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The representability number of a chain

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Abstract

For each pair of linear orderings (L, M), the representability number $\mathrm{repr}_M(L)$ of L in M is the least ordinal α such that L can be order-embedded into the lexicographic power $M^{\alpha}_{\mathrm{lex}}$. The case $M = \mathbb{R}$ is relevant to utility theory. The main results in this paper are as follows. (i) If κ is a regular cardinal that is not order-embeddable in M, then $\mathrm{repr}_M(\kappa) = \kappa$; as a consequence, $\mathrm{repr}_{\mathbb{R}}(\kappa) = \kappa$ for each $\kappa \geqslant \omega_1$. (ii) If M is an uncountable linear ordering with the property that $A \times_{\mathrm{lex}} 2$ is not order-embeddable in M for each uncountable $A \subseteq M$, then $\mathrm{repr}_M(M^{\alpha}_{\mathrm{lex}}) = \alpha$ for any ordinal α; in particular, $\mathrm{repr}_{\mathbb{R}}(\mathbb{R}^{\alpha}_{\mathrm{lex}}) = \alpha$. (iii) If L is either an Aronszajn line or a Souslin line, then $\mathrm{repr}_{\mathbb{R}}(L) = \omega_1$.

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1. Introduction

In this paper we deal with representations of linear orderings (also called chains) in ways that are useful in the field of mathematical economics called *utility theory* (see [6] for an overview of this topic). A key notion in utility theory is that of representability: a chain $(L, \prec)$ is *representable* (in $\mathbb{R}$) if there exists a map $u : L \to \mathbb{R}$, called a *utility function*,

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which is an order-embedding (i.e., $x \prec y$ if and only if $u(x) < u(y)$ for all $x, y \in L$). If we interpret $x \prec y$ as "y is preferred to x", then a utility function on L measures preferences quantitatively. In the traditional approach much attention has been given to characterizations of representable chains. A well-known result in this sense is the following (see, e.g., [2]). (Recall that a *jump* in a chain L is a pair $(a, b) \in L^2$ such that $a \prec b$ and the open interval (a, b) is empty.)

Theorem 1.1. *A chain is representable* (*in* $\mathbb{R}$) *if and only if it is separable in the order topology and has at most countably many jumps.*

A more recent approach to the problem of representability focuses on finding structural obstructions to the representability of a chain among its subchains (see [1,3]). Classical examples of chains for which representability fails are the real plane endowed with the lexicographic order $\mathbb{R}^2_{\mathrm{lex}}$, the first uncountable ordinal ω_1 and its reverse ordering ${\omega_1}^*$. Recall that a chain L is *short* if neither ω_1 nor ${\omega_1}^*$ order-embed into L, and it is *long* otherwise; further, an *Aronszajn line* is an uncountable chain that is short and does not contain any uncountable representable subchain. The next result (from [1]) gives a subordering characterization of non-representable chains.

Theorem 1.2. *A chain L is non-representable* (*in* $\mathbb{R}$) *if and only if* (i) *it is long, or* (ii) *it order-embeds a non-representable subchain of the lexicographic plane, or* (iii) *it order-embeds an Aronszajn line.*

Our objective is to give a more descriptive classification of non-representable chains (and, more generally, of all chains). In this paper we begin to pursue this goal by classifying chains according to a measure of their "lexicographic complexity". To this aim we take the point of view that a chain which can be order-embedded in the lexicographically ordered real plane is representable, even if in a weaker sense. Such an ordering is realized in a way that is more complex than for suborderings of $\mathbb{R}$, but which still fits within the general utility concept. This is based on the observation that an order-embedding of $(L, \prec)$ into $\mathbb{R}^2_{\mathrm{lex}}$ corresponds to two functions $u_1, u_2 : L \to \mathbb{R}$ with the property that for all $x, y \in L$, we have $x \prec y$ if and only if either $u_1(x) < u_1(y)$, or $u_1(x) = u_1(y)$ and $u_2(x) < u_2(y)$. In other words, preference in the sense of L corresponds to preference according to u_1 and u_2 together, but with u_1 being given higher priority.

More generally, we say that a chain $(L, \prec)$ is *α-representable* (in $\mathbb{R}$) if it can be order-embedded into the lexicographic power $\mathbb{R}^\alpha_{\mathrm{lex}}$, where α is an ordinal number. This corresponds to having a representation of the preference ordering $\prec$ by a well-ordered family of utility functions $u_\xi : L \to \mathbb{R}$ indexed by the ordinals $\xi < \alpha$; for any $x, y \in L$ one has $x \prec y$ if and only if $u_\beta(x) < u_\beta(y)$ holds, where β is the least ordinal below α at which $u_\beta(x)$ and $u_\beta(y)$ differ. One can think of the ordinal indices as determining the relative importance of the utility functions u_ξ.

The least ordinal α for which a chain L is α-representable is called the *representability number of L* (in $\mathbb{R}$). More generally, for any pair of chains (L, M), we define the *representability number of L in M* as the least ordinal α such that L can be order-embedded into M^α_{lex}; this ordinal is denoted by $\mathrm{repr}_M(L)$. In this paper we determine $\mathrm{repr}_M(L)$ for some pairs of chains (L, M). Our goal is to classify chains that are non-representable in $\mathbb{R}$; thus, we focus on the case $M = \mathbb{R}$.

Long chains are not α-representable (in $\mathbb{R}$) for any countable ordinal α (see [4]). Therefore the family of all chains can be partitioned in the following three disjoint classes: (i) long chains; (ii) short chains with uncountable representability number; (iii) chains with countable representability number. Surprisingly, class (ii) is very rich in variety. In fact, there exists a hierarchy of short chains that do not embed an Aronszajn line, and yet have uncountable representability number (see [8, Chapter 5]). Further, some chains in this class are rather complicated: for example, in this paper we prove that Aronszajn lines belong to class (ii).

The paper is organized as follows. In Section 2 we introduce some basic terminology and prove some easy results for lexicographic products. In Section 3 we study the representability of cardinal numbers; for example, we show that if κ is a regular cardinal that is not order-embeddable in M, then $\mathrm{repr}_M(\kappa) = \kappa$. In Section 4 we prove that if M is an uncountable chain such that $A \times_{\mathrm{lex}} 2$ is not order-embeddable in M for each uncountable set $A \subseteq M$, then $\mathrm{repr}_M(M^{\alpha}_{\mathrm{lex}}) = \alpha$ for any ordinal α; thus, $\mathrm{repr}_{\mathbb{R}}(\mathbb{R}^{\alpha}_{\mathrm{lex}}) = \alpha$ for each ordinal α. Finally, in Section 5 we use the (known) technique of lexicographic linearization of a tree to prove some facts about order-homomorphisms of lexicographically ordered ω_1-trees; then we deduce that the representability number in $\mathbb{R}$ of an Aronszajn line and of a Souslin line is ω_1.

2. Preliminaries

By $\mathbb{R}$ and $\mathbb{Q}$ we mean the chains $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$, respectively; the chain $(\mathbb{N}, <)$ can be denoted either by $\mathbb{N}$ or by the ordinal number ω. As usual, an ordinal α is identified with the set of all ordinals below it. A cardinal is an initial ordinal, and the first cardinal greater than a cardinal κ is denoted by κ^+. Thus, for example, $|\alpha|^+$ denotes the first cardinal greater than the cardinality of the ordinal α. The unique chain with exactly one element is denoted by $\mathbf{1}$. Further, for any chain L, the symbol L^* denotes the reverse ordering of L. For all undefined set-theoretic notions the reader is referred to [9].

Let $(L, \prec)$ and $(M, \prec)$ be two chains. A map $f : L \to M$ such that $x \prec y$ implies $f(x) \preceq f(y)$ for all $x, y \in L$ is said to be an *order-homomorphism* (or, simply, a *homomorphism*). In particular, an *embedding* (respectively, *isomorphism*) is an injective (respectively, bijective) homomorphism. The notation $L \hookrightarrow M$ stands for embeddability of the chain L into the chain M, whereas $L \cong M$ denotes the existence of an isomorphism between L and M. For operations and basic properties of linear orderings the reader is referred to [12].

Next we recall the definitions of some cardinal invariants for a chain $(L, \prec)$. The *density* $\mathrm{d}(L)$ of L is the density of the topological space $(L, \tau_\prec)$, where $\tau_\prec$ is the order topology induced by $\prec$. The *perfect density* $\mathrm{d}'(L)$ of L is the least infinite cardinal κ such that there exists $D \subseteq L$, which has size $\leqslant \kappa$ and intersects every closed interval in L containing at least two points; in particular, L is *perfectly separable* if $\mathrm{d}'(L) = \omega$. Note that $(L, \prec)$ is perfectly separable if and only if it is representable if and only if $(L, \tau_\prec)$ is second countable. A chain is *dense-in-itself* if it has no jumps. The set of jumps in L is denoted by $\mathrm{Jump}(L)$; further, we let $\mathrm{j}(L) = |\mathrm{Jump}(L)|$. The *cellularity* $\mathrm{c}(L)$ of L is the least infinite cardinal κ such that every family of pairwise disjoint nonempty open intervals of L has cardinality $\leqslant \kappa$; in particular, L has the *c.c.c.* (*countable chain condition*) if $\mathrm{c}(L) = \omega$. A *Souslin line* is a chain that has the c.c.c. but is not separable; the existence of Souslin lines is independent from the usual axioms of set theory (see [9]). Note that for any chain L, we have $\mathrm{c}(L) \leqslant \mathrm{d}(L) \leqslant (\mathrm{c}(L))^+$ and $\mathrm{d}(L) \leqslant \mathrm{d}'(L)$; in particular, a chain that does not satisfy the c.c.c. is not representable. All chains that have the c.c.c. are short (e.g., $\mathbb{R}$ and Souslin lines); on the other hand, there exist chains that are short, yet they do not satisfy the c.c.c. (e.g., some Aronszajn lines).

