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Logarithmically improved regularity criterion for the nematic liquid crystal flows in $\dot{B}^{-1}_{\infty,\infty}$ space

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ABSTRACT

In this work, we study the regularity criterion of the three-dimensional nematic liquid crystal flows. It is proved that if the vorticity satisfies

$$\int_0^T \frac{\|\omega(t,\cdot)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1+\log(e+\|\omega(t,\cdot)\|_{\dot{B}^{-1}_{\infty,\infty}})}dt < \infty,$$

where $\dot{B}^{-1}_{\infty,\infty}$ denotes the critical Besov space, then the solution (u, d) becomes a regular solution on $(0, T]$. This result extends the recent regularity criterion obtained by Fan and Ozawa (2012) [11].

1. Introduction

This paper deals with the blow-up criterion for the hydrodynamic system modeling the flow of nematic liquid crystal materials in $\mathbb{R}^3$:

$$u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla\pi = -\lambda\nabla\cdot(\nabla d\odot\nabla d), \tag{1.1}$$

$$d_t + (u\cdot\nabla)d = \gamma(\Delta d - f(d)), \tag{1.2}$$

$$\nabla\cdot u = 0, \tag{1.3}$$

$$(u, d)(x, 0) = (u_0, d_0)(x). \tag{1.4}$$

Here $u = u(x, t)$ denotes the velocity field of the flow, the direction field $d = d(x, t)$ represents the orientation parameter of the liquid crystal and $\pi = \pi(x, t)$ is the pressure of the flow, while ν, λ, γ are positive physical constants. The term $\nabla d\odot\nabla d$ denotes the 3×3 matrix whose (i, j)-th element is given by $\partial_i d\cdot\partial_j d$ (for $1\le i, j\le 3$). In addition, $f(d) = \frac{1}{\eta^2}(|d|^2-1)d$ ($\eta > 0$ a constant) is a Ginzburg–Landau approximation function whose primitive function is clearly $F(d) = \frac{1}{4\eta^2}(|d|^2-1)^2$. For the sake of simplicity, we will take $\nu = \lambda = \gamma = \eta = 1$ since their sizes do not play any role in our analysis.

This model can be seen as a variant of the 3D Navier–Stokes equation, where the finite time blow-up for nematic liquid crystal flow being a very important problem. This model was introduced by Lin in [1]. It is a simplified version of the Ericksen–Leslie model (cf. [2–4]) of the liquid crystal flow. In [5], Lin and Liu proved local-in-time existence of classical solutions and global-in-time existence of weak solutions. Later in [6] they also considered regularity of weak solutions and proved that the one dimensional space time Hausdorff measure of the singular set of the "suitable" weak solutions is zero.

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When $d = 0$, the nematic liquid crystal flows reduces to the incompressible Navier–Stokes equations. For the Navier–Stokes equations, different criteria for regularity of the weak solutions have been proposed. In 2005, Montgomery-Smith [7] (see also [8–10] and references therein) showed that if

$$\int_0^T \frac{\|u(t)\|_{L^q}^{\alpha}}{1+\ln\left(e+\|u(t)\|_{L^q}\right)}dt < \infty \quad \text{with } \frac{2}{\alpha}+\frac{3}{q}=1,\ 3<q<\infty,$$

then u is regular. Note that the log improvement is here, in time only. This can be seen as a natural Gronwall type extension of the Prodi–Serrin conditions. Recently, Fan and Ozawa [11] obtained a finite time blow-up criterion, which says that the local smooth solution for initial data $(u_0, d_0) \in H^3(\mathbb{R}^3) \times H^4(\mathbb{R}^3)$, blows up at T if

$$\int_0^T \frac{\|\nabla u(t)\|_{\dot{B}^{-(s+1)}_{\infty,\infty}}^{\frac{2}{1-s}}}{1+\log(e+\|\nabla u(t)\|_{\dot{B}^{-(1+s)}_{\infty,\infty}})}dt = \infty \quad \text{with } 0<s<1.$$

In this work, we shall consider the critical Besov space $\dot{B}^{-1}_{\infty,\infty}$ $(s=0)$ and we concentrate on the blow-up criterion under the condition

$$\int_0^T \frac{\|\nabla u(t)\|_{\dot{B}^{-1}_{\infty,\infty}}^{2}}{1+\log(e+\|\nabla u(t)\|_{\dot{B}^{-1}_{\infty,\infty}})}dt < \infty.$$

2. Preliminaries and main result

We begin this section with some notations and lemmas used later. Let $e^{t\Delta}$ denote the heat semi-group defined by

$$e^{t\Delta}f = K_t * f, \qquad K_t(x) = (4\pi t)^{-\frac{3}{2}}\exp\left(-\frac{|x|^2}{4t}\right)$$

for $t > 0$ and $x \in \mathbb{R}^3$, where $*$ means convolution of functions defined on $\mathbb{R}^3$.

