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Uniformly resolvable $\mathcal{H}$-designs with $\mathcal{H} = \{P_3, P_4\}$

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Abstract

In this paper we consider the problem regarding the existence of uniformly resolvable decompositions of the complete graph K_v into subgraphs such that each resolution class contains only blocks isomorphic to the same graph. We completely determine the spectrum for the case in which all the resolution classes are either P_3 or P_4.

1 Introduction

Given a collection of graphs $\mathcal{H}$, an $\mathcal{H}$-*decomposition* of a graph G is a decomposition of the edges of G into isomorphic copies of graphs from $\mathcal{H}$; the copies of $H \in \mathcal{H}$ in the decomposition are called *blocks*. Such a decomposition is called *resolvable* if it is possible to partition the blocks into *classes* $\mathcal{P}_i$ such that every vertex of G appears in exactly one block of each $\mathcal{P}_i$.

A resolvable $\mathcal{H}$-decomposition of G is sometimes also referred to as an $\mathcal{H}$-*factorization of* G, a class can be called an $\mathcal{H}$-*factor of* G. The case where $\mathcal{H}$ is a single edge (K_2) is known as a 1-*factorization of* G and it is well known to exist for $G = K_v$ if and only if v is even.

In many cases we wish to impose further constraints on the classes of an $\mathcal{H}$-decomposition. For example, a class is called *uniform* if every block of the class is isomorphic to the same graph from $\mathcal{H}$.

A uniformly resolvable $\mathcal{H}$-design of order v is a uniformly resolvable $\mathcal{H}$-decomposition of K_v. Of particular note is the result of Rees [12] which finds necessary and sufficient conditions for the existence of uniformly resolvable $\{K_2, K_3\}$-designs of order v. Uniformly resolvable decompositions of K_v have also been studied in [4], [6], [9], [10], [11], [15], [16], [17], and [18]. In what follows, we will denote by $[a_1, \ldots, a_k]$, $k \geq 2$, the path P_k having vertex set $\{a_1, \ldots, a_k\}$ and edge set $\{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{k-1}, a_k\}\}$. The existence problem of resolvable P_k-designs

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of order v was solved by Horton [8] for $k = 3$ and by Bermond, Heinrich and Yu [3] for $k \geq 4$.

If v is a multiple of 12 we make the followings definitions for the purposes of this paper.

- Let (P_3, P_4)-URD$(v; r, s)$ denote a uniformly resolvable decomposition of K_v into r classes containing only copies of paths P_3 and s classes containing only copies of paths P_4.
- Let $J(v)$= $\{(6 + 9x, 2 + \frac{2(v-12)}{3} - 8x), x = 0, \ldots, \frac{v-12}{12}\}$.
- Let URD$(v; P_3, P_4)$ denote the set of all pairs (r, s) such that there exists a uniformly resolvable decomposition of K_v into r classes containing only copies of P_3 and s classes containing only copies of P_4.

In this paper, the main purpose is to investigate the existence problem of a (P_3, P_4)-URD$(v; r, s)$ of K_v. We completely solve the spectrum problem for such design; i.e., characterize the existence of uniformly resolvable $\{P_3, P_4\}$-designs of order v, by proving the following result:

Main Theorem. *For every integer* $v \equiv 0 \pmod{12}$*,* $v \geq 4$*, the set* $URD(v; P_3, P_4)$ *is identical to the set* $J(v)$*.*

2 Preliminaries and necessary conditions

In this section we introduce some useful definitions, results and give necessary conditions for the existence of a uniformly resolvable decomposition of K_v into P_3 and P_4 graphs. A (resolvable)$\mathcal{H}$-decomposition of the complete multipartite graph with u parts each of size g is known as a resolvable group divisible design $\mathcal{H}$-RGDD of type g^u; the parts of size g are called the groups of the design. When $\mathcal{H} = \{K_n\}$ we will call it an n-(R)GDD.

A (P_3, P_4)-URGDD (r, s) of type g^u is a uniformly resolvable decomposition of the complete multipartite graph with u parts each of size g into r classes containing only copies of paths P_3 and s classes containing only copies of paths P_4.

If the blocks of an $\mathcal{H}-$RGDD of type g^u can be partitioned into partial parallel classes, by which we mean that each contains all points except none of those of one group, we refer to the decomposition as a *frame*. When $\mathcal{H} = \{K_n\}$ we will call it an *n-frame* and it is easy to deduce that the number of partial parallel classes missing a specified group G is $\frac{|G|}{n-1}$. There exists a 2-frame of type g^u if and only if $u \geq 3$ and $g(u - 1)$ is even [14].

