

PRIVATE AND SOCIAL INCENTIVES TOWARDS INVESTMENT IN PRODUCT DIFFERENTIATION

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We consider a dynamic oligopoly where firms invest to increase product differentiation and an externality effect operates in the R&D activity. We compare the steady state solutions under alternative decision rules, namely, the open-loop and the closed-loop Nash equilibrium. Significant differences emerge, concerning the effect of the number of firms upon the optimal degree of product differentiation. We also compare the private optima with the social optimum, and derive implications concerning the social desirability of different decision rules.

Keywords: Differential games; research and development, product differentiation, spillovers

JEL Classification code: C73, L13, O31

1. Introduction

We adopt a differential game approach to study the optimal effort of oligopolistic firms concerning the R&D investment aimed at product innovation. In particular, we consider a dynamic model where firms produce symmetrically differentiated goods, and where the degree of differentiation is the outcome of R&D investments by firms over time. Since the differentiation between goods is symmetric, it can be interpreted as a public good; thus, the activity of R&D aimed at product differentiation can be interpreted as the private provision of a public good.

The basics of the present model are the same as in Cellini and Lambertini (2002), but in the present paper we do not confine ourselves to the open-loop solution concept. On the contrary, we adopt the closed-loop Nash equilibrium concept. It is well-known that these two solution concepts may be interpreted as two different choice rules, corresponding to different information sets. Under the open-loop rule, firms precommit their decisions on the control variables to a path over time, and the resulting equilibrium is only (usually) weakly time-consistent.^a By contrast, under

^aThere exist classes of games where the open-loop solution is strongly time consistent. For an

the closed-loop rule, firms do not precommit on any path and their strategies at any instant may depend on all the preceding history, even if - in such a situation - the information set used by firms in setting their strategies at any given time is often simplified to be only the current value of the state variables. The relevant equilibrium concept is the so-called “memoryless closed-loop” Nash equilibrium, and it is strongly time-consistent or Markov-perfect (see Basar and Olsder, 1982, and Dockner *et al.*, 2000).

The comparison between the steady states under these different equilibrium concepts provides us with a view of some interesting properties of the optimal behaviour of oligopolistic firms under alternative decision rules. In particular, under the open-loop solution concept an Arrowian conclusion is reached, according to which the amount of resources invested by the industry in product differentiation is increasing in the number of firms, i.e., in the intensity of market competition (Arrow, 1962). By contrast, under the closed-loop time consistent solution concept, the Schumpeterian conclusion may be reached, according to which the amount of resources invested by the industry in product differentiation is decreasing in the intensity of market competition, as captured by the number of firms (Schumpeter, 1942).

Moreover, the privately optimal allocations under the aforementioned equilibrium concepts are compared with the social optimum. In particular, we show that the steady state equilibrium under the open-loop decision rules is closer to the steady state social optimum, than the closed-loop steady state equilibrium is.

The outline of the paper is as follows. Section 2 illustrates the basics of the model. Section 3 develops the private optimum under the closed-loop decision rule, and compares the results with the open-loop decision rules, with particular emphasis on profits and social welfare. Section 4 derives the social optimum allocation and proposes some considerations about the social desirability of market regulation under alternative private decision rules. Section 5 briefly concludes.

2. The Setup

We consider a market where $n > 1$ single-product firms sell differentiated products over the time period $[0, \infty)$. Market competition takes place *à la* Cournot. The demand structure is borrowed from Spence (1976). In each period $t \in [0, \infty)$, the inverse demand function for product i is:

$$p_i(t) = A - Bq_i(t) - D(t) \sum_{j \neq i} q_j(t) \quad (2.1)$$

where $D(t) \in [0, B]$ is the symmetric degree of substitutability between any pair of products. If $D(t) = B$, products are completely homogeneous. If $D(t) = 0$, products are completely independent and each firm is a monopolist.^b At any time t , the output level $q_i(t)$ is produced at constant returns to scale, for a given $D(t)$, so that we define the individual total operative cost per period as $C_i(t) = cq_i(t)$, where $c \in (0, A)$.

exhaustive overview, see Dockner *et al.* (2000, ch. 7).

^bThe idea that D depends upon the behaviour of firms has been investigated in static models by Harrington (1995); Lambertini and Rossini (1998); Lambertini, Poddar and Sasaki (1998).