We now recall the definition of the homogeneous Besov space with negative indices $\dot{B}^{-\alpha}_{\infty,\infty}$ on $\mathbb{R}^3$ with $\alpha > 0$. It is known [12, p. 192] that $f \in \mathcal{S}'(\mathbb{R}^3)$ belongs to $\dot{B}^{-\alpha}_{\infty,\infty}(\mathbb{R}^3)$ if and only if $e^{t\Delta} \in L^\infty$ for all $t > 0$ and $t^{\frac{\alpha}{2}}\left\|e^{t\Delta}f\right\|_\infty \in L^\infty(0,\infty;L^\infty)$. The norm of $\dot{B}^{-\alpha}_{\infty,\infty}$ is defined, up to equivalence, by

$$\|f\|_{\dot{B}^{-\alpha}_{\infty,\infty}} = \sup_{t>0}\left(t^{\frac{\alpha}{2}}\left\|e^{t\Delta}f\right\|_\infty\right).$$

The following lemma is essentially due to Meyer–Gerard–Oru [13], which plays an important role for the proof of our theorem.

Lemma 2.1. *Let $1 < p < q < \infty$ and $s = \alpha\left(\frac{q}{p}-1\right) > 0$. Then there exists a constant depending only on α, p and q such that the estimate*

$$\|f\|_{L^q} \le C\left\|(-\Delta)^{\frac{s}{2}}f\right\|_{L^p}^{\frac{p}{q}}\|f\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{1-\frac{p}{q}} \tag{2.1}$$

holds for all $f \in \dot{H}^s_p(\mathbb{R}^3) \cap \dot{B}^{-\alpha}_{\infty,\infty}(\mathbb{R}^3)$, where $\dot{H}^s_p$ denotes the homogeneous Sobolev space.

In particular, for $s = 1$, $p = 2$ and $q = 4$, we get $\alpha = 1$ and

$$\|f\|_{L^4} \le C\|f\|_{\dot{H}^1}^{\frac{1}{2}}\|f\|_{\dot{B}^{-1}_{\infty,\infty}}^{\frac{1}{2}} \tag{2.2}$$

for all $f \in \dot{H}^1(\mathbb{R}^3) \cap \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$.

Let us recall by Biot–Savart law, for the solenoidal vectors u, the following representation:

$$\frac{\partial u}{\partial x_j} = \mathcal{R}_j(\mathcal{R}\times\omega), \quad j=1,2,3, \text{ where } \omega = \nabla\times u. \tag{2.3}$$

where $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$ and $\mathcal{R}_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$ denote the Riesz transforms. It is known by Jawerth [14] that

$$\|\nabla u\|_{\dot{B}^{-\alpha}_{\infty,\infty}} \le C\|\omega\|_{\dot{B}^{-\alpha}_{\infty,\infty}}. \tag{2.4}$$

Notice that

$$f \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^3) \Longleftrightarrow \vec{\nabla} f \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3).$$

Now our result reads as follows.

Theorem 2.2. *Let (u, d) be a smooth solution to (1.1)–(1.4) with initial data $(u_0, d_0) \in H^3(\mathbb{R}^3) \times H^4(\mathbb{R}^3)$. Suppose that the corresponding vorticity field $\omega = \operatorname{curl} u$ satisfies*

$$\int_0^T \frac{\|\omega(t,\cdot)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1+\log(e+\|\omega(t,\cdot)\|_{\dot{B}^{-1}_{\infty,\infty}})} dt < \infty. \tag{2.5}$$

Then, the solution (u, d) can be smoothly extended after time T. In other words, if the solution blows up at $t = T$, then

$$\int_0^T \frac{\|\omega(t,\cdot)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1+\log(e+\|\omega(t,\cdot)\|_{\dot{B}^{-1}_{\infty,\infty}})} dt = \infty.$$

Remark 2.1. This result says that the velocity field of the fluid plays a more dominant role than the direction vector d modeling the orientation of the crystal molecules in the nematic liquid crystal. The theorem is still true, if we replace $\omega = \operatorname{curl} u$ by ∇u, due to the boundedness operator in $\dot{B}^{-1}_{\infty,\infty}$.

Remark 2.2. Since the Besov space $\dot{B}^{-1}_{\infty,\infty}$ is much wider than the Lebesgue space L^3 and $\dot{X}_1$, hence our result covers the results contained in [15–17] and [18].

As a consequence we have the following result.