An incomplete uniformly resolvable (P_3, P_4)-decomposition of K_{v+h} with a hole of size h is a (P_3, P_4)-decomposition of $K_{v+h} - K_h$ in which there are two types of classes, full classes and partial classes which cover every point except those in the hole (the points of K_h are referred to as the hole). Specifically a (P_3, P_4)-IURD$(v + h, h; [r_1, s_1], [\bar{r}_1, \bar{s}_1])$ is a uniformly resolvable $(P_3, P_4)-$decomposition of $K_{v+h} - K_h$

into r_1 partial classes of paths P_3 and s_1 partial classes of paths P_4 which cover only the points not in the hole, $\bar{r}_1$ full classes of paths P_3 and $\bar{s}_1$ full classes of paths P_4 which cover every point of K_{v+h}.

We also need the following definitions. Let (s_1, t_1) and (s_2, t_2) be two pairs of non-negative integers. Define $(s_1, t_1) + (s_2, t_2) = (s_1 + s_2, t_1 + t_2)$. If X and Y are two sets of pairs of non-negative integers, then $X + Y$ denotes the set $\{(s_1, t_1) + (s_2, t_2) : (s_1, t_1) \in X, (s_2, t_2) \in Y\}$. If X is a set of pairs of non-negative integers and h is a positive integer, then $h * X$ denotes the set of all pairs of non-negative integers which can be obtained by adding any h elements of X together (repetitions of elements of X are allowed).

Lemma 2.1. *If* $(r, s) \in URD(v; P_3, P_4)$*, with* $r > 0$ *and* $s > 0$*, then* $v \equiv 0 \pmod{12}$ *and* $(r, s) \in J(v)$.

Proof. Assume that there exists a (P_3, P_4)-URD$(v; r, s)$ D of K_v, with $r > 0$ and $s > 0$. By resolvability it follows that $v \equiv 0 \pmod{12}$. Counting the edges of K_v that appear in D we obtain

$$\frac{2rv}{3} + \frac{3sv}{4} = \frac{v(v-1)}{2},$$

and hence that

$$8r + 9s = 6(v-1). \tag{1}$$

This equation implies that $8r \equiv 6(v-1) \pmod 9$ and $9s \equiv 6(v-1) \pmod 8$. Then we obtain

$$r \equiv 6 \pmod 9 \quad \text{and} \quad s \equiv 2 \pmod 8.$$

Now letting $r = 6 + 9x$ and $s = 2 + 8y$, the equation (1) yields

$$x + y = \frac{v-12}{12}. \tag{2}$$

The equation (2) yields $x = 0, \ldots, \frac{v-12}{12}$ and $y = \frac{v-12}{12} - x$. Hence $(r, s) \in \{(6+9x, 2+\frac{2(v-12)}{3} - 8x), x = 0, \ldots, \frac{v-12}{12}\}$. This complete the proof of the lemma.

□

To establish the existence of some small (P_3, P_4)-designs we need the following results.

Theorem 2.2. *[11, Lemma 4.1]. Let* $v \equiv 0 \pmod 3$, $v \geq 9$. *The union of any two edge-disjoint parallel classes of 3-cycles of* K_v *can be decomposed into three parallel classes of* P_3.

Lemma 2.3. *Let* $v \equiv 0 \pmod{12}$. *If there exists a* (K_3, K_4)*-URD*$(v; 2r, s)$ K *of* K_v*, then there exists a* (P_3, P_4)*-URD*$(v; 3r, 2s)$ P *of* K_v.

Proof. Let K be a (K_3, K_4)-URD$(v; 2r, s)$. By Theorem 2.2 the $2r$ classes of K of size 3 can be decomposed into $3r$ classes of paths P_3. Filling in each block of K of size 4 with the same (P_3, P_4)-URD$(4; 0, 2)$, we obtain the result.

□

3 Small cases

Lemma 3.1. *There exists a* (P_3, P_4)*-URGDD*$(0, 4)$ *of type* 6^2.

Proof. Take the groups to be $\{0, 1, 2, 3, 4, 5\}, \{6, 7, 8, 9, 10, 11\}$ and the classes as listed below:
$\{[7, 0, 10, 3], [5, 6, 2, 9], [1, 8, 4, 11]\}$, $\{[0, 6, 1, 7], [2, 8, 3, 9], [4, 10, 5, 11]\}$,
$\{[1, 9, 0, 8], [3, 11, 2, 10], [5, 7, 4, 6]\}$, $\{[0, 11, 1, 10], [2, 7, 3, 6], [4, 9, 5, 8]\}$.