We assume that $D(0) = B$, so that, at the initial instant, firms may produce the same homogeneous good through a technology which is public domain. Product differentiation may increase, i.e., parameter D may decrease, through firms' R&D investment according to the following law, borrowed from Cellini and Lambertini (2002):

$$\dot{D}(t) = \frac{dD(t)}{dt} = -\frac{K(t)}{1+K(t)} \cdot D(t) \equiv -\frac{k_i(t) + \sum_{j \neq i} k_j(t)}{1 + [k_i(t) + \sum_{j \neq i} k_j(t)]} \cdot D(t); \quad (2.2)$$

with $k_i(t) \geq 0 \forall i$. The above dynamics of product differentiation can be interpreted as follows. The industry overall R&D expenditure is $K(t) = \sum_{i=1}^n k_i(t)$, where $k_i(t)$ is individual investment of firm i .^c Given the symmetric nature of product differentiation in this model, there exists a complete spillover effect in the R&D process. Notice that the externality effect we consider here entails that the outcome of R&D activity is public domain via the demand function. On the contrary, the externality effects usually considered in the literature are associated with information leakage or transmission (see, *inter alia*, d'Aspremont and Jacquemin, 1988). The dynamic equation (2.2) can be interpreted as a production function whose output is $-\dot{D}(t)/D(t)$, obtained through an R&D input represented by capital. This technology can be shown to exhibit decreasing returns to scale w.r.t. $K(t)$. As a result, $D(t)$ is non-increasing over time, and approaches zero as $K(t)$ tends to infinity.

The instantaneous profit of firm i is $\pi_i(t) = [p_i(t) - c]q_i(t) - k_i(t)$. Each firm aims at maximizing the discounted value of its flow of profits:

$$\Pi_i = \int_0^{\infty} e^{-\rho t} \pi_i(t) dt$$

under the dynamic constraint (2.2) concerning the state variable $D(t)$. The firm's control variables are $q_i(t)$ and $k_i(t)$.

3. The Private Optimum

Suppose firms choose non-cooperatively both R&D efforts and output levels. The solution concept we adopt is the closed-loop memoryless (or Markovian) Nash equilibrium. The objective function of firm i is:

$$\Pi_i = \int_0^{\infty} e^{-\rho t} \left\{ q_i(t) \cdot \left[A - c - Bq_i(t) - D(t) \sum_{j \neq i} q_j(t) \right] - k_i(t) \right\} dt$$

to be maximised w.r.t. $q_i(t)$ and $k_i(t)$, s.t. (2.2). The Hamiltonian is:

$$\mathcal{H}(t) = e^{-\rho t} \cdot \left\{ [A - c]q_i(t) - B(q_i(t))^2 - D(t)q_i(t) \sum_{j \neq i} q_j(t) - k_i(t) \right.$$

^cIn eq. (2.2), we keep $k_i(t)$ separated from $\sum_{j \neq i} k_j(t)$ in order to put in evidence the control variable of firm i .

$$+ \lambda_i(t) \left[- \frac{k_i(t) + \sum_{j \neq i} k_j(t)}{1 + [k_i(t) + \sum_{j \neq i} k_j(t)]} \cdot D(t) \right] \Bigg\},$$

where $\lambda_i(t) = \mu_i(t)e^{\rho t}$, $\mu_i(t)$ being the current value costate variable associated with $D(t)$.

The necessary conditions for a path to be optimal are:^d

$$\frac{\partial \mathcal{H}(t)}{\partial q_i(t)} = A - c - 2Bq_i(t) - D(t) \sum_{j \neq i} q_j(t) = 0 \Rightarrow \quad (3.7)$$

$$q_i^*(t) = \frac{A - c - D(t) \sum_{j \neq i} q_j(t)}{2B};$$

$$\frac{\partial \mathcal{H}(t)}{\partial k_i(t)} = -1 - D(t)\lambda_i(t) \frac{1}{\left(1 + k_i(t) + \sum_{j \neq i} k_j(t)\right)^2} = 0 \Rightarrow \quad (3.9)$$

$$k_i^*(t) = -1 - \sum_{j \neq i} k_j(t) + \sqrt{-D(t)\lambda_i(t)};$$

$$-\frac{\partial \mathcal{H}(t)}{\partial D(t)} - \sum_{j \neq i} \frac{\partial \mathcal{H}(t)}{\partial q_j(t)} \frac{\partial q_j^*(t)}{\partial D(t)} - \sum_{j \neq i} \frac{\partial \mathcal{H}(t)}{\partial k_j(t)} \frac{\partial k_j^*(t)}{\partial D(t)} = \frac{d\mu_i(t)}{dt} \Rightarrow \quad (3.11)$$

$$\begin{aligned} \frac{d\lambda_i(t)}{dt} &= q_i(t) \sum_{j \neq i} q_j(t) - \left[\sum_{j \neq i} D(t)q_j(t) \sum_{m \neq j} \frac{q_m(t)}{2B} \right. \\ &+ \left. \sum_{j \neq i} \frac{\lambda_i(t) \sqrt{\lambda_i(t) D(t)}}{2 \left[1 + k_i(t) + \sum_{j \neq i} k_j(t)\right]^2} \right] + \lambda_i(t) \left(\frac{k_i(t) + \sum_{j \neq i} k_j(t)}{1 + [k_i(t) + \sum_{j \neq i} k_j(t)]} + \rho \right); \\ \lim_{t \rightarrow \infty} \mu_i(t) \cdot D(t) &= 0 \end{aligned}$$

We introduce the following:

Assumption 3.1. $q_i(t) = q_j(t) = q(t)$, and $k_i(t) = k_j(t) = k(t)$.