Corollary 2.3. *Let $(u_0, d_0) \in H^3(\mathbb{R}^3) \times H^4(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$. Suppose that (u, d) is a smooth solution to the liquid crystal flow (1.1)–(1.4) on the time interval $[0, T)$ for some $0 < T < \infty$. If u satisfies*

$$\int_0^T \frac{\|u(t,\cdot)\|^2_{\dot{B}^{0}_{\infty,\infty}}}{1+\log(e+\|u(t,\cdot)\|_{\dot{B}^{0}_{\infty,\infty}})} dt < \infty, \tag{2.6}$$

then (u, d) can be extended beyond T.

It is well-known that

$$L^\infty(\mathbb{R}^3) \subset \mathrm{BMO}(\mathbb{R}^3) \subset \dot{B}^0_{\infty,\infty}(\mathbb{R}^3),$$

where $\mathrm{BMO}(\mathbb{R}^3)$ is the space of the bounded mean oscillations defined by

$$\mathrm{BMO}(\mathbb{R}^3) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^3) : \sup_{x,R} \frac{1}{|B(x,R)|} \int_{B(x,R)} \left| f(y) - \bar{f}_{B(x,R)} \right| dy < \infty \right\}$$

with

$$\bar{f}_{B(x,R)} = \frac{1}{|B(x,R)|} \int_{B(x,R)} f(y) dy.$$

Thus the conclusion of Theorem 2.2 remains true if (2.6) is replaced by the condition

$$\int_0^T \frac{\|u(t,\cdot)\|^2_{\mathrm{BMO}}}{1+\log(e+\|u(t,\cdot)\|_{\mathrm{BMO}})} dt < \infty.$$

To prove Theorem 2.2, we need the following lemma.

Lemma 2.4. *With the assumptions of Theorem 2.2, we have*

$$\sup_{0\le t<T} \|d(.,t)\|_{L^\infty} \le 1 + \|d_0\|_{L^\infty}.$$

Proof. See [11]. For the reader's convenience and completeness, we give the proof. It suffices to show that

$$\sup_{0\le t<T} \|d(.,t)\|_{L^\infty} \le \max\left(1, \|d_0\|_{L^\infty}\right)$$

where we use the inequalities

$$\max\left(|a|, |b|\right) \le \sqrt{|a|^2 + |b|^2} \le |a| + |b|.$$

Without loss of generality, we may assume $\|d_0\|_{L^\infty} \ge 1$ and denote

$$h(x,t) = |d(x,t)|^2 - \|d_0\|^2_{L^\infty}.$$

Then, multiplying Eq. (1.2) by d, it follows that

$$\frac{1}{2}\partial_t |d|^2 + \frac{1}{2}(u\cdot\nabla)|d|^2 - d\Delta d + |d|^2\left(|d|^2-1\right) = 0.$$

By the identities

$$\frac{1}{2}\partial_t \|d_0\|^2_{L^\infty} + \frac{1}{2}(u\cdot\nabla)\|d_0\|^2_{L^\infty} - \frac{1}{2}\Delta(\|d_0\|^2_{L^\infty}) + (\|d_0\|^2_{L^\infty}-1)|d|^2 = (\|d_0\|^2_{L^\infty}-1)|d|^2,$$

and

$$d\Delta d = \frac{1}{2}\Delta(|d|^2) - |\nabla d|^2,$$

it is easy to deduce that

$$\partial_t h + (u\cdot\nabla)h - \Delta h + 2|d|^2 h = -2(\|d_0\|^2_{L^\infty}-1)|d|^2 - 2|\nabla d|^2 \le 0.$$

Now taking the inner product in $L^2(\mathbb{R}^3)$ with h, using that

$$\int_{\mathbb{R}^3}(u\cdot\nabla)h\cdot h dx = \int_{\mathbb{R}^3} u\cdot\nabla\left(\frac{|h|^2}{2}\right)dx = 0,$$

we obtain the inequality

$$\frac{d}{dt}\|h\|^2_{L^2} + \|\nabla h\|^2_{L^2} + 2\|d\cdot h\|^2_{L^2} \le 0$$

and hence

$$\|h(.,t)\|_{L^2} \le \|h(.,0)\|_{L^2}.$$

Therefore, in $t=0$, we have

$$h(.,0) = |d_0(.)|^2 - \|d_0\|^2_{L^\infty} \le 0,$$

and by the maximum principle, yields $h(x,t)\le 0$, that is $|d(x,t)| \le \|d_0\|_{L^\infty}$ for all $(x,t)\in\mathbb{R}^3\times[0,T)$, which proves the lemma. □

In the proof of the main result, we frequently employ the following Gagliardo–Nirenberg inequality having fractional derivatives contained in [19].