Let $(L_i, \prec)_{i \in I}$ be a family of chains indexed by a well-ordered set $(I, <)$. The *lexicographic product* of this family is the chain $(\prod_{i \in I} L_i, \prec_{\mathrm{lex}})$, where the relation of total order is defined as follows: for each $x = (x_i)_{i \in I},\ y = (y_i)_{i \in I} \in \prod_{i \in I} L_i$, let $x \prec_{\mathrm{lex}} y$ if there exists an index $j \in I$ with the property that $x_j \prec y_j$ and for each $i \in I$ such that $i < j$, $x_i = y_i$; this chain is denoted by $\prod_{i \in I}^{\mathrm{lex}} L_i$. For any $j \in I$, denote by $\pi_j : \prod_{i \in I}^{\mathrm{lex}} L_i \to L_j$ the projection onto the jth component; observe that if $j \neq \min I$, then π_j fails in general to be a homomorphism. Further, for $j \neq \min I$, let $\hat{\pi}_j : \prod_{i \in I}^{\mathrm{lex}} L_i \to \prod_{i<j}^{\mathrm{lex}} L_i$ be the projection onto the first j components (which is always a homomorphism). If the well-ordered set I is an ordinal α, the corresponding lexicographic product is denoted by $\prod_{\xi<\alpha}^{\mathrm{lex}} L_\xi$; in particular, the lexicographic product of the two chains L and M is denoted by $L \times_{\mathrm{lex}} M$. Further, the lexicographic power $(L^\alpha, \prec_{\mathrm{lex}}) = \prod_{\xi<\alpha}^{\mathrm{lex}} L$ is denoted by L^α_{lex}; in particular, $L^1_{\mathrm{lex}} = L$ and $L^0_{\mathrm{lex}} = \mathbf{1}$. The empty set is a chain (it is the ordinal 0), but in this paper we assume that all chains are nonempty. The next result collects some simple facts about lexicographic products.

Lemma 2.1. *Let Z be a chain, and $(L_\xi)_{\xi<\alpha}$, $(M_\xi)_{\xi<\alpha}$ two families of chains indexed by an ordinal α. We have*:

(i) $\prod_{\xi<\alpha}^{\mathrm{lex}} Z^{\beta_\xi}_{\mathrm{lex}} \cong Z^{\gamma}_{\mathrm{lex}}$, *where $(\beta_\xi)_{\xi<\alpha}$ is a family of ordinals and γ their ordinal sum*;
(ii) $L_\xi \hookrightarrow M_\xi$ *for all $\xi < \alpha$ implies* $\prod_{\xi<\alpha}^{\mathrm{lex}} L_\xi \hookrightarrow \prod_{\xi<\alpha}^{\mathrm{lex}} M_\xi$;
(iii) *for any $I \subseteq \alpha$,* $\prod_{i \in I}^{\mathrm{lex}} L_i \hookrightarrow \prod_{\xi<\alpha}^{\mathrm{lex}} L_\xi$.

Now we introduce the notion of representability number of a chain relative to another chain.

Definition 2.2. Let L and M be chains, with $|M| \geqslant 2$. For any ordinal α, we say that L is *α-representable in M* if L can be embedded into the lexicographic power M^α_{lex}; the chain M is called the *base* of the representation. The *representability number of L in M* is the least ordinal α such that L is α-representable in M; this ordinal is denoted by $\mathrm{repr}_M(L)$. The representability number of L in $\mathbb{R}$ is simply called the *representability number of L* and is denoted by $\mathrm{repr}(L)$.

Whenever we write $\mathrm{repr}_M(L)$, we assume that the base M of the representation is a chain with at least two elements. Observe that $\mathrm{repr}_M(L) = 0$ if and only if $L = \mathbf{1}$. Further, if $N \hookrightarrow M$ then $\mathrm{repr}_M(L) \leqslant \mathrm{repr}_N(L)$; in particular, $\mathrm{repr}_M(L) \leqslant \mathrm{repr}_2(L)$ for each M. The next result ensures that $\mathrm{repr}_M(L)$ is always well-defined.

Lemma 2.3. *For all chains L and M,* $\mathrm{repr}_M(L) \leqslant \mathrm{repr}_2(L) \leqslant \min\{\mathrm{d}'(L), \mathrm{d}(L)+1\}$.

Proof. Since L embeds into $2_{\mathrm{lex}}^{\mathrm{d}(L)+1}$ (see [5]), it suffices to prove that L embeds into $2_{\mathrm{lex}}^{\mathrm{d}'(L)}$. Let D be a perfectly dense subset of L such that $|D| = \mathrm{d}'(L) = \kappa$, and let $f : \kappa \to D$ be a bijection. It is enough to show that $L \hookrightarrow 3_{\mathrm{lex}}^{\kappa}$. Define a map $\iota : L \to 3_{\mathrm{lex}}^{\kappa}$ by

$$\iota(x)(\alpha) := \begin{cases} 0 & \text{if } x \prec f(\alpha), \\ 1 & \text{if } x = f(\alpha), \\ 2 & \text{if } x \succ f(\alpha) \end{cases}$$

where $x \in L$ and $\alpha \in \kappa$. The map ι is an embedding. □

The case in which the base of the representation is $\mathbb{R}$ is relevant in applications to economics. In fact, $\mathrm{repr}(L) \leqslant 1$ if and only if L is representable in the sense of utility theory.

Example 2.4. We have:

(i) $\mathrm{repr}(\mathbb{Q}_{\mathrm{lex}}^{\omega}) = 1$;
(ii) $\mathrm{repr}(\mathbb{R} \times_{\mathrm{lex}} 2) = 2$;
(iii) $\mathrm{repr}(\omega_1) = \mathrm{repr}(\omega_1^*) = \omega_1$.

Parts (i) and (ii) are a consequence of Theorem 1.1; in fact, $\mathbb{Q}_{\mathrm{lex}}^{\omega}$ is separable and has no jumps, whereas $\mathbb{R} \times_{\mathrm{lex}} 2$ has uncountably many jumps. For (iii), see [4].

Example 2.5. Let $\prod_{\xi<\alpha}^{\mathrm{lex}} L_\xi$ be the lexicographic product of the family of chains $(L_\xi)_{\xi<\alpha}$, where $\alpha \geqslant 1$ and for each $\xi < \alpha$, $L_\xi \neq \mathbf{1}$. Then, $\mathrm{repr}(\prod_{\xi<\alpha}^{\mathrm{lex}} L_\xi) = 1$ if and only if either (i) $\alpha \leqslant \omega$ and L_ξ is countable for each $\xi < \alpha$, or (ii) $\alpha < \omega$, L_ξ is countable for each $\xi < \alpha - 1$, and $L_{\alpha-1}$ is uncountable but representable (see [7]).

In the remainder of this section we prove some miscellaneous facts about the representability number. We begin with some results related to reverse orderings.

Lemma 2.6. *Let L and M be chains. We have*:

(i) *for each ordinal α, $(L_{\mathrm{lex}}^{\alpha})^* = (L^*)_{\mathrm{lex}}^{\alpha}$*;
(ii) $\mathrm{repr}_M(L) = \mathrm{repr}_{M^*}(L^*)$.

Proof. The underlying sets of $(L_{\mathrm{lex}}^{\alpha})^* = (L^{\alpha}, (\prec_{\mathrm{lex}})^*)$ and of $(L^*)_{\mathrm{lex}}^{\alpha} = ((L, \prec^*)^{\alpha}, \prec_{\mathrm{lex}}) = (L^{\alpha}, (\prec^*)_{\mathrm{lex}})$ are the same. It is easy to show that the orders $(\prec_{\mathrm{lex}})^*$ and $(\prec^*)_{\mathrm{lex}}$ coincide. Thus (i) holds. Part (ii) is a consequence of (i). □

If $(Z_i)_{i\in I}$ is a family of chains indexed by a chain $(I,<)$, then the *sum of* $(Z_i)_{i\in I}$ is the chain $(\bigcup_{i\in I}\{i\}\times Z_i,\prec)$, where the order is defined as follows: for each $(j,z_j),(k,z_k)\in\bigcup_{i\in I}\{i\}\times Z_i$, let $(j,z_j)\prec(k,z_k)$ if either $j<k$ or $j=k$ and $z_j\prec z_k$ in Z_j. This chain is denoted by $\sum_{i\in I}Z_i$. Note that a lexicographic product of two chains can be written as a sum of chains; namely, $L\times_{\mathrm{lex}}M=\sum_{x\in L}M_x$, where $M_x:=M$ for each $x\in L$.

Lemma 2.7. *Let $L=\sum_{i\in I}Z_i$ and $M=\sum_{i\in I^*}Z_i$, where I and $(Z_i)_{i\in I}$ are chains. Then L embeds into $I\times_{\mathrm{lex}}M$. In particular, if I embeds into M, then* $\mathrm{repr}_M(L)\leqslant 2$.

Proof. The map $\varphi:L\to I\times_{\mathrm{lex}}M$, defined by $\varphi(i,z_i):=(i,(i,z_i))$ for each $(i,z_i)\in L$, is an embedding. □

The next result gives an upper bound to the representability number of lexicographic products.

Lemma 2.8. *For any family of chains $(L_\xi)_{\xi<\alpha}$,* $\mathrm{repr}_M(\prod^{\mathrm{lex}}_{\xi<\alpha}L_\xi)\leqslant\sum_{\xi<\alpha}\mathrm{repr}_M(L_\xi)$.

Proof. The statement is a consequence of Lemma 2.1. □

The equality $\mathrm{repr}_M(\prod^{\mathrm{lex}}_{\xi<\alpha}L_\xi)=\sum_{\xi<\alpha}\mathrm{repr}_M(L_\xi)$ does not hold in general.

Example 2.9. Let $L:=\mathbb{R}\times_{\mathrm{lex}}2$. By Example 2.4, $\mathrm{repr}(L)+\mathrm{repr}(L)=4$. On the other hand, $L^2_{\mathrm{lex}}\hookrightarrow\mathbb{R}\times_{\mathrm{lex}}L$ and so $\mathrm{repr}(L^2_{\mathrm{lex}})\leqslant 3$. (In fact, $\mathrm{repr}(L^2_{\mathrm{lex}})=3$, see Example 2.11.)

We conclude the section by determining the representability number for some pairs of chains.

Proposition 2.10. *Let L and M be chains, and let Z be an uncountable chain that is dense-in-itself and has the c.c.c. For any homomorphism $f:Z\times_{\mathrm{lex}}L\to Z\times_{\mathrm{lex}}M$, there exist a co-countable set $A\subseteq Z$, a homomorphism $g:A\to Z$, and a family of homomorphisms $(h_a:L\to M)_{a\in A}$ such that $f(a,l)=(g(a),h_a(l))$ for each $(a,l)\in A\times L$. Further, if f is an embedding, then we may also require that h_a is an embedding for each $a\in A$.*

Proof. Let $f:Z\times_{\mathrm{lex}}L\to Z\times_{\mathrm{lex}}M$ be a homomorphism. Denote by $f_0:Z\times_{\mathrm{lex}}L\to Z$ the homomorphism $f_0=\pi_0\circ f$, where $\pi_0:Z\times_{\mathrm{lex}}M\to Z$ is the projection onto the first component. Consider the following subset of Z:

$$A:=\{a\in Z:\ f_0\restriction\{a\}\times L\text{ is constant}\}.$$

We claim that $Z\setminus A$ is countable. Indeed, if $z\in Z\setminus A$, then $f_0[\{z\}\times L]$ is a subset of Z containing more than one point. Let U_z denote the interior of the convex hull of $f_0[\{z\}\times L]$. Observe that for each $z\in Z$, U_z is nonempty, because Z is dense-in-itself. Further, if x and y are two distinct points of $Z\setminus A$, then $|f_0[\{x\}\times L]\cap f_0[\{y\}\times L]|\leqslant 1$, whence $U_x\cap U_y$ is empty. Thus, $\mathcal{U}:=\{U_z:\ z\in Z\setminus A\}$ is a set of nonempty pairwise disjoint open sets in Z. Since Z has the c.c.c., it follows that $\mathcal{U}$ must be countable. This proves the claim.