This is a usual symmetry assumption involving no loss of generality as long as one adopts the Nash equilibrium as the solution concept.^e In particular, it implies $\sum_{j \neq i} q_j(t) = (n-1)q(t)$ and $\sum_{j \neq i} k_j(t) = (n-1)k(t)$.

^dThe appropriate second-order conditions for a maximum are fulfilled. They are omitted for brevity. For the same reason, in the first-order conditions we also omit the indication of exponential discounting.

^eUnder non-cooperative behaviour, such a symmetry assumption can be introduced only after the computation of the first-order conditions. Imposing symmetry *ex ante* would be appropriate in order to solve the game where firms build up full cartelisation in quantities and R&D investments.

Then, from (3.7) we derive the equilibrium per firm output:

$$q(t) = \frac{A - c}{2B + (n - 1)D(t)} \quad (3.15)$$

which coincides with the standard outcome of Cournot models with product differentiation (see, e.g., Singh and Vives, 1984; Majerus, 1988; Lambertini, 1997; Cellini and Lambertini, 1998). Given that Assumption 3.1 also implies the symmetry condition $\lambda_i(t) = \lambda(t) \forall i$, equation (3.9) rewrites as

$$-\lambda(t) = \frac{[1 + nk(t)]^2}{D(t)}. \quad (3.16)$$

By symmetry, and using (3.16), (3.11) simplifies as follows:

$$\dot{\lambda}(t) = \frac{d\lambda(t)}{dt} = \frac{D(t)(n - 1) [q(t)]^2 [2B - D(t)(n - 1)]}{2BD(t)} + \quad (3.17)$$

$$-\frac{B(1 + nk) [2nk + n - 1 + 2\rho(1 + nk)]}{2BD(t)}. \quad (3.18)$$

From (3.16) we obtain $k(t)$, which can be differentiated w.r.t. t :

$$\dot{k}(t) = \frac{1}{2n\sqrt{-\lambda(t)D(t)}} \left[-\dot{\lambda}(t) \cdot D(t) - \lambda(t) \cdot \dot{D}(t) \right] \quad (3.19)$$

Then, plugging (3.17) and (3.16) into (3.19) and rearranging, one obtains:

$$\dot{k}(t) = \frac{B [1 + nk(t)] [2\rho(1 + nk(t)) + n - 1] - D(t)(n - 1)[q(t)]^2 [2B - D(t)(n - 1)]}{2n [1 + nk(t)]} \quad (3.20)$$

The expression in (3.20) is valid for all $D(t) \in (0, B]$. If $D(t) = 0$, optimal per-period investment is $k(t) = 0$. Likewise, one can also exclude the monopoly case ($n = 1$), where strategic interaction between goods is absent by definition. Therefore, in the remainder we focus on $n \geq 2$.

We are now in a position to assess the overall dynamic properties of the model, fully characterised by (3.19) and $\dot{D}(t) = -nk(t)D(t)/(1+nk(t))$. The latter equation establishes that $\dot{D}(t) < 0$ for all $k(t) \in (0, \infty)$ and for all $D(t) \in (0, B]$; while $\dot{D}(t) = 0$ if $k(t) = 0$ or $D(t) = 0$. In the latter circumstances, it is immediate to verify that $\dot{k}(t)$ is also zero.

Moreover, from (3.20), using the equilibrium value of the output level from (3.15), we obtain that

$$\begin{aligned} \dot{k}(t) = 0 \text{ if} \\ 2B\rho n^2[\alpha(t)]^2 [k(t)]^2 + Bn[\alpha(t)]^2 (4\rho + n - 1) k(t) + \\ + B[\alpha(t)]^2 (2\rho + n - 1) - (n - 1)(A - c)^2 D(t) [2B - (n - 1)D(t)] = 0, \end{aligned} \quad (3.23)$$

where

$$\alpha(t) \equiv 2B + (n - 1)D(t).$$

The roots of equation (3.23) are:

$$k(t) = \frac{-B \cdot \alpha(t) \cdot (4\rho + n - 1) \pm \sqrt{\Omega}}{4B\rho n\alpha(t)}$$

$$\Omega \equiv 8(A - c)^2 D(t)\alpha(t)\rho - B^2(n - 1)[2B - D(t)]^2 + Bn \left\{ 4B[B + (n - 2)D(t)] + [D(t)]^2 (n^2 - 3n + 3) \right\}.$$