Lemma 2.5. *Let $1 < p, p_0, p_1 \le \infty$, $s, r\in\mathbb{R}$, $0\le\alpha\le 1$. Then, there exists a constant C such that*

$$\|f\|_{\dot{H}^s_p} \le C\|f\|^{1-\alpha}_{L_{p_0}}\|f\|^{\alpha}_{\dot{H}^r_{p_1}},$$

where

$$\frac{1}{p} - \frac{s}{3} = \frac{1-\alpha}{p_0} + \alpha\left(\frac{1}{p_1} - \frac{r}{3}\right), \quad s \le \alpha r.$$

Using Lemma 2.5, we have

$$\|\nabla u\|_{L^3} \le C\|\nabla u\|^{\frac{3}{4}}_{L^2}\|\Lambda^3 u\|^{\frac{1}{4}}_{L^2},$$

$$\|\Lambda^3 u\|_{L^3} \le C\|\nabla u\|^{\frac{1}{6}}_{L^2}\|\Lambda^4 u\|^{\frac{5}{6}}_{L^2},$$

$$\|\nabla d\|_{L^\infty} \le C\|\Delta d\|^{\frac{3}{4}}_{L^2}\|\Lambda^4 d\|^{\frac{1}{4}}_{L^2},$$

$$\|\Lambda^4 d\|_{L^2} \le C\|\Delta d\|^{\frac{1}{3}}_{L^2}\|\Lambda^5 d\|^{\frac{2}{3}}_{L^2}.$$

Hereafter, C will denote a generic dimensionless constant.

3. Proof of Theorem 2.2

For any $T > 0$ we suppose that u is a smooth solution to (1.1)–(1.4) on $\mathbb{R}^3 \times (0, T)$ and will establish a priori bounds that will allow us to extend u for all time under (2.5). If (2.5) holds, one can deduce that for any small $\epsilon > 0$, there exists $T_* < T$ such that

$$\int_{T_*}^{T} \frac{\|\omega(t)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1 + \log\left(e + \|\omega(t)\|_{\dot{B}^{-1}_{\infty,\infty}}\right)} dt \le \epsilon.$$

We shall establish the following a priori estimate

$$\limsup_{t\to T^-}(\|\nabla^3 u(t,\cdot)\|^2_{L^2} + \|\nabla^4 d(t,\cdot)\|^2_{L^2}) < \infty. \tag{3.1}$$

We first recall some conservative identities of the system (1.1)–(1.4) and their immediate consequences. First, it is easy to see that (see e.g. [5,17])

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(|u|^2 + |\nabla d|^2 + 2F(d))dx + \int_{\mathbb{R}^3}(|\nabla u|^2 + |\Delta d - f(d)|^2)dx = 0, \tag{3.2}$$

which implies

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} \le C. \tag{3.3}$$

Moreover, multiplying (1.2) by $|d|^2 d$ and integrating by parts, we have

$$\frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}^3}|d|^4(t,x)dx + \int_{\mathbb{R}^3}\left(d^2|\nabla d|^2 + \frac{1}{2}|\nabla |d|^2|^2 + |d|^6\right)(t,x)dx = \int_{\mathbb{R}^3}|d|^4(t,x)dx,$$

which implies

$$\|d(t,\cdot)\|_{L^\infty(0,T;L^4)} + \int_0^t\int_{\mathbb{R}^3}(|d|^2|\nabla d|^2 + |d|^6)(\tau,x)dxd\tau \le C. \tag{3.4}$$

Thanks to (3.2) and (3.4), we obtain

$$\|d\|_{L^\infty(0,T;H^1)} + \|d\|_{L^2(0,T;H^2)} \le C. \tag{3.5}$$

Besides, multiplying (1.2) by $|d|^4 d$ and integrating over $\mathbb{R}^3$, we have

$$\frac{1}{6}\frac{d}{dt}\int_{\mathbb{R}^3}|d|^6(t,x)dx + \int_{\mathbb{R}^3}(|d|^4|\nabla d|^2 + |\nabla |d|^2|^2|d|^2 + |d|^8)(t,x)dx = \int_{\mathbb{R}^3}|d|^6(t,x)dx.$$

This implies that

$$\|d(t,\cdot)\|_{L^\infty(0,T;L^6)} \le C\|d_0\|_{L^6} \le C\|d_0\|_{H^1}. \tag{3.6}$$

Taking the operation curl on both sides of Eq. (1.1), we obtain

$$\omega_t + (u\cdot\nabla)\omega - \Delta\omega = (\omega\cdot\nabla)u + \sum_{k=1}^{3}\nabla\Delta d_k\cdot\nabla d_k, \tag{3.7}$$

where $\omega = \operatorname{curl} u$. Multiplying (3.7) by ω and integrating it over $\mathbb{R}^3$, we find after integration by part