Note that $f_0 \restriction A \times_{\text{lex}} L$ depends only on the first component. Thus, if $(a, l) \in A \times L$, then the map $g : A \to Z$ given by $g(a) := f_0(a, l)$ is a well-defined homomorphism. Next observe that for each $(a, l) \in A \times L$, if $f(a, l) = (z, m) \in Z \times M$, then $g(a) = f_0(a, l) = z$. Therefore, for any fixed $a \in A$, we can define a map $h_a : L \to M$ by $h_a(l) := m$, where $m \in M$ is such that the equality $f(a, l) = (g(a), m)$ holds. The function h_a is a homomorphism for each $a \in A$. Finally, if f is injective, then so is its restriction $f \restriction \{a\} \times L$. Thus, since $f(a, l) = (g(a), h_a(l))$ for each $l \in L$, it follows that also h_a is an embedding. □

Example 2.11. $\operatorname{repr}(\mathbb{R} \times_{\text{lex}} 2)^2 = 3$. By Example 2.9, it suffices to show that $\operatorname{repr}(\mathbb{R} \times_{\text{lex}} 2)^2 > 2$. Otherwise, we have $(\mathbb{R} \times_{\text{lex}} 2)^2_{\text{lex}} \hookrightarrow \mathbb{R}^2_{\text{lex}}$, hence Proposition 2.10 yields that $2 \times_{\text{lex}} \mathbb{R} \times_{\text{lex}} 2 \hookrightarrow \mathbb{R}$, which is a contradiction.

Corollary 2.12. *If Z is an uncountable chain that is dense-in-itself and has the c.c.c., then* $\operatorname{repr}_Z(Z^\alpha_{\text{lex}}) = \alpha$ *for each ordinal $\alpha \leqslant \omega$.*

Proof. The equality $\operatorname{repr}_Z(Z^n_{\text{lex}}) = n$ can be proved by induction on $n < \omega$, using Proposition 2.10. To prove that $\operatorname{repr}(Z^\omega_{\text{lex}}) = \omega$, assume by contradiction that $\operatorname{repr}(Z^\omega_{\text{lex}}) = n < \omega$. It follows that $Z^{n+1}_{\text{lex}} \hookrightarrow Z^\omega_{\text{lex}} \hookrightarrow Z^n_{\text{lex}}$, which contradicts $\operatorname{repr}_Z(Z^{n+1}_{\text{lex}}) = n + 1$. □

In particular, Corollary 2.12 yields that for each $\alpha \leqslant \omega$, $\operatorname{repr}(\mathbb{R}^\alpha_{\text{lex}}) = \alpha$ and $\operatorname{repr}_S(S^\alpha_{\text{lex}}) = \alpha$, where S is a dense-in-itself Souslin line (cf. [10, Corollary 2.4]). These results will be strengthened later (see Corollary 4.14).

3. Representability of cardinal numbers

In this section we deal with special types of homomorphisms, which are useful to study the representability of cardinal numbers. In particular, we prove that if κ is a regular cardinal that does not embed into M, then $\operatorname{repr}_M(\kappa) = \kappa$. As a consequence, if M is a short chain and κ is an uncountable cardinal, then $\operatorname{repr}_M(\kappa) = \kappa$.

Definition 3.1. Let X be an infinite set. A set $X' \subseteq X$ is *small in X* if $|X'| < |X|$; it is *co-small in X* if its complement is small in X. We use the notation $X' \circledast X$ to indicate that X' is a subset of X that is co-small in X. A homomorphism $f : L \to M$ from an infinite chain L into a chain M is *almost-constant* if there exists $L' \circledast L$ such that $f \restriction L'$ is constant.

Lemma 3.2. *A homomorphism $f : \kappa \to M$ from an infinite cardinal κ into a chain M is almost-constant if and only if it is eventually constant.*

Proof. If $A \circledast \kappa$ is such that $f \restriction A$ is constant, then f is constant on the convex hull of A. It follows that f is eventually constant. Conversely, if $\alpha < \kappa$ is such that $f \restriction (\kappa \setminus \alpha)$ is constant, then $\kappa \setminus \alpha$ is co-small in κ. Thus f is almost-constant. □

Definition 3.3. Let L and M be two chains, where $|L| \geqslant \omega$ and $|M| \geqslant 2$. The pair (L, M) is called an *a.c.-pair* (*almost-constant pair*) if any homomorphism $f : L \to M$ is almost-constant. Further, (L, M) is called an *h.a.c.-pair* (*hereditary almost-constant pair*) if for all $L' \circledast L$, (L', M) is an a.c.-pair.

We first analyze pairs of chains of the type $(L, 2)$. If X and Y are subsets of a chain $(L, \prec)$, the notation $X \prec Y$ means that $x \prec y$ for each $x \in X$ and $y \in Y$.

Lemma 3.4. *The following statements are equivalent for an infinite chain* L:

(i) $(L, 2)$ *is an a.c.-pair*;
(ii) $(L, 2)$ *is an h.a.c.-pair*;
(iii) *there exists no partition* $L = X \cup Y$ *of* L *such that* $X \prec Y$ *and* $|X| = |Y|$.

Proof. The proof is easy and is left to the reader. □

Example 3.5. We call an infinite ordinal α a *quasi-cardinal* if it is of the form $\alpha = |\alpha| + \gamma$, where $\gamma < |\alpha|$. For each infinite ordinal α, $(\alpha, 2)$ is an a.c.-pair if and only if α is a quasi-cardinal. In particular, $(\kappa, 2)$ is an a.c.-pair for each cardinal $\kappa \geqslant \omega$.

More generally, let α and β be infinite ordinals. Then $(\alpha + \beta^*, 2)$ is an a.c.-pair if and only if either (i) $|\alpha| > |\beta|$ and α is a quasi-cardinal, or (ii) $|\alpha| < |\beta|$ and β is a quasi-cardinal.

Example 3.6. Let L_0 and L_1 be disjoint subsets of a chain $(L, \prec)$. We say that L_0 and L_1 are *mutually cofinal* (respectively, *mutually coinitial*) if for each $x_0 \in L_0$ and $x_1 \in L_1$, there exist $x_0' \in L_0$ and $x_1' \in L_1$ such that $x_0 \prec x_1'$ and $x_1 \prec x_0'$ (respectively, $x_1' \prec x_0$ and $x_0' \prec x_1$). Furthermore, if $\mathcal{F} = (L_\xi)_{\xi<\gamma}$ is a family of pairwise disjoint subsets of $(L, \prec)$ such that any two chains in $\mathcal{F}$ are mutually cofinal (respectively, mutually coinitial), then we say that $\mathcal{F}$ is a *mutually cofinal family* (respectively, *mutually coinitial family*) of subsets of L.

Let $(\alpha_\xi)_{\xi<\gamma}$ be a family of infinite ordinals and L a chain for which there exists a partition $L = \bigcup_{\xi<\gamma} L_\xi$ such that for each $\xi < \gamma$, either $L_\xi = \alpha_\xi$ or $L_\xi = {\alpha_\xi}^*$. Assume that $\mathrm{cf}(|L|) > \gamma$. Set $A := \{\xi < \gamma\colon L_\xi = \alpha_\xi \wedge |L_\xi| = |L|\}$ and $B := \{\xi < \gamma\colon L_\xi = {\alpha_\xi}^* \wedge |L_\xi| = |L|\}$. (Note that $A \cup B$ is nonempty.) Then $(L, 2)$ is an a.c.-pair if and only if one of the following two conditions holds: (i) $B = \emptyset$, $A \neq \emptyset$, α_ξ is a quasi-cardinal for each $\xi \in A$, and $(|\alpha_\xi|)_{\xi\in A}$ is a mutually cofinal family of subsets of L; (ii) $A = \emptyset$, $B \neq \emptyset$, α_ξ is a quasi-cardinal for each $\xi \in B$, and $(|\alpha_\xi|^*)_{\xi\in B}$ is a mutually coinitial family of subsets of L.

An h.a.c.-pair is an a.c.-pair, but the converse does not hold in general.

Example 3.7. $(\omega + 1, \omega)$ is an a.c.-pair, which fails to be an h.a.c.-pair. $(\omega_1, \mathbb{R})$ is an h.a.c.-pair.

Under certain conditions on L, the pair (L, M) is an a.c.-pair if and only if it is an h.a.c.-pair.

Definition 3.8. An infinite chain L is *almost-reflexive* if for each $L' \Subset L$ there exists $L'' \Subset L$ such that $L'' \subseteq L'$ and L'' is a homomorphic image of L.

Example 3.9. All infinite cardinals are almost-reflexive in a strong sense. In fact, if κ is an infinite cardinal and $B \subseteq \kappa$ is unbounded in κ (in particular, if B is co-small in κ), then the map $f : \kappa \to B$, defined by $\alpha \mapsto \min\{\beta \in B\colon \alpha \leqslant \beta\}$, is a homomorphism of κ onto B. On the other hand, all quasi-cardinals fail to be almost-reflexive.

Note that if L is almost-reflexive and L' is co-small in L, then L' is almost-reflexive.

Lemma 3.10. *Assume that L is almost-reflexive. For each chain M, (L, M) is an a.c.-pair if and only if it is an h.a.c.-pair.*

Proof. Assume that (L, M) is an a.c.-pair and let L' be a co-small subset of L. To prove the claim, it suffices to show that (L', M) is an a.c.-pair. Let $g : L' \to M$ be a homomorphism. By hypothesis there exists a homomorphism $f : L \to L'$ such that $\operatorname{ran} f \Subset L'$. Since the homomorphism $g \circ f : L \to M$ is almost-constant, it follows that g is almost-constant as well. This shows that (L', M) is an a.c.-pair. □

Before stating the main results of this section, we prove some technical facts.