The smaller root corresponds to a locus where k has always negative values, and therefore can be disregarded, as being economically meaningless. Then, considering the larger root, we are interested in investigating the dynamics of the system in the positive quadrant of the space (D, k) , which is described in Figure 1. The locus $\dot{D}(t) = 0$ corresponds to the axes. The locus $\dot{k}(t) = 0$ is a curve over the admissible range of parameter D , which may or may not cross the horizontal axis within the same range, i.e., $D \in (0, B]$. If it does, the resulting candidate degree of substitutability in steady state is either

$$D_{cl1} = \frac{B \left[(A - c)^2 - 2B(2\rho + n - 1) - (A - c) \sqrt{(A - c)^2 - 8B(2\rho + n - 1)} \right]}{(n - 1) [(A - c)^2 + B(2\rho + n - 1)]} \quad (3.25)$$

or

$$D_{cl2} = \frac{B \left[(A - c)^2 - 2B(2\rho + n - 1) + (A - c) \sqrt{(A - c)^2 - 8B(2\rho + n - 1)} \right]}{(n - 1) [(A - c)^2 + B(2\rho + n - 1)]}$$

with $D_{cl1,2} \in R$ iff $B \in (0, \bar{B})$, $\bar{B} \equiv (A - c)^2 / [8(2\rho + n - 1)]$. Subscript cl stands for closed-loop.

Moreover, it can quickly be verified that

$$D_{cl2} > D_{cl1} > 0$$

in the whole admissible parameter range. Now it remains to check the conditions ensuring that $D_{cl1,2} < B$.

It is easily established that $D_{cl1} < B$ for all $n \geq 2$. As to the larger root, we have:^f

$$D_{cl2} < B \text{ for all } B > \hat{B} \equiv \frac{(A - c)^2 (n - 3)(1 - n)}{(n + 1)^2 (2\rho + n - 1)},$$

with

$$\begin{aligned} \bar{B} &> \hat{B} > 0 \text{ iff } n = 2; \\ \bar{B} &> \hat{B} = 0 \text{ iff } n = 3; \\ \bar{B} &> 0 > \hat{B} \text{ for all } n \geq 4. \end{aligned}$$

^fThe equation $D_1^* = B$ has another root: $B = -(A - c)^2 / (2\rho + n - 1)$, which can be disregarded as it is always negative.

We can now formulate the following

Lemma 3.1. *If $n = 2$ and $B \in (0, \widehat{B})$, there exists only one steady state at D_{cl1} . If $n = 2$ and $B \in (\widehat{B}, \overline{B})$, there exist two steady state levels of product substitutability, D_{cl1} and D_{cl2} , with $B > D_{cl2} > D_{cl1} > 0$. For all $n \geq 3$ and all $B \in (0, \overline{B})$, there exist two steady state levels of product substitutability, D_{cl1} and D_{cl2} , with $B > D_{cl2} > D_{cl1} > 0$.*

The alternative situations identified by Lemma 3.1 are illustrated in Figures 1-2.

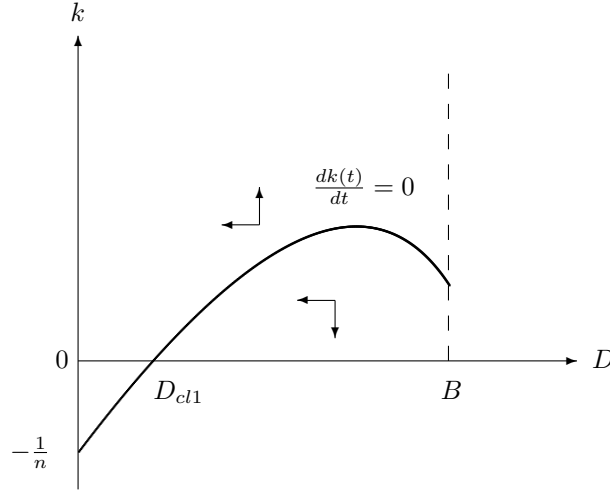


Fig. 1. Dynamics in the space (D, k) for $n = 2$ and $B \in (0, \widehat{B})$.

Considering the stability of the system, the system of dynamic equations (2.2) and (3.19) can be linearised around the steady state points $(D_{cl1}, 0)$ (whenever $D_{cl1} \in (0, B)$) and $(D_{cl2}, 0)$, to prove that the sign of the determinant of the Jacobian matrix is the sign of

$$3D(n - 1) - 2B,$$

which:

- in $(D_{cl1}, 0)$ is negative for all $B \in (0, \overline{B})$;
- in $(D_{cl2}, 0)$ is positive for all $B \in (0, \overline{B})$.

Since in $(D_{cl2}, 0)$ the trace of the Jacobian matrix is positive over the admissible range of parameters, we can state the following:[§]

[§]The detailed calculations involved in the assessment of the stability properties are omitted for the sake of brevity. They are available from the authors upon request.

Proposition 3.1. *The steady state $(D_{cl1}, 0)$ is a saddle point. The steady state $(D_{cl2}, 0)$, whenever it exists, is an unstable focus.*

It should be stressed that the saddle point $(D_{cl1}, 0)$ can be approached only along the north-east branch of the saddle point path.