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\omega|^2dx + \int_{\mathbb{R}^3}|\nabla\omega|^2dx = \int_{\mathbb{R}^3}(\omega\cdot\nabla u)\cdot\omega dx + \sum_{k=1}^{3}\int_{\mathbb{R}^3}(\nabla\Delta d_k\cdot\nabla d_k)\omega dx. \tag{3.8}$$

Applying Δ to (1.2), multiplying the resulting equation by Δd, we find that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\Delta d|^2dx + \int_{\mathbb{R}^3}|\nabla\Delta d|^2dx = -\int_{\mathbb{R}^3}\Delta((u\cdot\nabla)d)\cdot\Delta d dx - \int_{\mathbb{R}^3}\Delta f(d)\Delta d dx. \tag{3.9}$$

Summing up (3.8) and (3.9), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left(|\omega|^2 + |\Delta d|^2\right)dx + \int_{\mathbb{R}^3}(|\nabla\omega|^2 + |\nabla\Delta d|^2)dx &= \int_{\mathbb{R}^3}(\omega\cdot\nabla u)\cdot\omega dx + \sum_{k=1}^{3}\int_{\mathbb{R}^3}(\nabla\Delta d_k\cdot\nabla d_k)\omega dx\\ &\quad - \int_{\mathbb{R}^3}\Delta((u\cdot\nabla)d)\cdot\Delta d dx - \int_{\mathbb{R}^3}\Delta f(d)\Delta d dx\\ &= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4.\end{aligned} \tag{3.10}$$

We estimate the first term I_1 by using Hölder's inequality, the interpolation inequality $\|\omega\|_{L^4}^2 \le C\|\omega\|_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla\omega\|_{L^2}$ and the Young inequality as follows:

$$\begin{aligned}\mathrm{I}_1 &\le C\|\omega\|_{L^4}^2\|\nabla u\|_{L^2}\\ &\le C\|\omega\|_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla\omega\|_{L^2}\|\omega\|_{L^2}\\ &\le \frac{1}{2}\|\nabla\omega\|_{L^2}^2 + C\|\omega\|^2_{\dot{B}^{-1}_{\infty,\infty}}\|\omega\|_{L^2}^2.\end{aligned} \tag{3.11}$$

For I_2, we have

$$\begin{aligned}\mathrm{I}_2 &\le \|\omega\|_{L^4}\|\nabla\Delta d\|_{L^2}\|\nabla d\|_{L^4}\\ &\le C\|\omega\|^{\frac{1}{2}}_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla\omega\|^{\frac{1}{2}}_{L^2}\|\nabla\Delta d\|_{L^2}\|\Delta d\|^{\frac{1}{2}}_{L^2}\|d\|^{\frac{1}{2}}_{L^\infty}\\ &\le C\|\omega\|^{\frac{1}{2}}_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla\omega\|^{\frac{1}{2}}_{L^2}\|\nabla\Delta d\|_{L^2}\|\Delta d\|^{\frac{1}{2}}_{L^2}\\ &= \left(\|\nabla\Delta d\|^2_{L^2}\right)^{\frac{1}{2}}\left(C\|\omega\|^2_{\dot{B}^{-1}_{\infty,\infty}}\|\Delta d\|^2_{L^2}\right)^{\frac{1}{4}}\left(\|\nabla\omega\|^2_{L^2}\right)^{\frac{1}{4}}\\ &\le \frac{1}{4}\|\nabla\Delta d\|^2_{L^2} + \frac{1}{4}\|\nabla\omega\|^2_{L^2} + C\|\omega\|^2_{\dot{B}^{-1}_{\infty,\infty}}\|\Delta d\|^2_{L^2}\\ &\le \frac{1}{4}\|\nabla\Delta d\|^2_{L^2} + \frac{1}{4}\|\nabla\omega\|^2_{L^2} + C\|\omega\|^2_{\dot{B}^{-1}_{\infty,\infty}}\left(\|\omega\|^2_{L^2} + \|\Delta d\|^2_{L^2}\right).\end{aligned}$$