Lemma 3.11. *If $(L, 2)$ is an a.c.-pair and M is a chain such that $|M| < \mathrm{cf}(|L|)$, then (L, M) is an h.a.c.-pair.*

Proof. Let $(L, \prec)$ be an infinite chain and assume that there exists a chain M, with $2 \leqslant |M| < \mathrm{cf}(|L|)$, such that (L, M) is not an h.a.c.-pair; we show that $(L, 2)$ fails to be an a.c.-pair. By hypothesis, there exist $L' \Subset L$ and a homomorphism $f : L' \to M$ such that for any $m \in M$, $f^{-1}\{m\}$ is not co-small in L'. Set

$$P := \{m \in M\colon \, |f^{-1}\{m\}| = |L'|\}.$$

Then P is nonempty, because $L' = \bigcup_{m \in M} f^{-1}\{m\}$ and $|M| < \mathrm{cf}(|L|) = \mathrm{cf}(|L'|)$. Select $p \in P$, and denote $L_0 := \{l \in L\colon \{l\} \prec f^{-1}\{p\}\}$ and $L_1 := \{l \in L\colon \{l\} \succ f^{-1}\{p\}\}$. Observe that $L = L_0 \cup f^{-1}\{p\} \cup L_1$, $|f^{-1}\{p\}| = |L|$ and $|L \setminus f^{-1}\{p\}| = |L|$. It follows that either $|L_0| = |L|$ or $|L_1| = |L|$; without loss of generality, assume that $|L_1| = |L|$. Set $X := L_0 \cup f^{-1}\{p\}$ and $Y := L_1$. Then $L = X \cup Y$ is a partition of L such that $X \prec Y$ and $|X| = |Y|$, and so Lemma 3.4 yields that $(L, 2)$ is not an a.c.-pair. □

Lemma 3.12. *If (L, M) is an h.a.c.-pair and α is an ordinal such that $\alpha < \mathrm{cf}(|L|)$, then $(L, M^{\alpha}_{\mathrm{lex}})$ is an h.a.c.-pair.*

Proof. Assume that (L, M) is an h.a.c.-pair and α is an ordinal such that $0 < \alpha < \mathrm{cf}(|L|)$. Let L' be co-small in L and $f : L' \to M^{\alpha}_{\mathrm{lex}}$ a homomorphism; we shall find $L'' \Subset L'$ such that $f \restriction L''$ is constant. For each $\beta < \alpha$, let $f_\beta = \pi_\beta \circ f : L' \to M$, where $\pi_\beta : M^{\alpha}_{\mathrm{lex}} \to M$ is the projection onto the βth component. Note that if $A \subseteq L'$ is such that $f_\gamma \restriction A$ is constant for each $\gamma < \beta$, then $f_\beta \restriction A$ is a homomorphism. In the following we define by recursion a decreasing sequence $(L'_\gamma)_{\gamma<\alpha}$ of subsets of L' such that for each $\gamma<\alpha$, the following two properties hold: (a) L'_γ is co-small in L'; (b) $f_\gamma \restriction L'_\gamma$ is constant.

To build the sequence, observe that the map $f_0 : L' \to M$ is a homomorphism defined on a co-small subset of L, hence it is almost-constant by hypothesis; thus, there exists $L'_0 \circledast L'$ such that $f_0 \restriction L'_0$ is constant. Next, assume that L'_γ satisfying (a) and (b) has been constructed. Since the restriction $f_{\gamma+1} \restriction L'_\gamma$ is a homomorphism, the hypothesis implies that there exists a set $L'_{\gamma+1} \circledast L'_\gamma$ such that $f_{\gamma+1} \restriction L'_{\gamma+1}$ is constant; then $L'_{\gamma+1}$ satisfies both (a) and (b). Finally, let $\gamma<\alpha$ be a limit ordinal, and assume that L'_δ satisfying (a) and (b) has been constructed for all $\delta<\gamma$. Observe that $|L' \setminus \bigcap_{\delta<\gamma} L'_\delta| < |L'|$, because $\gamma < \alpha < \mathrm{cf}(|L|) = \mathrm{cf}(|L'|)$. Therefore, the homomorphism $f_\gamma \restriction \bigcap_{\delta<\gamma} L'_\delta$ is almost-constant, and there exists a set $L'_\gamma \subseteq L'$ such that (a) and (b) hold. This completes the definition of the sequence $(L'_\gamma)_{\gamma<\alpha}$.

Set $L'' := \bigcap_{\gamma<\alpha} L'_\gamma$. Since $\alpha < \mathrm{cf}(|L'|)$, property (a) implies that $|L' \setminus L''| < |L'|$, and so $L'' \circledast L'$. Furthermore, property (b) yields that $f \restriction L''$ is constant. This shows that $(L, M^\alpha_{\mathrm{lex}})$ is an h.a.c.-pair. □

Corollary 3.13. *Let L and M be chains, and α an ordinal such that $\alpha < \mathrm{cf}(|L|)$.*

(i) *If L is almost-reflexive and (L, M) is an a.c.-pair, then $(L, M^\alpha_{\mathrm{lex}})$ is an h.a.c.-pair.*
(ii) *If $|M| < \mathrm{cf}(|L|)$ and $(L, 2)$ is an a.c.-pair, then $(L, M^\alpha_{\mathrm{lex}})$ is an h.a.c.-pair.*

Proof. Part (i) follows from Lemmas 3.10 and 3.12, part (ii) from Lemmas 3.11 and 3.12. □

Corollary 3.14. *Let κ be a cardinal, M a chain and α an ordinal such that $\alpha < \mathrm{cf}(\kappa)$. If $\mathrm{cf}(\kappa)$ does not embed into M, then $(\kappa, M^\alpha_{\mathrm{lex}})$ is an h.a.c.-pair.*

Proof. By Example 3.9, κ is almost-reflexive. Further, if $\mathrm{cf}(\kappa)$ does not embed into M, then (κ, M) is an a.c.-pair. Therefore, Corollary 3.13(i) implies that $(\kappa, M^\alpha_{\mathrm{lex}})$ is an h.a.c.-pair. □

Corollary 3.15. *Let β be a quasi-cardinal, M a chain and α an ordinal such that $\alpha < \mathrm{cf}(|\beta|)$. If $|M| < \mathrm{cf}(|\beta|)$, then $(\beta, M^\alpha_{\mathrm{lex}})$ is an h.a.c.-pair.*

Proof. By Example 3.5, $(\beta, 2)$ is an a.c.-pair. The claim follows from Corollary 3.13(ii). □

Corollary 3.16. *Let κ be a regular cardinal and M a chain.*

(i) *If κ does not embed into M, then $\mathrm{repr}_M(\kappa) = \kappa$.*
(ii) *If κ^* does not embed into M, then $\mathrm{repr}_M(\kappa^*) = \kappa$.*

Proof. To prove (i), we argue by contradiction. Assume that $\kappa \not\hookrightarrow M$ but $\mathrm{repr}_M(\kappa) = \alpha < \kappa$. Then $|\alpha|^+$ embeds into M^α_{lex}, and so $(|\alpha|^+, M^\alpha_{\mathrm{lex}})$ fails to be an h.a.c.-pair. By Corollary 3.14, it follows that $|\alpha|^+$ embeds into M. Thus the hypothesis implies that $|\alpha|^+ < \kappa$.

Now another application of Corollary 3.14 yields that $(\kappa, M_{\text{lex}}^{|\alpha|^+})$ is an h.a.c.-pair, which contradicts the fact that κ embeds into M_{lex}^{α}.

For (ii), note that $\kappa^* \not\hookrightarrow M$ implies $\kappa \not\hookrightarrow M^*$. Thus, $\text{repr}_M(\kappa^*) = \text{repr}_{M^*}(\kappa) = \kappa$, using Lemma 2.6 and part (i). □

Corollary 3.16 does not hold for arbitrary cardinals.

Example 3.17. Let M be the chain $\sum_{n\in\omega^*} \omega_n$. Then ω_ω does not embed into M, and yet $\text{repr}_M(\omega_\omega) = 2$, using Lemma 2.7.

Recall that the *well-ordering number* of a chain L, denoted by $\text{wo}(L)$, is the supremum of the set of all cardinals κ such that either κ or κ^* embeds into L. (Thus, L is short if and only if $\text{wo}(L) \leqslant \omega$.) The following weak version of Corollary 3.16 holds for all cardinals.

Corollary 3.18. *Let κ be a cardinal and M a chain. If* $\text{wo}(M) < \kappa$, *then* $\text{repr}_M(\kappa) = \text{repr}_M(\kappa^*) = \kappa$. *In particular,* $\text{repr}(\kappa) = \text{repr}(\kappa^*) = \kappa$ *for each cardinal* $\kappa \geqslant \omega_1$.

Proof. If κ is regular, then the claim follows from Corollary 3.16. Next, let κ be a singular cardinal such that $\text{wo}(M) < \kappa$. To prove that $\text{repr}_M(\kappa) = \kappa$, we argue by contradiction. Assume that $\text{repr}_M(\kappa) = \alpha < \kappa$. Let $(\kappa_\xi)_{\xi<\text{cf}(\kappa)}$ be an increasing transfinite sequence of regular cardinals such that $\sup\{\kappa_\xi\colon \xi < \text{cf}(\kappa)\} = \kappa$. Then there exists $\eta < \text{cf}(\kappa)$ such that $\kappa_\eta > \max\{\text{wo}(M), \alpha\}$. Since κ_η is a regular cardinal $> \text{wo}(M)$, we obtain

$$\text{repr}_M(\kappa) \geqslant \text{repr}_M(\kappa_\eta) = \kappa_\eta > \alpha$$

which contradicts the hypothesis. Therefore $\text{repr}_M(\kappa) = \kappa$. The proof that $\text{repr}_M(\kappa^*) = \kappa$ is similar. □

4. Representability of unsplittable chains

In this section we study homomorphisms between lexicographic products. We show that under certain conditions on the chain M, we have $\text{repr}_M(M_{\text{lex}}^{\alpha}) = \alpha$ for each ordinal α. In particular, we obtain that $\text{repr}(\mathbb{R}_{\text{lex}}^{\alpha}) = \alpha$ and $\text{repr}_S(S_{\text{lex}}^{\alpha}) = \alpha$, where S is a Souslin line with at most countably many jumps. This generalizes to arbitrary ordinals a result obtained at the end of Section 2 (cf. Corollary 2.12).