□

Lemma 3.2. *There exists a* (P_3, P_4)*-URGDD*$(6, 0)$ *of type* 4^3.

Proof. Let $\{a_0, \dots, a_3\}$, $\{b_0, \dots, b_3\}$ and $\{c_0, \dots, c_3\}$ be the groups and the classes as listed below:
$\{[c_i, b_i, a_{i+1}], i \in Z_4\}$, $\{[c_{i+2}, b_i, a_{i+2}], i \in Z_4\}$, $\{[c_i, a_i, b_i], i \in Z_4\}$,
$\{[c_{i+1}, a_i, b_{i+1}], i \in Z_4\}$, $\{[b_{i+1}, c_i, a_{i+1}], i \in Z_4\}$, $\{[b_{i+3}, c_i, a_{i+2}], i \in Z_4\}$.

□

Lemma 3.3. *There exists a* (P_3, P_4)*-URGDD*(r, s) *of type* 12^2 *with* $(r, s) \in \{(9, 0), (0, 8)\}$.

Proof. The case $(9, 0)$ corresponds to a (P_3, P_4)-URGDD$(9, 0)$ of type 12^2 which is known to exist [19]. For the case $(0, 8)$, take a 2-RGDD of type 2^2 with 2 parallel classes of edges. Expand each point 6 times and replace the blocks of a given resolution class with the same (P_3, P_4)-URGDD$(0, 4)$ of type 6^2, which exists by Lemma 3.1.

□

Lemma 3.4. *URD*$(12; P_3, P_4) = J(12) = \{(6, 2)\}$.

Proof. Take a (P_3, P_4)-URGDD$(6, 0)$ of type 4^3, which exists by Lemma 3.2. Fill each of the groups of size 4 with the same (P_3, P_4)-URD$(4; 0, 2)$. This completes the proof.

□

Lemma 3.5. *URD*$(36; P_3, P_4) = J(36)$.

Proof. Take a (K_3, K_4)-URD$(36; r, s)$ of K_{36} with $(r, s) \in \{(16, 1), (10, 5), (4, 9)\}$, which exists by Lemma 3.2 of [18]. Applying Lemma 2.3 we obtain the result.

□

Lemma 3.6. *URD*$(60; P_3, P_4) = J(60)$.

Proof. Take a (K_3, K_4)-URD$(60; r, s)$ of K_{60}, with $(r, s) \in \{(28, 1), (22, 5), (16, 9), (10, 13), (4, 17)\}$, which exists by Lemma 3.4 of [18]. Applying Lemma 2.3 we obtain the result.

□

Lemma 3.7. *There exists a* (P_3, P_4)*-URGDD*(r, s) *of type* 12^3, *with* $(r, s) \in \{(18, 0), (9, 8), (0, 16)\}$.

Proof.

- $(18, 0)$.
  Take a 3-RGDD G of type 3^3 which exists by [13]. Give weight 4 to all points and replace the blocks of a given resolution class with the same (P_3, P_4)-URGDD$(6, 0)$ of type 4^3, which exists by Lemma 3.2. Since G contains three parallel classes we obtain the result.

- $(0, 16)$.
  Let $\mathcal{F}=\{F_1, \ldots, F_4\}$ be a 1-factorization of the complete graph $K_{2,2,2}$ [7]. Give weight 6 to all points and place on each edge of a given 1-factor of $\mathcal{F}$ the same (P_3, P_4)-URGDD$(0, 4)$ of type 6^2, which exists by Lemma 3.1. Since $\mathcal{F}$ contains four 1-factors we obtain the result.

- $(9, 8)$.
  Let $A = \{a_1, a_2, a_3, \bar{a}_1, \bar{a}_2, \bar{a}_3\}$, $B = \{b_1, b_2, b_3, \bar{b}_1, \bar{b}_2, \bar{b}_3\}$,
  $X = \{x_1, x_2, x_3, \bar{x}_1, \bar{x}_2, \bar{x}_3\}$, $Y = \{y_1, y_2, y_3, \bar{y}_1, \bar{y}_2, \bar{y}_3\}$,
  $Z = \{z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3\}$, $T = \{t_1, t_2, t_3, \bar{t}_1, \bar{t}_2, \bar{t}_3\}$.
  Take the groups to be $\{a_1, a_2, a_3, \bar{a}_1, \bar{a}_2, \bar{a}_3, b_1, b_2, b_3, \bar{b}_1, \bar{b}_2, \bar{b}_3\}$,
  $\{x_1, x_2, x_3, \bar{x}_1, \bar{x}_2, \bar{x}_3, y_1, y_2, y_3, \bar{y}_1, \bar{y}_2, \bar{y}_3\}$, $\{z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3, t_1, t_2, t_3, \bar{t}_1, \bar{t}_2, \bar{t}_3\}$.