For I_3, we have

$$\begin{aligned}|\mathrm{I}_3| &\le \left|\int_{\mathbb{R}^3}(\Delta u\cdot\nabla)d\cdot\Delta d dx\right| + \sum_{k=1}^{3}\left|\int_{\mathbb{R}^3}(\partial_k u\cdot\nabla)\partial_k d\cdot\Delta d dx\right| + \left|\int_{\mathbb{R}^3}(u\cdot\nabla)\Delta d\cdot\Delta d dx\right|\\ &\le \|\Delta u\|_{L^2}\|\nabla d\|_{L^4}\|\Delta d\|_{L^4} + \|\nabla u\|_{L^2}\|\Delta d\|^2_{L^4}\\ &\le C\|\nabla\omega\|_{L^2}\|\Delta d\|^{\frac{1}{2}}_{L^2}\|d\|^{\frac{1}{2}}_{L^\infty}\|\nabla\Delta d\|^{\frac{1}{2}}_{L^2}\|\nabla d\|^{\frac{1}{2}}_{L^\infty} + C\|\omega\|_{L^2}\|d\|_{L^\infty}\|\nabla\Delta d\|_{L^2}\\ &\le C\|\nabla\omega\|_{L^2}\|\Delta d\|^{\frac{1}{2}}_{L^2}\|\nabla\Delta d\|^{\frac{1}{2}}_{L^2} + C\|\omega\|_{L^2}\|\nabla\Delta d\|_{L^2}\\ &= \left(\|\nabla\Delta d\|^2_{L^2}\right)^{\frac{1}{4}}\left(C\|\Delta d\|^2_{L^2}\right)^{\frac{1}{4}}\left(\|\nabla\omega\|^2_{L^2}\right)^{\frac{1}{2}} + \frac{1}{8}\|\nabla\Delta d\|^2_{L^2} + C\|\omega\|^2_{L^2}\\ &\le \frac{1}{4}\|\nabla\Delta d\|^2_{L^2} + \frac{1}{4}\|\nabla\omega\|^2_{L^2} + C\|\Delta d\|^2_{L^2} + C\|\omega\|^2_{L^2}\\ &\le \frac{1}{4}\|\nabla\Delta d\|^2_{L^2} + \frac{1}{4}\|\nabla\omega\|^2_{L^2} + C\left(\|\Delta d\|^2_{L^2} + \|\omega\|^2_{L^2}\right)\left(1 + \|\omega\|^2_{\dot{B}^{-1}_{\infty,\infty}}\right).\end{aligned}$$

Here we have used the following Gagliardo–Nirenberg inequality:

$$\|\Delta d\|^2_{L^4} \le C\|d\|_{L^\infty}\|\nabla\Delta d\|_{L^2}$$

and

$$\int_{\mathbb{R}^3}(u\cdot\nabla)\Delta d\cdot\Delta d dx = 0.$$

By the Hölder's inequality and (3.6), I_4 is bounded as follows:

$$\begin{aligned}\mathrm{I}_4 &= -\int_{\mathbb{R}^3}\Delta(|d|^2 d)\Delta d dx + \int_{\mathbb{R}^3}\Delta d\Delta d dx\\ &= 3\int_{\mathbb{R}^3}|d|^2\nabla d\cdot\nabla\Delta d dx + \|\Delta d\|^2_{L^2}\\ &\le 3\|d\|^2_{L^6}\|\nabla d\|_{L^6}\|\nabla\Delta d\|_{L^2} + \|\Delta d\|^2_{L^2}\\ &\le \frac{1}{4}\|\nabla\Delta d\|^2_{L^2} + C\|d\|^4_{L^6}\|\Delta d\|^2_{L^2} + \|\Delta d\|^2_{L^2}\\ &\le \frac{1}{4}\|\nabla\Delta d\|^2_{L^2} + C\|\Delta d\|^2_{L^2},\end{aligned} \tag{3.12}$$

where we used the inequality $ab \leq \frac{a^2}{2\kappa} + \frac{\kappa b^2}{2}$. Combining (3.11)–(3.12) into (3.10) and from the embedding $L^\infty\left(\mathbb{R}^3\right) \subset \dot{B}^0_{\infty,\infty}\left(\mathbb{R}^3\right)$ and the fact that

$$u \in \dot{B}^0_{\infty,\infty}\left(\mathbb{R}^3\right) \Longleftrightarrow \nabla u \in \dot{B}^{-1}_{\infty,\infty}\left(\mathbb{R}^3\right),$$

we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\omega\|^2_{L^2} + \|\Delta d\|^2_{L^2}) + (\|\nabla\omega\|^2_{L^2} + \|\nabla\Delta d\|^2_{L^2}) \leq C(\|\omega\|^2_{\dot{B}^{-1}_{\infty,\infty}} + 1)\left(\|\nabla u\|^2_{L^2} + \|\Delta d\|^2_{L^2}\right)\\
&\quad\leq C\left(1 + \frac{\|\omega\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1+\log(e+\|\omega\|_{\dot{B}^{-1}_{\infty,\infty}})}\right)(1+\log(e+\|\omega\|_{\dot{B}^{-1}_{\infty,\infty}}))\left(\|\omega\|^2_{L^2} + \|\Delta d\|^2_{L^2}\right)\\
&\quad\leq C\left(1 + \frac{\|\omega\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1+\log(e+\|\omega\|_{\dot{B}^{-1}_{\infty,\infty}})}\right)(1+\log(e+\|u\|_{\dot{B}^{0}_{\infty,\infty}}))\left(\|\omega\|^2_{L^2} + \|\Delta d\|^2_{L^2}\right)\\
&\quad\leq C\left(1 + \frac{\|\omega\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1+\log(e+\|\omega\|_{\dot{B}^{-1}_{\infty,\infty}})}\right)(1+\log(e+\|u\|_{L^\infty}))\left(\|\omega\|^2_{L^2} + \|\Delta d\|^2_{L^2}\right).
\end{aligned}$$