To begin we recall some basic terminology. A *tree* is a poset $(T, \preceq)$ such that for each $t \in T$, the initial segment $\{x \in T\colon x \prec t\}$ is well-ordered by $\preceq$. A tree is *rooted* if it has a minimum element, called the *root*; all trees considered in this paper are rooted. A *subtree* of T is a subposet $T' \subseteq T$, which is *downward closed* (i.e., for each $t, t' \in T$, if $t \preceq t'$ and $t' \in T'$, then $t \in T'$).

Notation 4.1. Let $L = \prod_{\xi<\alpha}^{\text{lex}} L_\xi$. For each ordinal $\beta \leqslant \alpha$, let

$$L\restriction\beta := \prod_{\xi<\beta}^{\text{lex}} L_\xi.$$

Observe that $L\restriction\alpha = L$. For each $\beta < \alpha$, $b \in L\restriction\beta$ and $x \in L_\beta$, denote by $b^\frown x$ the *concatenation* of b and x, i.e., the element of $L\restriction(\beta+1)$ such that $b^\frown x\restriction\beta = b$ and $b^\frown x(\beta) = x$. Let $L\downarrow$ be the collection of all restrictions of elements of L, i.e.,

$$L\downarrow := \bigcup_{\xi\leqslant\alpha} L\restriction\xi.$$

For each $u, v \in L\downarrow$, we write $u \sqsubseteq v$ if u is a restriction of v (v is an extension of u). Note that $(L\downarrow, \sqsubseteq)$ is a tree.

Let $C \subseteq L\downarrow$. Define the *downward closure* $C\downarrow$ and the *upward closure* $C\uparrow$ of C by

$$C\downarrow := \{u \in L\downarrow\colon\ \exists c \in C (u \sqsubseteq c)\} \quad\text{and}\quad C\uparrow := \{u \in L\downarrow\colon\ \exists c \in C (c \sqsubseteq u)\}.$$

For $C = \{c\}$, we simplify the notation to $c\downarrow$ and $c\uparrow$, respectively. Observe that $(C\downarrow, \sqsubseteq)$ and $(C\downarrow \cup C\uparrow, \sqsubseteq)$ are subtrees of $(L\downarrow, \sqsubseteq)$. A set $C \subseteq L\downarrow$ is *downward closed* if $C = C\downarrow$, i.e., if it is a subtree of $L\downarrow$. The *top* of C is the (possibly empty) set $\partial C := C \cap L$.

For each $\beta \leqslant \alpha$, define on L an equivalence relation $\sim_\beta$ as follows: for all $x, y \in L$, let $x \sim_\beta y$ if $x\restriction\beta = y\restriction\beta$. Thus, each element $b \in L\restriction\beta$ determines an equivalence class in L, namely, $\partial(b\uparrow) = \{x \in L\colon\ x\restriction\beta = b\}$.

The next fact is immediate.

Lemma 4.2. *Let $L = \prod^{\text{lex}}_{\xi<\alpha} L_\xi$, $A \subseteq L$ and $C \subseteq L\downarrow$. Assume that for each $c \in C$, the set $\partial(c\uparrow \cap C)$ is nonempty. Then $\partial C \subseteq A$ if and only if $C \subseteq A\downarrow$.*

Now we define a particular kind of subtree of $L\downarrow$, where $L = \prod^{\text{lex}}_{\xi<\alpha} L_\xi$. We use this notion only when each factor L_ξ is uncountable.

Definition 4.3. Let $(L_\xi)_{\xi<\alpha}$ be a family of uncountable chains, $L = \prod^{\text{lex}}_{\xi<\alpha} L_\xi$ and $C \subseteq L\downarrow$. For each $\beta < \alpha$ and $c \in C \cap (L\restriction\beta)$, define

$$C(c) := \{u \in L_\beta\colon\ c^\frown u \in C\}.$$

We say that C is *nearly-full* if the following conditions hold:

(F.1) C is a nonempty subtree of $L\downarrow$;
(F.2) for each $\beta < \alpha$ and $c \in C \cap (L\restriction\beta)$, the sets $C(c)$ are co-countable in L_β;
(F.3) for each $x \in L$ and limit ordinal $\beta \leqslant \alpha$, if $x\restriction\gamma \in C$ for all $\gamma < \beta$, then $x\restriction\beta \in C$.

Lemma 4.4. *Let $L = \prod^{\text{lex}}_{\xi<\alpha} L_\xi$ be a lexicographic product of uncountable chains, C a nearly-full subtree of $L\downarrow$, β an ordinal $< \alpha$ and c_β an element of $C \cap (L\restriction\beta)$. For each $x \in C(c_\beta)$, there exists $c^x \in \partial C$ such that $c^x\restriction(\beta+1) = c_\beta{}^\frown x$.*

Proof. Fix $x \in C(c_\beta)$. We construct a sequence $(c^x_\gamma)_{\beta<\gamma\leqslant\alpha}$ such that the following conditions are verified: (a) $c^x_{\beta+1} = c_\beta{}^\frown x$; (b) $c^x_\gamma \in C \cap (L\restriction\gamma)$ for all γ such that $\beta < \gamma \leqslant \alpha$; (c) $c^x_\delta = c^x_\gamma\restriction\delta$ for all δ and γ such that $\beta < \delta < \gamma \leqslant \alpha$. The element $c^x := c^x_\alpha \in \partial C$ satisfies the claim.

To start, set $c^x_{\beta+1} := c_\beta{}^\frown x$. For the successor case, assume that c^x_ξ satisfying (a)–(c) has been constructed for all ξ such that $\beta < \xi \leqslant \gamma < \alpha$. Using (F.2), select an element $y \in C(c^x_\gamma)$ and define $c^x_{\gamma+1} := c^x_\gamma{}^\frown y \in C \cap (L \restriction (\gamma + 1))$; by the induction hypothesis, (a)–(c) hold for $c^x_{\gamma+1}$.

Finally, if γ be a limit ordinal such that $\beta < \gamma \leqslant \alpha$, set $c^x_\gamma := \bigcup_{\beta<\xi<\gamma} c^x_\xi$. By (F.3), c^x_γ is a well-defined element of $C \cap (L \restriction \gamma)$ such that if $\beta < \delta < \gamma$ then $c^x_\delta = c^x_\gamma \restriction \delta$. This completes the definition of the sequence. □

In the next result we list some basic properties of nearly-full subtrees.

Lemma 4.5. *Let $L = \prod^{\text{lex}}_{\xi<\alpha} L_\xi$, where each factor L_ξ is an uncountable chain. Further, let C be a nearly-full subtree of $L\downarrow$. We have*:

- (i) *for each $\beta \leqslant \alpha$, the set $C \cap (L \restriction \beta)$ is nonempty*; *in particular, if $\beta > 0$, then $C \cap (L \restriction \beta)$ is uncountable*;
- (ii) $C = (\partial C)\downarrow$;
- (iii) *for each $c \in C \setminus \partial C$, the set $\partial(c\uparrow) \cap \partial C$ is uncountable*;
- (iv) *if $(C_n)_{n\in\omega}$ is a family of nearly-full subtrees of $L\downarrow$, then $\bigcap_{n\in\omega} C_n$ is also nearly-full.*

Proof. To prove (i), observe that the empty function c_0 belongs to $C \cap (L \restriction 0)$. Lemma 4.4 yields that for all $x \in C(c_0)$, there exists $c^x \in \partial C$ such that $c^x(0) = x$. Note that for all $\beta \leqslant \alpha$, $c^x \restriction \beta$ belongs to C. Thus, if $\beta > 0$, then $\{c^x \restriction \beta\colon\ x \in C(c_0)\}$ is an uncountable subset of $C \cap (L \restriction \beta)$.

For (ii), assume that $c_\beta \in C \cap (L \restriction \beta)$ for some $\beta \leqslant \alpha$. By Lemma 4.4, there exists $c \in \partial C$ such that $c \restriction \beta = c_\beta$. Thus $C \subseteq (\partial C)\downarrow$, using Lemma 4.2. The other inclusion follows from the fact that C is downward closed.

For (iii), let $c \in C \setminus \partial C$; thus, $c \in C \cap (L \restriction \beta)$ for some $\beta < \alpha$. By Lemma 4.4, there exists an uncountable set $A_c := \{c^x\colon\ x \in C(c)\} \subseteq \partial C$ such that $c^x \restriction \beta = c$ for all $x \in C(c)$. Thus, $|\partial(c\uparrow) \cap \partial C| \geqslant |A_c| > \omega$.

To prove (iv), let $(C_n)_{n\in\omega}$ be a family of nearly-full subtrees of $L\downarrow$; it suffices to show that (F.2) holds for $D := \bigcap_{n\in\omega} C_n$. Let $\beta < \alpha$ and $d \in D \cap (L \restriction \beta)$. Then, $D(d) = \bigcap_{n\in\omega} C_n(d)$ is co-countable in L_β, because so are all the sets $C_n(d)$. □

Next we introduce a notion of "large" set in a lexicographic product of uncountable chains.

Definition 4.6. Let $L = \prod^{\text{lex}}_{\xi<\alpha} L_\xi$, where each factor L_ξ is an uncountable chain. A set $A \subseteq L$ is *large* in L if there exists $B \subseteq A$ such that $B\downarrow$ is a nearly-full subtree of $L\downarrow$ (equivalently, if there exists a nearly-full subtree $C \subseteq L\downarrow$ such that $\partial C \subseteq A$). We denote by $\mathrm{Large}(L)$ the family of all large subsets of L.

Lemma 4.7. *Let $L = \prod^{\text{lex}}_{\xi<\alpha} L_\xi$, where each factor L_ξ is an uncountable chain. We have*:

- (i) *if A is large in L and C is a nearly-full subtree of $L\downarrow$ contained in $A\downarrow$, then for each $c \in C \setminus \partial C$, the set $\partial(c\uparrow) \cap A$ is uncountable*;

(ii) *if* $(A_n)_{n\in\omega}$ *is a subfamily of* $\mathrm{Large}(L)$ *and for each* $n \in \omega$, C_n *is a nearly-full subtree of* $L\downarrow$ *contained in* $A_n\downarrow$, *then* $(\bigcap_{n\in\omega} A_n)\downarrow \supseteq \bigcap_{n\in\omega} C_n$;
(iii) *the set* $\mathrm{Large}(L)$ *is a* σ*-complete filter on* L.