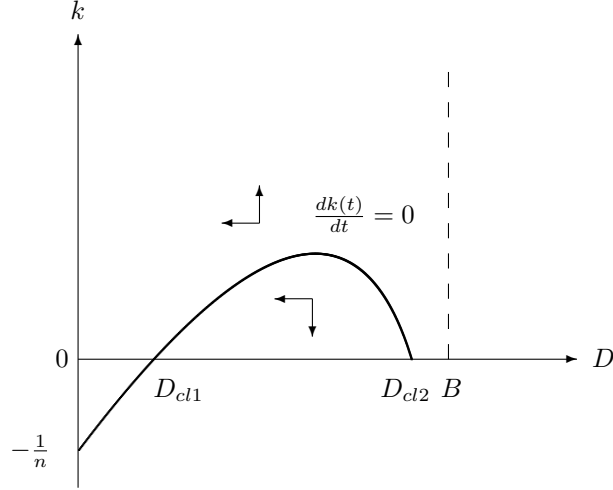


Fig. 2 : Dynamics in the space (D, k) for either $n = 2$ and $B \in (\widehat{B}, \overline{B})$, or $n \geq 3$ and all $B \in (0, \overline{B})$.

3.1. Comparative Evaluation of Open-Loop and Closed-Loop Equilibria

Comparative statics exercises can be carried out on D_{cl1} to describe the effects of market size $(A-c)^2/B$, the discount rate ρ , and the number of firms n on the optimal amount of product substitutability in the closed-loop equilibrium. The derivative $\partial D_{cl1}/\partial n$ is a quartic expression in n , and can be evaluated only numerically, but a few examples suffice to characterise the behaviour of optimal substitutability as n changes. For instance, consider the case $A - c = 10$, $\rho = 1/10$ and $B = 1/2$ to verify numerically that

$$\frac{\partial D_{cl1}}{\partial n} < 0 \text{ for all } n \in [2, 4.3616); \quad (3.30)$$

$$\frac{\partial D_{cl1}}{\partial n} > 0 \text{ for all } n > 4.3616. \quad (3.31)$$

In view of the integer constraint, this means that D_{cl1} is decreasing in n for $n \in [2, 4)$ and then increasing in n for all $n \geq 5$. By increasing B , it can be seen that the interval wherein $\partial D_{cl1}/\partial n < 0$ shrinks. If $B = 2.2425$, the closed-loop equilibrium degree of substitutability is the same at $n = 2$ and $n = 3$. D_{cl1} takes a minimum

at $n = 2.431$, where $D_{cl1} = 0.1297$. However, due to the integer constraint, the behaviour of D_{cl1} over $n \in [2, 3)$ is irrelevant; therefore, if $B \in (2.2425, \bar{B})$, product differentiation is decreasing for all $n \geq 2$ in the closed-loop equilibrium. Of course, this exercise can be repeated for many other admissible values of parameters $\{A, c, \rho\}$. In general, there exists a threshold \tilde{B} such that D_{cl1} is everywhere increasing in n for all $B \in (\tilde{B}, \bar{B})$. Thus, there exists a parameter region where, whenever a real solution does exist, optimal product differentiation is decreasing in the number of firms in equilibrium. The intuition behind this result is that, as the number of firms increases and the market becomes more competitive, the feedback effects characterising the closed-loop solution induce a firm to invest less in product differentiation in that, in the present model, differentiation is a public good whose benefits are more difficult to internalise as n grows larger.

In the light of the debate on the relationship between market structure and the incentive to invest (either in process or in product innovation), that can be traced back to Schumpeter (1942) and Arrow (1962),^h the comparative statics in (3.30-3.31) lead to the following:

Proposition 3.2. *Under the closed-loop information structure, an increase in market power generated by a decrease in the number of firms entails an increase in product differentiation in equilibrium, for $n \in [2, 4)$; the opposite holds for $n \geq 5$.*

Hence, the closed-loop equilibrium has a strong Schumpeterian flavour, which sharply contrasts with the conclusions drawn from the open-loop formulation of the same model.

From Cellini and Lambertini (2002) we know that the opposite holds in an open-loop equilibrium, where the optimal amount of product substitutability is:

$$D_{ol} = \frac{(A - c)^2 - 4B\rho - (A - c)\sqrt{(A - c)^2 - 8B\rho}}{2(n - 1)\rho}; \quad (3.32)$$

$$D_{ol} \in (0, B) \text{ for all } B < \tilde{B} \equiv \frac{(A - c)^2(n - 1)}{\rho(n + 1)^2},$$

with subscript *ol* standing for *open-loop*. Since $\tilde{B} > \bar{B}$ for all $n \geq 2$, both D_{ol} and D_{cl1} belong to $(0, B)$ for all $B \in (0, \bar{B})$. Clearly, $\partial D_{ol}/\partial n < 0$ always, i.e., the open-loop degree of product differentiation is increasing in the intensity of market competition. Increasing the number of firms reduces equilibrium profits, but this effect can be offset by increasing R&D efforts so as to decrease the degree of substitutability among products.