  * 9 classes of paths P_3
  Take 3 classes of paths P_3 listed below:
  $\{[a_1, z_j, y_1], [a_2, z_{j+1}, y_2], [a_3, z_{j+2}, y_3], [\bar{a}_1, \bar{z}_j, \bar{y}_1], [\bar{a}_2, \bar{z}_{j+1}, \bar{y}_2], [\bar{a}_3, \bar{z}_{j+2}, \bar{y}_3],$
  $[x_1, b_j, t_1], [x_2, b_{j+1}, t_2], [x_3, b_{j+2}, t_3], [\bar{x}_1, \bar{b}_j, \bar{t}_1], [\bar{x}_2, \bar{b}_{j+1}, \bar{t}_2], [\bar{x}_3, \bar{b}_{j+2}, \bar{t}_3]\},$
  $\{[\bar{z}_1, a_j, x_1], [\bar{z}_2, a_{j+1}, x_2], [\bar{z}_3, a_{j+2}, x_3], [z_1, \bar{a}_j, \bar{x}_1], [z_2, \bar{a}_{j+1}, \bar{x}_2], [z_3, \bar{a}_{j+2}, \bar{x}_3],$
  $[\bar{b}_1, t_j, y_1], [\bar{b}_2, t_{j+1}, y_2], [\bar{b}_3, t_{j+2}, y_3], [b_1, \bar{t}_j, \bar{y}_1], [b_2, \bar{t}_{j+1}, \bar{y}_2], [b_3, \bar{t}_{j+2}, \bar{y}_3]\},$
  $\{[\bar{t}_1, y_j, \bar{z}_1], [\bar{t}_2, y_{j+1}, \bar{z}_2], [\bar{t}_3, y_{j+2}, \bar{z}_3], [t_1, \bar{y}_j, z_1], [t_2, \bar{y}_{j+1}, z_2], [t_3, \bar{y}_{j+2}, z_3],$
  $[\bar{b}_1, x_j, \bar{a}_1], [\bar{b}_2, x_{j+1}, \bar{a}_2], [\bar{b}_3, x_{j+2}, \bar{a}_3], [b_1, \bar{x}_j, a_1], [b_2, \bar{x}_{j+1}, a_2], [b_3, \bar{x}_{j+2}, a_3]\}.$
  From the above classes, for $j = 1, 2, 3$, we obtain 9 classes of P_3.

  * 8 classes of paths P_4
  Take on $K_{2,2,2}$, with $V(K_{2,2,2}) = \{a, b\} \cup \{x, y\} \cup \{z, t\}$, the following two 1-factors $F_1 = \{[a, t], [b, y], [z, x]\}$, $F_2 = \{[a, y], [b, z], [t, x]\}$. Expand each point 6 times and replace each edge of F_1 and F_2 with the same (P_3, P_4)-URGDD$(0, 4)$ of type 6^2, which exists by Lemma 3.1. This gives 8 parallel classes of paths P_4.

□

Lemma 3.8. *There exists a* (P_3, P_4)*-IURD*$(36, 12; [6, 2], [r, s])$ *with* $(r, s) \in \{(18, 0),$ $(9, 8), (0, 16)\}$.

Proof. Start with a 3-partite graph G of type 12^3 with groups H, H_1 and H_2. We will construct our design on $H_1 \cup H_2 \cup H$ in such a way that the hole covers the points of H. Place on G a copy of a $(P_3, P_4)-$URGDD(r, s) of type 12^3 with $(r, s) \in \{(18, 0), (9, 8), (0, 16)\}$, which exists by Lemma 3.7. This gives the full classes. The partial classes can be obtained by filling the groups H_1 and H_2 of sizes 12 with the same (P_3, P_4)-URD$(12; 6, 2)$ of K_{12}, which exists by Lemma 3.4, and joining the resultant classes. This completes the proof.