Proof. Part (i) follows from Lemma 4.5(iii). To prove (ii), for each $n \in \omega$, let C_n be a nearly-full subtree of $L\downarrow$ such that $A_n \supseteq \partial C_n$. Then, $\bigcap_{n\in\omega} A_n \supseteq \bigcap_{n\in\omega}(\partial C_n) = \partial(\bigcap_{n\in\omega} C_n)$, and so $(\bigcap_{n\in\omega} A_n)\downarrow \supseteq (\partial(\bigcap_{n\in\omega} C_n))\downarrow = \bigcap_{n\in\omega} C_n$. For (iii), it suffices to show that if $(A_n)_{n<\omega}$ is a countable subfamily of $\mathrm{Large}(L)$, then $\bigcap_{n\in\omega} A_n \in \mathrm{Large}(L)$. This is a consequence of Lemma 4.5(iv) and part (ii). $\square$

Finally we introduce the notion of unsplittable chains.

Definition 4.8. Let L be an uncountable chain. We say that L is *splittable* if there exists an uncountable set $A \subseteq L$ such that the chain $A \times_{\mathrm{lex}} 2$ embeds into L. A chain is *unsplittable* if it is not splittable.

More generally, let L and M be two uncountable chains. We say that M is L*-splittable* if there exists an uncountable set $A \subseteq L$ such that the chain $A \times_{\mathrm{lex}} 2$ embeds into M; otherwise, M is L*-unsplittable*. An *unsplittable pair* is a pair of uncountable chains (L, M) such that both L is M-unsplittable and M is L-unsplittable.

Note that L is unsplittable if and only if (L, L) is an unsplittable pair.

Example 4.9. *A chain with uncountably many jumps is splittable. In particular,* α *and* α^* *are splittable for any ordinal* $\alpha \geqslant \omega_1$. Let L be a chain such that $\mathrm{j}(L) > \omega$; without loss of generality, assume that $\mathrm{j}(L) = \omega_1$. We claim that there exists a set $\mathcal{F} \subseteq \mathrm{Jump}(L)$ with cardinality $\mathrm{j}(L)$ such that any two jumps in $\mathcal{F}$ have no common endpoint. To prove this, define an equivalence relation $\sim$ on $\mathrm{Jump}(L)$ as follows: for any two jumps (x, y), (v, w) in L, let $(x, y) \sim (v, w)$ if the interval with endpoints x and v is finite. Since each equivalence class is at most countable and $\mathrm{j}(L) = \omega_1$, there are $\mathrm{j}(L)$ equivalence classes. Thus we can select one jump from each equivalence class and form a set $\mathcal{F} \subseteq \mathrm{Jump}(L)$ that satisfies the claim. If we denote $\mathcal{F} := \{(a_\xi, b_\xi)\colon\ \xi < \omega_1\}$, then $A := \{a_\xi \in L\colon\ (a_\xi, b_\xi) \in \mathcal{F}\}$ is an uncountable subset of L. Endow A with the induced order. Then the correspondence $(a_\xi, 0) \mapsto a_\xi$ and $(a_\xi, 1) \mapsto b_\xi$ gives an embedding $A \times_{\mathrm{lex}} 2 \hookrightarrow L$. This proves that L is splittable.

Example 4.10. $\mathbb{R}$ *and any Souslin line with at most countably many jumps are unsplittable*. If $X \subseteq \mathbb{R}$ is an uncountable set, then $X \times_{\mathrm{lex}} 2$ has uncountably many jumps, and so it does not embed into $\mathbb{R}$ by Theorem 1.1; this proves that $\mathbb{R}$ is unsplittable. Similarly, if S is a Souslin line such that $\mathrm{j}(S) \leqslant \omega$ and X is an uncountable subset of S, then $X \times_{\mathrm{lex}} 2$ is not embeddable in S, because S has the c.c.c. and $\mathrm{j}(S)$ is countable.

Observe that there exist Aronszajn lines that are dense-in-themselves and splittable (e.g., $A \times_{\mathrm{lex}} \mathbb{Q}$, where A is any Aronszajn line).

Example 4.11. *The following are unsplittable pairs* (A *is any Aronszajn line*):

(i) $(\omega_1, \mathbb{R})$;
(ii) $(A, \mathbb{R})$;
(iii) (ω_1, A).

For (i), let $Z \subseteq \omega_1$ and $X \subseteq \mathbb{R}$ be uncountable sets. Since $Z \cong \omega_1$, it follows that $Z \times_{\text{lex}} 2 \not\hookrightarrow \mathbb{R}$. On the other hand, $X \times_{\text{lex}} 2 \not\hookrightarrow \omega_1$, because if $f : X \times_{\text{lex}} 2 \hookrightarrow \omega_1$ is an embedding, then $\operatorname{ran} f$ is an uncountable tail of ω_1; thus $X \times_{\text{lex}} 2 \cong \omega_1$, which is impossible. Part (ii) is immediate. The proof of (iii) is similar to that of part (i).

Theorem 4.12. *Let $(L_\xi)_{\xi<\alpha}$ and $(M_\xi)_{\xi<\alpha}$ be two families of uncountable chains such that M_ξ is L_ξ-unsplittable for each $\xi < \alpha$. For any homomorphism $f : \prod^{\text{lex}}_{\xi<\alpha} L_\xi \to \prod^{\text{lex}}_{\xi<\alpha} M_\xi$, there exists $A \in \operatorname{Large}(\prod^{\text{lex}}_{\xi<\alpha} L_\xi)$ such that for each $\beta < \alpha$ and for each $a, a' \in A$, if $a \restriction \beta = a' \restriction \beta$, then $f(a) \restriction \beta = f(a') \restriction \beta$.*

Proof. Set $L := \prod^{\text{lex}}_{\xi<\alpha} L_\xi$ and $M := \prod^{\text{lex}}_{\xi<\alpha} M_\xi$. For each $\beta \leqslant \alpha$, let $f_\beta : L \to M \restriction \beta$ be the homomorphism defined by $f_\beta := \hat{\pi}_\beta \circ f$, where $\hat{\pi}_\beta : M \to M \restriction \beta$ is the projection onto the first β components. Define by transfinite recursion on $\beta \leqslant \alpha$ a sequence of sets $(A_\beta)_{\beta\leqslant\alpha}$ as follows:

$$A_\beta := \bigl\{x \in L \restriction \beta\colon\ \exists y \in M \restriction \beta \bigl(f_\beta\bigl[\partial(x\uparrow)\bigr] = \{y\}\bigr) \wedge \forall \gamma < \beta (x \restriction \gamma \in A_\gamma)\bigr\}.$$

Set $C := \bigcup_{\beta\leqslant\alpha} A_\beta$. Note that $C \subseteq L\downarrow$ and for each $\beta \leqslant \alpha$, $C \cap (L \restriction \beta) = A_\beta$; in particular, $\partial C = A_\alpha$. In the sequel we show that C is a nearly-full subtree of $L\downarrow$.

Property (F.1) is immediate. To prove (F.2), let $\beta < \alpha$ and $c \in A_\beta$; we show that $C(c)$ is co-countable. Note that c is an element of $L \restriction \beta$ such that $f_\gamma \restriction \partial((c \restriction \gamma)\uparrow)$ is constant for each $\gamma \leqslant \beta$. Then, for any $x \in L_\beta$, we have: $x \in C(c)$ if and only if $c^\frown x \in A_{\beta+1}$ if and only if $f_\gamma \restriction \partial(((c^\frown x) \restriction \gamma)\uparrow)$ is constant for each $\gamma \leqslant \beta + 1$ if and only if $f_{\beta+1} \restriction \partial((c^\frown l)\uparrow)$ is constant. It follows that the equality

$$C(c) = \bigl\{x \in L_\beta\colon\ f_{\beta+1} \restriction \partial\bigl((c^\frown x)\uparrow\bigr) \text{ is constant}\bigr\}$$

holds. Now assume by way of contradiction that $C(c)$ is not co-countable; i.e., there exists an uncountable set $R_\beta \subseteq L_\beta$ such that for all $r \in R_\beta$, $f_{\beta+1} \restriction \partial((c^\frown r)\uparrow)$ fails to be constant. Thus, for each $r \in R_\beta$, we can find two elements $y^r = (y^r_\xi)_{\xi<\alpha}$ and $z^r = (z^r_\xi)_{\xi<\alpha}$ in $f[\partial((c^\frown r)\uparrow)] \subseteq M$ such that $y^r \restriction \beta = z^r \restriction \beta$, but $y^r_\beta \prec z^r_\beta$. The correspondence $(r, 0) \mapsto y^r_\beta$ and $(r, 1) \mapsto z^r_\beta$ gives an embedding of $R_\beta \times_{\text{lex}} 2$ into M_β, which contradicts the fact that M_β is L_β-unsplittable.

Finally we show that (F.3) holds. Let $\beta \leqslant \alpha$ be a limit ordinal and $x \in L$ such that for each $\gamma < \beta$, $x \restriction \gamma \in C$. To prove that $x \restriction \beta \in A_\beta$, it suffices to show that $f_\beta \restriction \partial((x \restriction \beta)\uparrow)$ is constant. Assume by contradiction that $f_\beta \restriction \partial((x \restriction \beta)\uparrow)$ is not constant, i.e., there exist $y, z \in f[\partial((x \restriction \beta)\uparrow)] \subseteq M$ such that $y \restriction \beta \neq z \restriction \beta$. Since β is a limit ordinal, there exists $\delta < \beta$ such that $y \restriction \delta \neq z \restriction \delta$. This is impossible, because $x \restriction \delta \in A_\delta$, and so $f_\delta \restriction \partial((x \restriction \delta)\uparrow)$ is constant. This proves that $C \subseteq L\downarrow$ is nearly-full.