Comparing (3.25) and (3.32), it is straightforward to verify that

$$D_{cl1} - D_{ol} = 0 \text{ at } B_1 = -\frac{(A - c)^2}{2\rho + n - 1}; B_2 = -\frac{(n - 1)(A - c)^2}{(2\rho + n - 1)^2}; B_3 = 0,$$

with $D_{cl1} - D_{ol} > 0$ for all $B > B_3$.

Consequently, the following holds:

^hFor an exhaustive overview of this literature, see Reinganum (1989).

Proposition 3.3. *Ceteris paribus, the difference $D_{cl1} - D_{ol}$ is positive and increasing in n for all $B \in (0, \bar{B})$.*

This leads us to compare profits and social welfare levels in the two settings. Concerning profits, observing that (i) optimal output in (3.15) is the same under both information structures; and (ii) equilibrium steady state investments are zero in both cases, suffices to prove that the following corollary to Proposition 3.3 holds:

Corollary 3.1. *Ceteris paribus, the individual firm's profits as well as industry profits, are lower in the closed-loop equilibrium than in the open-loop equilibrium.*

This is a result that confirms the existing literature on dynamic investment in oligopoly, be such investment related to R&D, productive capacity or other long run variables affecting firms' size (see, *inter alia*, Reynolds, 1987; and the related discussion in Fudenberg and Tirole, 1991, ch. 13).

As to the welfare assessment, we proceed as follows. The utility function of the representative consumer is:

$$U(t) = A \sum_i q_i(t) - \frac{1}{2} \left[B \sum_{i=1}^N (q_i(t))^2 + D(t) \sum_i \sum_{j \neq i} q_i(t) q_j(t) \right]$$

whose maximization under the budget constraint $Y(t) \geq \sum_i p_i(t) q_i(t)$, where $Y(t)$ is nominal income, yields demand functions (2.1). Accordingly, consumer surplus is measured by $CS(t) \equiv U(t) - \sum_i p_i(t) q_i(t)$ and social welfare is $SW(t) \equiv \sum_i \pi_i(t) + CS(t)$. Under the symmetry assumption, and given that $k = 0$ in steady state, consumer surplus writes as

$$CS = \frac{nq^2}{2} [B + D(n-1)], \quad (3.36)$$

while welfare is given by:

$$SW = n(A - c)q - \frac{nq^2}{2} [B + D(n-1)]. \quad (3.37)$$

Now one can plug the equilibrium output (3.15) into (3.36) and (3.37), and differentiate w.r.t. D , to verify that $\partial CS / \partial D$ and $\partial SW / \partial D$ are everywhere negative, which implies a second corollary to Proposition 3.3:

Corollary 3.2. *Ceteris paribus, social welfare is higher in the open-loop equilibrium than in the closed-loop equilibrium, due to both higher industry profits and higher consumer surplus.*

When compared to existing contributions in oligopoly theory, this result sounds quite peculiar. Usually, if firms' profits increase, consumer surplus decreases and the overall welfare effect is *a priori* ambiguous. In the present setting, however, given that first-order conditions on output levels are the same under both solution concepts, an increase in product differentiation benefits both the firms and the

consumers alike. One reason is that firms and consumers have a common “taste for differentiation”,ⁱ which generates the above result.

This considerations raise the issue of whether firms can be expected to adopt open-loop decision rules, either spontaneously or because they are induced to do so by a regulator trying to increase social welfare. The latter case seems quite implausible, since R&D policy is usually implemented through subsidies or patent laws, while in this case it should compel firms to initially declare their investment plans and stick to them forever. In our view, the former case is more realistic, since firms may decide to follow open-loop decision rules, though only weakly time consistent, for the following reasons. First, the open-loop rule requires a minimum amount of information, as compared to the closed-loop rule. Second, in the case of investment choices, the stickiness of investment makes it difficult to believe that a firm can react “period by period” to the rivals’ choices, as the closed-loop rule would require.

4. The Social Optimum

The objective of a benevolent social planner consists in maximising the discounted value of social welfare under the dynamic constraint (2.2). Under the symmetry assumption, which in this setting can be imposed on outputs and R&D efforts from the outset, the instantaneous social welfare simplifies as follows:

$$SW(t) = n(A - c)q(t) - \frac{n[B + D(n - 1)][q(t)]^2}{2} - nk(t)$$

The Hamiltonian function corresponding to the maximisation of the present value of the social welfare is:

$$\mathcal{H}_{sp}(t) = e^{-\rho t} n \left[(A - c)q(t) - \frac{[B + D(t)(n - 1)][q(t)]^2}{2} - k(t) - \frac{\lambda(t)D(t)k(t)}{1 + nk(t)} \right]$$