Since it is well know that the Sobolev space $H^s\left(\mathbb{R}^3\right)$ with $s > \frac{3}{2}$ is continuously embedded into $L^\infty\left(\mathbb{R}^3\right)$, this yields

$$\frac{d}{dt}F(t) \leq C\left(1 + \frac{\|\omega\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1+\log(e+\|\omega\|_{\dot{B}^{-1}_{\infty,\infty}})}\right)(1+\log(e+y(t)))\,F(t),$$

where $y(t)$ is defined by

$$y(t) = \sup_{T_*\leq\tau\leq t}\left(\left\|\Lambda^3 u(\tau,.)\right\|_{L^2} + \left\|\Lambda^3\nabla d(\tau,.)\right\|_{L^2}\right) \quad \text{for all } T_* \leq t < T$$

and

$$F(t) = \|\omega(t)\|^2_{L^2} + \|\Delta d(t)\|^2_{L^2}.$$

By Gronwall's lemma on the interval $[T_*, t]$, one has

$$\begin{aligned}
F(t) &\leq F(T_*)\exp\left(C(1+\log(e+y(t)))\int_{T_*}^t \frac{\|\omega(s)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1+\log\left(e+\|\omega(s)\|_{\dot{B}^{-1}_{\infty,\infty}}\right)}\,ds\right)\\
&\leq C_0\exp\left(C\epsilon\left(1+\log(e+y(t))\right)\right)\\
&\leq C_0\exp\left(2C\epsilon\log(e+y(t))\right)\\
&= C_0\left(e+y(t)\right)^{2C\epsilon},
\end{aligned}$$

where $C_0 = \|\omega(.,T_*)\|^2_{L^2} + \|\Delta d(.,T_*)\|^2_{L^2}$.

$$F(t) \leq F(T_*)(e+y(t))^{K\epsilon} \quad \forall t \in [0,T]. \tag{3.13}$$

We are now ready to study the estimate in $H^3 \times H^4$ norm. Taking the operation $\Lambda^3 = (-\Delta)^{\frac{3}{2}}$ on both sides of (1.1), then multiplying them by $\Lambda^3 u$, and integrating over $\mathbb{R}^3$, we have

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^3 u(t)\right\|^2_{L^2} + \left\|\Lambda^4 u(t)\right\|^2_{L^2} = -\int_{\mathbb{R}^3}\Lambda^3(u\cdot\nabla u)\,\Lambda^3 u dx + \int_{\mathbb{R}^3}\Lambda^3(\nabla d\odot\nabla d) : \Lambda^3\nabla u dx. \tag{3.14}$$

Noting that $\nabla\cdot u = 0$ and integrating by parts, we write (3.14) as

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^3 u(t)\right\|^2_{L^2} + \left\|\Lambda^4 u(t)\right\|^2_{L^2} = -\int_{\mathbb{R}^3}\Lambda^3\left(u\cdot\nabla u - u\nabla\Lambda^3 u\right)\Lambda^3 u dx + \int_{\mathbb{R}^3}\Lambda^3(\nabla d\odot\nabla d) : \Lambda^3\nabla u dx. \tag{3.15}$$

In what follows, we will use the following commutator and product estimates due to Kato and Ponce [20]:

$$\|\Lambda^\alpha(fg) - f\Lambda^\alpha g\|_{L^p} \leq C\left(\left\|\Lambda^{\alpha-1}g\right\|_{L^{q_1}}\|\nabla f\|_{L^{p_1}} + \|\Lambda^\alpha f\|_{L^{p_2}}\|g\|_{L^{q_2}}\right), \tag{3.16}$$

$$\|\Lambda^\alpha(fg)\|_{L^p} \leq C\left(\|\Lambda^\alpha g\|_{L^{q_1}}\|f\|_{L^{p_1}} + \|\Lambda^\alpha f\|_{L^{p_2}}\|g\|_{L^{q_2}}\right) \tag{3.17}$$