Set $A := A_\alpha = \partial C \in \mathrm{Large}(L)$; then A satisfies the claim of the theorem. Indeed, let $a, a' \in A$ and $\beta < \alpha$ be such that $a \restriction \beta = a' \restriction \beta = c \in L \restriction \beta$. Thus $c \in A_\beta$ by definition of A_α, and so $f_\beta \restriction \partial(c\uparrow)$ is constant. Since $a, a' \in \partial(c\uparrow)$, we obtain $f(a) \restriction \beta = f(a') \restriction \beta$. □

Corollary 4.13. *Let L, M be uncountable chains, and α, β ordinals such that $\beta < \alpha$. If M is L-unsplittable, then L^α_{lex} is not embeddable in M^β_{lex}. In particular, if L is unsplittable, then L^α_{lex} is not embeddable in L^β_{lex}.*

Proof. We prove that if M is L-unsplittable, then any homomorphism $g : L^\alpha_{\mathrm{lex}} \to M^\beta_{\mathrm{lex}}$ fails to be injective. Fix $z \in M^{\alpha-\beta}_{\mathrm{lex}}$ and define a map

$$f : L^\alpha_{\mathrm{lex}} \to M^\beta_{\mathrm{lex}} \times_{\mathrm{lex}} M^{\alpha-\beta}_{\mathrm{lex}}$$

by setting $f(x) := (g(x), z)$ for each $x \in L^\alpha_{\mathrm{lex}}$; then f is a homomorphism of L^α_{lex} into M^α_{lex}. Since M is L-unsplittable, Theorem 4.12 yields the existence of a set $A \in \mathrm{Large}(L^\alpha_{\mathrm{lex}})$ such that for each $a, a' \in A$, if $a \restriction \beta = a' \restriction \beta$, then $f(a) \restriction \beta = f(a') \restriction \beta$. Let C be a nearly-full subtree of $(L^\alpha_{\mathrm{lex}})\downarrow$ contained in $A\downarrow$. Lemmas 4.5(i) and 4.7(i) imply that there exists $c \in C \cap L^\beta_{\mathrm{lex}}$ such that $|\partial(c\uparrow) \cap A| > \omega$. In particular, we can select $a, a' \in A$ such that $a \neq a'$, and $a \restriction \beta = c = a' \restriction \beta$. On the other hand, $g(a) = f(a) \restriction \beta = f(a') \restriction \beta = g(a')$, so g is not injective. □

Corollary 4.14. *Let α be an ordinal, A an Aronszajn line and S a Souslin line with at most countably many jumps. We have*:

(i) $\mathrm{repr}_{\omega_1}(\mathbb{R}^\alpha_{\mathrm{lex}}) \geqslant \alpha$ *and* $\mathrm{repr}((\omega_1)^\alpha) \geqslant \alpha$;
(ii) $\mathrm{repr}_A(\mathbb{R}^\alpha_{\mathrm{lex}}) \geqslant \alpha$ *and* $\mathrm{repr}(A^\alpha_{\mathrm{lex}}) \geqslant \alpha$;
(iii) $\mathrm{repr}_{\omega_1}(A^\alpha_{\mathrm{lex}}) \geqslant \alpha$ *and* $\mathrm{repr}_A((\omega_1)^\alpha) \geqslant \alpha$;
(iv) $\mathrm{repr}_S(S^\alpha_{\mathrm{lex}}) = \alpha$;
(v) $\mathrm{repr}(\mathbb{R}^\alpha_{\mathrm{lex}}) = \alpha$.

5. Representability of Aronszajn lines and Souslin lines

In this section we prove some results about homomorphisms of a tree (ordered lexicographically) into a lexicographic power of $\mathbb{R}$. In particular, we show that the representability number of any Aronszajn line and Souslin line is ω_1.

To begin we establish some further terminology for a tree $(T, \preceq)$. (Note that the notation used here might conflict with standard terminology.) Elements of T are called *nodes*. For each $s, t \in T$, $s \perp t$ stands for $s \npreceq t$ and $t \npreceq s$. Also, we set $(\leftarrow, t) := \{x \in T\colon\ x \prec t\}$ and $(s, \rightarrow) := \{x \in T\colon\ x \succ s\}$; similarly we define $[s, \rightarrow)$ and $(\leftarrow, t]$. A *path* of T is a subtree P of T, which is linearly ordered by the induced order; the set of all paths in T is denoted by $\mathrm{Path}(T)$. A *branch* is a maximal path. The *height* of a node $t \in T$ is the order-type of the initial segment $(\leftarrow, t)$ and is denoted by $\mathrm{height}(t)$. The *αth level* of T is $\mathrm{Lev}_\alpha(T) := \{t \in T\colon\ \mathrm{height}(t) = \alpha\}$; further, we set $T \restriction \alpha := \bigcup_{\beta<\alpha} \mathrm{Lev}_\beta(T)$. The *height* of T is $\mathrm{height}(T) := \min\{\alpha\colon\ \mathrm{Lev}_\alpha(T) = \emptyset\}$.

Next we describe a procedure to extend the partial order $\preceq$ on a tree $(T, \preceq)$ to a total order $\preceq_{\text{lex}}$; we will follow the approach used in [13]. Define a map $\Upsilon : T \times T \to \text{Path}(T)$ by $\Upsilon(s, t) := (\leftarrow, s) \cap (\leftarrow, t)$ for all $s, t \in T$. The function Υ satisfies the following property (see [13]).

Lemma 5.1. *For any $s, t, u \in T$, the set $\{\Upsilon(s, t), \Upsilon(t, u), \Upsilon(s, u)\}$ has at most two elements.*

For each $s, t \in T$, let $s \sim t$ if $(\leftarrow, s) = (\leftarrow, t)$. Then $\sim$ is an equivalence relation; the set of equivalence classes is denoted by $\text{Block}(T)$, and its elements are called *blocks*. Note that each block B of T is a subset of $\text{Lev}_\alpha(T)$ for some α. Further, if P is a path in T that is not a branch, then there exists a unique block B_P such that $P \prec B_P$ and $P \cup B_P$ is a subtree of T; it follows that the correspondence $(s, t) \mapsto B_{\Upsilon(s,t)}$ gives a well-defined map from $T \times T$ into $\text{Block}(T)$. Finally, for any $B \in \text{Block}(T)$ and $t \in \bigcup_{s \in B}[s, \to)$, denote by t_B the unique element of $(\leftarrow, t] \cap B$; then the correspondence $(s, t) \mapsto (s_B, t_B)$, where $B = B_{\Upsilon(s,t)}$, gives a well-defined function from $T \times T$ into itself.

Definition 5.2. Let $(T, \preceq)$ be a tree and assume that for each block B in T, a linear order $\preceq_B$ is given on B. The collection $\mathcal{L} = \{\preceq_B\colon\ B \in \text{Block}(T)\}$ induces a linear order $\preceq_{\text{lex}}$ on T as follows: for each $s, t \in T$, set $s \preceq_{\text{lex}} t$ if either $s \preceq t$, or $s \perp t$ and $s_B \preceq_B t_B$, where $B = B_{\Upsilon(s,t)}$. (Equivalently, $s \preceq_{\text{lex}} t$ if $s \not\succ t$ and $s_B \preceq_B t_B$.) The chain $(T, \preceq_{\text{lex}})$ is called the *lexicographic linearization* (or, for short, the *linear tree*) of $(T, \preceq)$ induced by $\mathcal{L}$ and is denoted by T_{lex}. Sometimes, we speak of the chain T_{lex} as a linear tree, without mentioning the collection of linear orders that induces $\preceq_{\text{lex}}$. The *height* of a linear tree is the height of the original tree.

To distinguish intervals in the original tree $(T, \preceq)$ from intervals in the induced linear tree $(T, \preceq_{\text{lex}})$, we use the following notation: for each $s, t \in T$ such that $s \prec_{\text{lex}} t$, let $(s, t)_{\text{lex}}$ be the open interval in the chain T_{lex}; similarly, we denote by $[s, t)_{\text{lex}}$, $(s, t]_{\text{lex}}$ and $[s, t]_{\text{lex}}$ the other types of bounded intervals in T_{lex}. Further, $(\leftarrow, t)_{\text{lex}} = \{x \in T\colon\ x \prec_{\text{lex}} t\}$ denotes an open initial segment in T_{lex}; the notations $(\leftarrow, t]_{\text{lex}}$, $(s, \to)_{\text{lex}}$ and $[s, \to)_{\text{lex}}$ have similar meaning.

For any nodes $s, t \in T$, let $\sigma(s, t)$ be the ordinal defined as follows:

$$\sigma(s,t) := \begin{cases} \sup\bigl\{\text{height}(x)\colon\ x \in \Upsilon(s,t)\bigr\} & \text{if } s \perp t, \\ \text{height}(s) & \text{if } s \preceq t, \\ \text{height}(t) & \text{if } t \preceq s. \end{cases}$$

Note that $\sigma(s, t) \leqslant \min\{\text{height}(s), \text{height}(t)\}$.

Lemma 5.3. *Let T_{lex} be a linear tree and $s, t, u \in T$. If $u \in (s, t)_{\text{lex}}$, then* $\text{height}(u) > \sigma(s, t)$. *Thus, if $u \in [s, t]_{\text{lex}}$, then* $\text{height}(u) \geqslant \sigma(s, t)$.

Proof. We prove the contrapositive. Thus, we assume that $\text{height}(u) \leqslant \sigma(s, t)$, $s \prec_{\text{lex}} u$ and $u \neq t$, and we show that $u \succ_{\text{lex}} t$. It suffices to prove: (i) $u \perp t$, and (ii) $t_B \prec_B u_B$, where $B = B_{\Upsilon(u,t)}$. For (i), first note that $\text{height}(u) \leqslant \sigma(s, t) \leqslant \text{height}(t)$, hence $u \not\succ t$ holds. On the other hand, $s \prec_{\text{lex}} u$ and $\text{height}(u) \leqslant \text{height}(s)$ imply that $u \perp s$, whence $u \nprec t$ holds as well. Since $u \neq t$ by hypothesis, we obtain that $u \perp t$. For (ii), observe that since $u \perp s$ and $u \perp t$, it follows that $\Upsilon(s,u) \neq \Upsilon(s,t) \neq \Upsilon(t,u)$, and so $\Upsilon(s,u) = \Upsilon(t,u)$, using Lemma 5.1. Then $s \prec_{\text{lex}} u$ implies that $t_B = s_B \prec_B u_B$, where $B = B_{\Upsilon(s,u)} = B_{\Upsilon(u,t)}$. □

Now we introduce a notion of homogeneity for subsets of a tree.

Definition 5.4. Let $(T, \preceq)$ be a tree, H a subset of T, and α an ordinal such that $\alpha + 1 < \text{height}(T)$ (i.e., $\text{Lev}_\alpha(T)$ is not the maximum level of T). We say that H is *homogeneous above* α if for all $s, t \in T$, $\sigma(s,t) > \alpha$ implies "$s \in H \iff t \in H$". Also, we say that H is *eventually homogeneous* if it is homogeneous above α for some ordinal α with $\alpha + 1 < \text{height}(T)$.

For example, for any $t \in T$, if $\text{height}(t) + 1 < \text{height}(T)$ (i.e., the node t does not belong to the maximum level of T), then $(\leftarrow, t) \cup [t, \rightarrow)$ is eventually homogeneous.