where the subscript *sp* stands for *social planning*. The first-order conditions and adjoint equations are:

$$\frac{\partial \mathcal{H}_{sp}(t)}{\partial q(t)} = n(A - c) - nq(t)[B + D(t)(n - 1)] = 0; \quad (4.40)$$

$$\frac{\partial \mathcal{H}_{sp}(t)}{\partial k(t)} = -n \left\{ 1 + \frac{D(t)\lambda(t)}{[1 + nk(t)]^2} \right\} = 0; \quad (4.41)$$

$$-\frac{\partial \mathcal{H}_{sp}(t)}{\partial D(t)} = \frac{d\mu(t)}{dt} \Rightarrow \frac{d\lambda(t)}{dt} = \frac{n[q(t)]^2}{2} + \lambda(t) \left(\frac{nk(t)}{1 + nk(t)} + \rho \right); \quad (4.42)$$

$$\lim_{t \rightarrow \infty} \mu(t) \cdot D(t) = 0,$$

ⁱIt is worth stressing that consumers exhibit an additional “taste for variety”, that is, consumer surplus is increasing in the number of firms, *n*. This, of course, cannot hold for firms, as individual profits are decreasing in *n*.

with $\mu(t)$ being the costate variable associated with $D(t)$, and $\lambda(t) = \mu(t)e^{\rho t}$.

From (4.40) and (4.41), one obtains, respectively:

$$q(t) = \frac{A - c}{B + D(t)(n - 1)}; \quad (4.44)$$

$$-\lambda(t) = \frac{[1 + nk(t)]^2}{D(t)}. \quad (4.45)$$

The optimal quantity (4.44) reveals, as one would expect, that the planner sets the price equal to marginal cost. From (4.41), the evolution of $k(t)$ over time has the following dynamics:

$$\dot{k}(t) \propto -D(t) \cdot \dot{\lambda}(t) - \lambda(t) \cdot \dot{D}(t). \quad (4.46)$$

Using (4.42), (4.44) and (4.45), one can rewrite the right-hand side of (4.46) as follows:

$$\dot{k}(t) \propto 2\rho [1 + nk(t)]^2 - \frac{D(t)n(n - 1)(A - c)^2}{[B + D(t)(n - 1)]^2}. \quad (4.47)$$

The roots of the right-hand side of (4.47) are:

$$k(t) = -\frac{1}{n} \pm \frac{(A - c)\sqrt{2n(n - 1)\rho D(t)}}{2\rho n [B + D(t)(n - 1)]}. \quad (4.48)$$

The smaller root can be disregarded as it is always negative. The larger root is zero at

$$D_{sp1} = \frac{n(A - c)^2 - 4B\rho - (A - c)\sqrt{n[n(A - c)^2 - 8B\rho]}}{4(n - 1)\rho};$$

$$D_{sp2} = \frac{n(A - c)^2 - 4B\rho + (A - c)\sqrt{n[n(A - c)^2 - 8B\rho]}}{4(n - 1)\rho},$$

with

$$D_{sp1,2} \in R \text{ for all } B \in \left(0, \frac{n(A - c)^2}{8\rho}\right].$$

Moreover, simple calculations show that

$$D_{sp1} > 0 \text{ always};$$

$$D_{sp1} < B \text{ for all } B > \frac{(n - 1)(A - c)^2}{2n\rho};$$

$$D_{sp2} > B \text{ always}.$$

Noticing that

$$\frac{n(A - c)^2}{8\rho} \geq \frac{(n - 1)(A - c)^2}{2n\rho} \text{ for all } n \geq 2$$

we have proved the following:

Lemma 4.1. *The social planner reaches a steady state at $D_{sp1} \in (0, B)$ for all $B \in ((n - 1)(A - c)^2/2n\rho, n(A - c)^2/8\rho)$.*

The situation in Lemma 4.1 is illustrated in Figure 3.

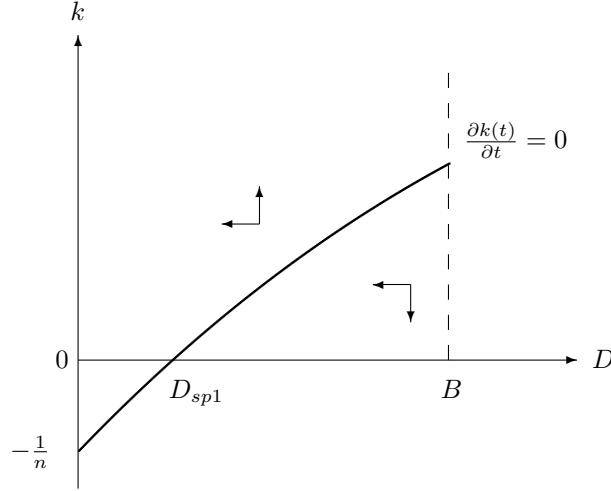


Fig. 3. Dynamics in the space (D, k) under social planning.

