \lambda = \frac{1}{2\mu}. \tag{1.5}$$

Let now $\lambda \in (0, \lambda^\star)$. From (b_3) of Theorem 1.1 (P_λ) has a unique solution, which, in view of the coercivity of $\mathcal{E}_\lambda$ in $H_0^1(\Omega)$, coincides with the global minimum u_λ. It has no solution for $\lambda = \lambda^\star$ (see (b_4) of Theorem 1.1). Then, if $\mu \in (\zeta, \eta)$, $\mathcal{F}_\mu$ has a unique global minimum, while the sets M_ζ and M_η are empty. Then, Theorem 1.3 applies with

$$\alpha = 0, \qquad \beta = +\infty.$$

So, if $h(\lambda) := \|u_\lambda\|^2$, then,

$$h(\lambda) = g\left(\frac{1}{2\lambda}\right)$$

(where g is from (d_2) of Theorem 1.3). Hence, h is increasing in $(0, \lambda^\star)$ with range equal to $(0, +\infty)$, so it is continuous and its inverse, still continuous and increasing satisfies

$$h^{-1}(r) = \frac{1}{2g^{-1}(r)}.$$

In our setting for every $r > 0$, one has also that $\hat{x}_r = u_{\frac{1}{2g^{-1}(r)}} = u_{h^{-1}(r)}$ (with $\hat{x}_r$ from (d_3) of Theorem 1.3) is the unique global maximum of $J|_{S_r}$ towards which every maximizing sequence in S_r converges; by applying $(d_4) - (d_5)$ of Theorem 1.3 we get that the mapping $r \to u_{h^{-1}(r)}$ is continuous in $(0, +\infty)$ and that the mapping $r \to J(u_{h^{-1}(r)})$ is of class C^1 with derivative equal to $\frac{1}{2h^{-1}(r)}$. This concludes the proof. ■

We would like to conclude this note with some examples of nonlinearities complying with our assumptions.

Example 1.1.

1. Let $a > 0$ and $\alpha \in (0, 1)$ such that $a > -\ln \alpha$. Let $f(s) = \alpha^s + as$ for every $s \geq 0$;
2. $f(s) = (s + 1) \arctan(s + 1)$ for every $s \geq 0$;
3. $f(s) = \sqrt{s^2 + s + 1}$ for every $s \geq 0$.

ACKNOWLEDGMENTS

The author would like to thank the anonymous referee for his/her suggestions.

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