□

4 Main results

Lemma 4.1. *For every $v \equiv 0 \pmod{24}$ $J(v) \subseteq URD(v; P_3, P_4)$.*

Proof. Let $v \equiv 0 \pmod{24}$, $v \geq 24$. Start with a 1-*factorization* $\mathcal{F}=\{F_1, \ldots, F_{\frac{v-12}{12}}\}$ of the complete graph $K_{\frac{v}{12}}$ [7]. Expand each point of $K_{\frac{v}{12}}$ 12 times and place on each edge of a given 1-factor of $\mathcal{F}$ the same (P_3, P_4)-URGDD(r, s) of type 12^2, with $(r, s) \in \{(9, 0), (0, 8)\}$, which exists by Lemma 3.3. Fill the groups of sizes 12 with the same (P_3, P_4)-URD$(12; 6, 2)$ of K_{12} which exists by Lemma 3.4. Since F contains $\frac{v-12}{12}$ 1-factors, the result is a (P_3, P_4)-URD$(v; r, s)$ of K_v for each $(r, s) \in \{(6, 2) + \frac{(v-12)}{12} * \{(9, 0), (0, 8)\}\}$. This implies

$$URD(v; P_3, P_4) \supseteq \{(6, 2) + \frac{v-12}{12} * \{(9, 0), (0, 8)\}\}.$$

Since $(\frac{v-12}{12}) * \{(9, 0), (0, 8)\} = \{(9x, 8\frac{v-12}{12} - 8x), x = 0, \ldots, \frac{v-12}{12}\}$, it easy to see that $\{(6, 2)+\frac{v-12}{12} * \{(18, 0), (9, 8), (0, 16)\}\} = J(v)$. This completes the proof.

□

Lemma 4.2. *For every $v \equiv 12 \pmod{24}$ $J(v) \subseteq URD(v; P_3, P_4)$.*

Proof. Let $v \equiv 12 \pmod{24}$. The cases $v = 12, 36, 60$ are covered by Lemmas 3.4, 3.5 and 3.6. For $v > 60$ start with a 2-frame F of type $2^{\frac{v-12}{24}}$ [17] with groups G_i, $i = 1, \ldots, \frac{v-12}{24}$. Let $p_{i,j}$, $j = 1, 2$ be the two partial parallel classes which miss the group G_i. Expand each point 12 times and add a set H of 12 ideal points $a_1, \ldots, a_{12}$. Fill H with a (P_3, P_4)-URD$(12; 6, 2)$ D of K_{12}, which exists by Lemma 3.4. For each $i = 1, \ldots, \frac{v-12}{24}$, place on $G_i \times \{1, \ldots, 12\} \cup H$ the same (P_3, P_4)-IURD$(36, 12; [6, 2], [x, y])$ D_i of $K_{36} - K_{12}$ with $(x, y) \in \{(18, 0), (9, 8), (0, 16)\}$, which exists by Lemma 3.8, in such a way that the hole covers the points of H. For $i = 1, \ldots, \frac{v-12}{24}$, place on each block of the two partial parallel classes $p_{i,j}$ the same (P_3, P_4)-URGDD(r_2, s_2) $D_{i,j}$ of type 12^2 with $(r_2, s_2) \in \{(9, 0), (0, 8)\}$, which exists by Lemma 3.3. Add the classes of D to the partial classes of D_i and form, on $\cup_{i=1}^{\frac{v-12}{24}} G_i \times \{1, \ldots, 12\} \cup H$, 6 classes of P_3-factors and two classes of P_4- factors. For each $i = 1, \ldots, \frac{v-12}{24}$, add the full classes of D_i to the two classes of $D_{i,j}$, $j = 1, 2$, and form r_3 classes of P_3-factors and s_3 classes of P_4-factors with $(r_3, s_3) \in \{(18, 0), (9, 8), (0, 16)\}$. Since each group G_i is missed by two partial parallel classes of F, we obtain a (P_3, P_4)-URD $(v; r, s)$ of K_v for each $(r, s) \in \{(6, 2) + \frac{v-12}{24} * \{(18, 0), (9, 8), (0, 16)\}\}$. This implies

$$URD(v; P_3, P_4) \supseteq \{(6, 2) + \frac{v-12}{24} * \{(18, 0), (9, 8), (0, 16)\}\}.$$

Since $(\frac{v-12}{24}) * \{(18, 0), (9, 8), (0, 16)\} = \{(9x, 8\frac{v-12}{12} - 8x), x = 0, \ldots, \frac{v-12}{12}\}$, it easy to see that $\{(6, 2) + \frac{v-12}{24} * \{(18, 0), (9, 8), (0, 16)\}\} = J(v)$. This completes the proof.

□

We are now in a position to prove the main result of the paper.

Theorem 4.3. *For every* $v \equiv 0 \pmod{12}$, *we have* $URD(v; P_3, P_4) = J(v)$.

Proof. Necessity follows from Lemma 2.1. Sufficiency follows from Lemmas 4.1 and 4.2. This completes the proof.

□

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