The dynamic properties of the system, as described by Figure 3, together with Lemma 4.1, can be used to prove the following:

Proposition 4.1. *For all $B \in ((n-1)(A-c)^2 2n\rho, n(A-c)^2 8\rho)$, the socially optimal steady state $(D_{sp1}, 0)$ is a saddle point, with each firm pricing at marginal cost.*

Evaluating D_{sp1} against D_{ol} reveals that $D_{sp1} < D_{ol}$ for all B such that $D_{sp1}, D_{ol} \in (0, B)$. The same holds if one compares D_{sp1} to D_{cl1} . Hence we have our final result:

Corollary 4.1. *$D_{sp1} < D_{ol} < D_{cl1}$ for all $B \in (0, B)$. This means that, at the private optima, product differentiation is too low as compared to social planning.*

By increasing product differentiation as compared to the private optima, the planner gets two eggs in one basket, because welfare (which coincides here with consumer surplus) increases due to both an increase in consumer utility U , which is everywhere decreasing in D for any given output level, and an increase in output levels.

In sharp contrast with the conventional wisdom (see, e.g., Tirole, 1988, ch. 7; and Eaton and Lipsey, 1989), in this model the private (i.e., firms') incentive to provide product differentiation is lower than the social incentive.¹ This is due to the fact

¹We have shown this result by contrasting the fully private equilibria with the first best where the planner controls both output and investment levels. It can be shown that the same conclusion obtains in the case where firms control output levels while a public agency interested in welfare maximisation controls investment levels.

that, in our model, product differentiation is partly a public good; therefore, its benefits spill over completely to all rivals, unlike what we see in so-called address models where firms choose locations (as in the literature stemming from Hotelling, 1929) or qualities (as in Shaked and Sutton, 1982; 1983), where the internalised benefit exceeds the external effect. From the policy-making perspective, the market failure originated from the externality and the public-good nature of product differentiation could be corrected through an appropriate R&D policy supporting the private efforts for investment in product differentiation. For instance, a policy-maker could introduce a subsidy $s \in (0, 1]$ rescaling downward the instantaneous investment cost, so that instantaneous profits would be $\pi_i(t) = [p_i(t) - c] q_i(t) - (1 - s) k_i(t)$. As a result, firms would invest more, and the associated equilibrium degree of product differentiation would be higher. Note, however, that R&D subsidization would not suffice to reach the first best, due to the distortion in output levels (as well as prices) at the oligopolistic equilibrium.

5. Conclusion

We have taken a differential game approach to analyse firms' R&D efforts to differentiate their products. Under our hypotheses, differentiation is, at least partly, a public good. We compared the open-loop Nash equilibrium strategy with the closed-loop decision rule. It was shown that, under the closed-loop decision rule, multiple steady states may exist, but only one is a saddle. We focused on this saddle point, and compared its properties with the properties of the saddle point emerging from the open-loop Nash equilibrium.

Some results are worth stressing. First, the link between market competition, as captured by the number of firms in the market, and optimal efforts for product differentiation is different under the two decision rules. Indeed, in open-loop equilibrium, a larger number of firms leads to a higher degree of differentiation, which in turn requires a higher aggregate effort. This conclusion has an anti-Schumpeterian flavour, lending support to the Arrowian idea that the harsher the competition, the higher the equilibrium level of R&D. In a closed-loop equilibrium, when the number of firms is sufficiently large, an increased number of firms leads to a smaller degree of differentiation, requiring a smaller aggregate R&D effort. This conclusion has a clear Schumpeterian flavour.

The "period by period" strategic reaction implied by the closed-loop decision rule leads to the result that the steady state individual firm's profits, as well as the steady state industry profits, are lower under the closed-loop than under the open-loop decision rule.

Moreover, the steady state degree of differentiation among goods under the open-loop is closer to the socially optimum level than the steady state degree of differentiation reached under the closed-loop rule. In both cases, the privately optimal product differentiation is too low as compared to social planning.

Finally, the social welfare in the steady state equilibrium under the closed-loop decision rule is smaller than that in open-loop equilibrium, due to higher industry profits and consumer surplus.

All the results are consistent with the fact that the differentiation is - in the present model - a public good. This assumption led to the conclusion that the

private incentive to produce differentiation is lower than the social incentive, which prompts for the adoption of an appropriate R&D policy, such as a subsidy, in order to increase private investments in product differentiation so as to ultimately enhance social welfare.

Acknowledgment

We thank Steffen Jørgensen, an anonymous referee and the audience at X International Symposium on Dynamic Games and Applications (St. Petersburg, July 8-11, 2002) for helpful comments and suggestions. The usual disclaimer applies. Roberto Cellini acknowledges financial support from the Young Researchers Programme of the University of Catania; Luca Lambertini acknowledges financial support from the University of Bologna and MIUR (60% funding scheme).

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