for $\alpha > 0$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Hence, from (3.16) and (3.17) with $\alpha = 3$, $p = \frac{3}{2}$, $p_1 = q_1 = p_2 = q_2 = 3$ and by using Lemma 2.5, we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\Lambda^3 u(t)\right\|_{L^2}^2 + \left\|\Lambda^4 u(t)\right\|_{L^2}^2 &\le C\|\nabla u\|_{L^3}\|\Lambda^3 u\|_{L^3}^2 + C\|\nabla d\|_{L^\infty}\|\Lambda^4 u\|_{L^2}\|\Lambda^4 d\|_{L^2} \\ &\le C\|\nabla u\|_{L^2}^{\frac{13}{12}}\|\Lambda^3 u\|_{L^2}^{\frac{1}{4}}\|\Lambda^4 u\|_{L^2}^{\frac{5}{3}} + C\|\nabla d\|_{L^\infty}^2\|\Lambda^4 d\|_{L^2}^2 + \frac{1}{8}\|\Lambda^4 u\|_{L^2}^2 \\ &\le \frac{1}{8}\|\Lambda^4 u\|_{L^2}^2 + C\|\omega\|_{L^2}^{\frac{13}{2}}\|\Lambda^3 u\|_{L^2}^{\frac{3}{2}} + C\|\Delta d\|_{L^2}^{\frac{3}{2}}\|\Lambda^4 d\|_{L^2}^{\frac{1}{2}}\|\Delta d\|_{L^2}^{\frac{2}{3}}\|\Lambda^5 d\|_{L^2}^{\frac{4}{3}} \\ &\le \frac{1}{8}\left(\|\Lambda^4 u\|_{L^2}^2 + \|\Lambda^5 d\|_{L^2}^2\right) + C\|\omega\|_{L^2}^{\frac{13}{2}}\|\Lambda^3 u\|_{L^2}^{\frac{3}{2}} + C\|\Delta d\|_{L^2}^{\frac{13}{2}}\|\Lambda^4 d\|_{L^2}^{\frac{3}{2}}. \end{aligned} \quad (3.18)$$

Taking the operation Λ^4 on both sides to the liquid crystal Eq. (1.2), then multiplying them by $\Lambda^4 d$, after integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\Lambda^4 d(t)\right\|_{L^2}^2 + \left\|\Lambda^5 d(t)\right\|_{L^2}^2 = -\int_{\mathbb{R}^3}\Lambda^4\left(u\cdot\nabla d - u\nabla\Lambda^4 d\right)\cdot\Lambda^4 d dx - \int_{\mathbb{R}^3}\Lambda^4 f(d)\cdot\Lambda^4 d dx \\ &\le C\|\nabla u\|_{L^3}\|\Lambda^4 d\|_{L^6}\|\Lambda^4 d\|_{L^2} + C\|\nabla d\|_{L^\infty}\|\Lambda^4 u\|_{L^2}\|\Lambda^4 d\|_{L^2} + C\|\Lambda^4 d\|_{L^2}^2 \\ &\le C\|\nabla u\|_{L^2}^{\frac{3}{4}}\|\Lambda^3 u\|_{L^2}^{\frac{1}{4}}\|\Delta d\|_{L^2}^{\frac{1}{3}}\|\Lambda^5 d\|_{L^2}^{\frac{5}{3}} + C\|\nabla d\|_{L^\infty}^2\|\Lambda^4 d\|_{L^2}^2 + \frac{1}{8}\|\Lambda^4 u\|_{L^2}^2 + C\|\Lambda^4 d\|_{L^2}^2 \\ &\le \frac{1}{8}\left(\|\Lambda^4 u\|_{L^2}^2 + \|\Lambda^5 d\|_{L^2}^2\right) + C\|\omega\|_{L^2}^{\frac{9}{2}}\|\Delta d\|_{L^2}^2\|\Lambda^3 u\|_{L^2}^{\frac{3}{2}} + C\|\Delta d\|_{L^2}^{\frac{13}{2}}\|\Lambda^4 d\|_{L^2}^{\frac{3}{2}} + C\|\Lambda^4 d\|_{L^2}^2. \end{aligned} \quad (3.19)$$

Combining (3.18) and (3.19), we easily get

$$\frac{d}{dt}\left(\left\|\Lambda^3 u(t)\right\|_{L^2}^2 + \left\|\Lambda^4 d(t)\right\|_{L^2}^2\right) \le C_0 C(e + y(t))^{\frac{3}{2}+\frac{13}{2}C\epsilon}. \quad (3.20)$$

Gronwall's inequality implies the boundedness of $H^3 \times H^4$-norm of (u, d) provided that $\epsilon < \frac{1}{13C}$, which can be achieved by the absolute continuous property of integral (2.5). This completes the proof of Theorem 2.2. □

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