Lemma 5.5. *Let T_{lex} be a linear tree and s, t two nodes in T such that* $\max\{\text{height}(s), \text{height}(t)\} + 1 < \text{height}(T)$. *If $s \prec_{\text{lex}} t$, then the interval $(s,t)_{\text{lex}}$ is eventually homogeneous as a subset of T.*

Proof. Set $\alpha := \max\{\text{height}(s), \text{height}(t)\}$; we prove that $(s,t)_{\text{lex}}$ is homogeneous above α. Let $u, v \in T$ such that $\sigma(u,v) > \alpha$. To prove that $u \in (s,t)_{\text{lex}}$ if and only if $v \in (s,t)_{\text{lex}}$, it suffices to show that $(u \succ_{\text{lex}} s \Longrightarrow v \succ_{\text{lex}} s)$ and $(u \prec_{\text{lex}} t \Longrightarrow v \prec_{\text{lex}} t)$. Indeed, Lemma 5.3 yields

$$\begin{aligned}\sigma(u,v) > \alpha &\Longrightarrow \neg(v \preceq_{\text{lex}} s \preceq_{\text{lex}} u) \wedge \neg(u \preceq_{\text{lex}} t \preceq_{\text{lex}} v)\\ &\Longrightarrow (u \succ_{\text{lex}} s \Longrightarrow v \succ_{\text{lex}} s) \wedge (u \prec_{\text{lex}} t \Longrightarrow v \prec_{\text{lex}} t)\end{aligned}$$

which proves the claim. □

The following immediate consequence of Lemma 5.5 is useful.

Corollary 5.6. *Let $f : T_{\text{lex}} \to L$ be a homomorphism. Further, let $a \prec b$ be two elements of L such that there exists $\alpha <$* $\text{height}(T)$ *with the property that both $f^{-1}\{a\} \cap (T \upharpoonright \alpha)$ and $f^{-1}\{b\} \cap (T \upharpoonright \alpha)$ are nonempty.*[1] *Then, there exists an open interval $(s,t)_{\text{lex}} \subseteq T_{\text{lex}}$ with the following properties*:

(i) $f^{-1}(a,b) \subseteq (s,t)_{\text{lex}}$;
(ii) $f[(s,t)_{\text{lex}}] \subseteq [a,b]$;
(iii) $(s,t)_{\text{lex}}$ *is eventually homogeneous.*

In particular, if $\text{height}(T)$ *is a limit ordinal, then for any $a, b \in \operatorname{ran} f$ such that $a \prec b$, there exists an open interval $(s,t)_{\text{lex}} \subseteq T_{\text{lex}}$ satisfying* (i)–(iii).

[1] I.e., $f^{-1}\{a\}$ and $f^{-1}\{b\}$ are nonempty, and they do not consist solely of elements in the maximum level of T.

Next we extend the notion of homogeneity to functions.

Definition 5.7. Let $f:(T,\preceq)\to X$ be any function of a tree into a nonempty set, and let α be an ordinal such that $\alpha+1<\text{height}(T)$. We say that f is *homogeneous above* α if for all $s,t\in T$, $\sigma(s,t)>\alpha$ implies $f(s)=f(t)$; further, f *is eventually homogeneous* if it is homogeneous above α for some ordinal α with $\alpha+1<\text{height}(T)$.

Note that $H\subseteq T$ is homogeneous above α if and only if its characteristic function $\chi_H:T\to 2$ is homogeneous above α; thus, H is eventually homogeneous if and only if so is χ_H.

Lemma 5.8. *Any homomorphism from a linear tree of height ω_1 into a representable chain L is eventually homogeneous.*

Proof. Let T_{lex} be a linear tree obtained from a tree $(T,\preceq)$ with height ω_1, L an infinite representable chain and $f:T_{\text{lex}}\to L$ a homomorphism. Since any subset of a representable chain is representable, we can assume without loss of generality that f is onto. By the representability of L, there exists a countable set of nonempty open intervals $\mathcal{B}=\{(a_n,b_n)\colon n\in\omega\}$ such that $\bigcap\overline{\mathcal{B}}_x=\{x\}$ for each $x\in L$, where $\overline{\mathcal{B}}_x:=\{[a_n,b_n]\colon x\in(a_n,b_n)\in\mathcal{B}\}$. Since $\text{height}(T)=\omega_1$, we can apply Corollary 5.6 for each $n\in\omega$. Thus, we get a sequence $((s_n,t_n)_{\text{lex}})_{n<\omega}$ of open intervals in the chain T_{lex} and a sequence $(\alpha_n)_{n<\omega}$ of countable ordinals satisfying the following properties: (i) $f^{-1}(a_n,b_n)\subseteq(s_n,t_n)_{\text{lex}}$; (ii) $f[(s_n,t_n)_{\text{lex}}]\subseteq[a_n,b_n]$; (iii) $(s_n,t_n)_{\text{lex}}$ is homogeneous above α_n. Set $\alpha:=\sup\{\alpha_n\colon n\in\omega\}$. In the sequel we show that f is homogeneous above α; since $\alpha<\omega_1$, this will end the proof.

Let $s,t\in T$ be such that $\sigma(s,t)>\alpha$. Assume by contradiction that $f(s)\prec f(t)$. Select $(a_k,b_k)\in\mathcal{B}$ such that $f(s)\in(a_k,b_k)$ and $f(t)\notin[a_k,b_k]$. Since $\alpha\geqslant\alpha_k$, condition (iii) implies that $s\in(s_k,t_k)_{\text{lex}}$ if and only if $t\in(s_k,t_k)_{\text{lex}}$. But then (i) and (ii) yield the following chain of implications:

$$f(t)\notin[a_k,b_k]\implies t\notin(s_k,t_k)_{\text{lex}}\implies s\notin(s_k,t_k)_{\text{lex}}\implies f(s)\notin(a_k,b_k)$$

which is a contradiction. Similarly, it cannot be $f(t)\prec f(s)$. Therefore $f(s)=f(t)$. This completes the proof. □

Recall that an ω_1-*tree* is a tree of height ω_1 such that all its levels are countable, and an *Aronszajn tree* is an ω_1-tree that has no branch of length ω_1. Observe that an eventually homogeneous homomorphism defined on a lexicographic linearization of an ω_1-tree has a countable range.

Theorem 5.9. *Every homomorphism from a lexicographic linearization of an ω_1-tree into a countable lexicographic power of $\mathbb{R}$ is eventually homogeneous.*

Proof. Let $(T,\preceq)$ be an ω_1-tree, T_{lex} a lexicographic linearization of T, α a countable ordinal and $f:T_{\text{lex}}\to\mathbb{R}^{\alpha}_{\text{lex}}$ a homomorphism. It suffices to show that there exists an ordinal $\beta<\omega_1$ with the property that for each $t\in T$ such that $\text{height}(t)\geqslant\beta$, $f\restriction[t,\to)$ is constant.

For each $\gamma < \alpha$, let $\pi_\gamma : \mathbb{R}^\alpha_{\text{lex}} \to \mathbb{R}$ be the projection onto the γth component; further, for each $1 \leqslant \gamma \leqslant \alpha$, denote by $\hat{\pi}_\gamma : \mathbb{R}^\alpha_{\text{lex}} \to \mathbb{R}^\gamma_{\text{lex}}$ the projection onto the first γ components. Note that: (i) $\pi_0 = \hat{\pi}_1$; (ii) $\hat{\pi}_{\gamma+1} = \hat{\pi}_\gamma \times \pi_\gamma$ for all $1 \leqslant \gamma < \alpha$; (iii) $\hat{\pi}_\alpha$ is the identity function on $\mathbb{R}^\alpha_{\text{lex}}$. We construct by recursion an increasing sequence $(\beta_\gamma)_{\gamma \leqslant \alpha}$ of countable ordinals such that for all $1 \leqslant \gamma \leqslant \alpha$ the following condition is satisfied:

$(*)_\gamma$ for each $t \in \text{Lev}_{\beta_\gamma}(T)$, the restriction of the homomorphism $\hat{\pi}_\gamma \circ f : T_{\text{lex}} \to \mathbb{R}^\gamma_{\text{lex}}$ to $[t, \to)$ is constant.

Then the countable ordinal $\beta = \beta_\alpha$ satisfies the claim.

To build the sequence, consider the homomorphism $\pi_0 \circ f : T_{\text{lex}} \to \mathbb{R}$. By Lemma 5.8, there exists a countable ordinal γ_0 such that $\pi_0 \circ f$ is homogeneous above γ_0. Set $\beta_0 := \gamma_0$ and $\beta_1 := \gamma_0 + 1$; then, $(*)_1$ holds. Next, assume that γ is a successor ordinal, say, $\gamma = \delta + 1$. Consider the set

$$H = \left\{ t \in \text{Lev}_{\beta_\delta}(T) \colon [t, \to) \text{ is an } \omega_1\text{-tree} \right\}.$$

Since T is an ω_1-tree, the set H is nonempty and countable; let $H = \{t^H_n \colon n \in \omega\}$ be an enumeration. Further, there exists an ordinal $\eta < \omega_1$ such that for all $t \in \text{Lev}_\eta(T)$, we have $t^H_n \preceq t$ for some $n \in \omega$. Fix $t^H_n \in H$ and denote by ψ_δ the restriction of the map $\pi_\delta \circ f$ to the interval $[t^H_n, \to)$. Since $(*)_\delta$ holds, the map $\hat{\pi}_\delta \circ f \upharpoonright [t^H_n, \to)$ is constant, and so ψ_δ is a homomorphism of an ω_1-tree into $\mathbb{R}$. Thus Lemma 5.8 yields the existence of a countable ordinal η_n such that for each $t \in \text{Lev}_{\eta_n}([t^H_n, \to))$, the map $\psi_\delta \upharpoonright [t, \to)$ is constant. Set $\eta_\omega := \sup\{\eta_n \colon n \in \omega\}$ and $\beta_\gamma := \max\{\eta, \beta_\delta + \eta_\omega\}$; then $(*)_\gamma$ holds. Finally, if γ is a limit ordinal, then $(*)_\gamma$ holds for $\beta_\gamma := \sup\{\beta_\delta \colon \delta < \gamma\}$. □

Corollary 5.10. *Every lexicographic linearization of an ω_1-tree has an uncountable representability number.*

Corollary 5.11. *The representability number of every Aronszajn line and of every Souslin line is ω_1.*

Proof. A Souslin line contains an Aronszajn line, which is dense in it (see [13, Proposition 3.9]). Thus it suffices to show that for each Aronszajn line A and Souslin line S, we have $\text{repr}(A) \geqslant \omega_1$ and $\text{repr}(S) \leqslant \omega_1$. Since A is isomorphic to a linear tree T_{lex} obtained from an Aronszajn tree (see [13]), Corollary 5.10 yields $\text{repr}(A) = \text{repr}(T_{\text{lex}}) \geqslant \omega_1$. On the other hand, $\text{repr}(S) \leqslant \omega_1$, because any short chain embeds into $2^{\omega_1}_{\text{lex}}$ (see [11]). □

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