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AN EXTENSION TO $\mathbb{R}^k$ OF A RESULT BY FEKETE AND MEIJER

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Abstract

Given a finite configuration of points A in $\mathbb{R}^k$ endowed with the Manhattan distance, we prove that the ratio of the sum of the distances from a centroid of A over the sum of the distances from the Steiner center of A is bounded by $1 + (k-1)/k$; further, this bound can be attained. This fact extends to an arbitrary finite dimension $k \geq 2$ a result proved by Fekete and Meijer for $k \in \{2, 3\}$.

1. Introduction

Given a base space S and a finite collection of points A in S, a rather

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common problem in discrete geometry is to find a point $\mathbf{s} \in S$ such that a suitable distance of A from **s** is minimized: usually, this distance to be minimized is the sum of the distances of points in A from **s**. If the point **s** that solves the minimization problem must belong to A, then **s** is called a *centroid* of A; on the other hand. If the solution **s** is allowed not to be in A, then **s** is called the *Steiner center* of A. Observe that there can be several centroids of A, whereas, in particular metric spaces, the Steiner center is unique.

Of course, the nature of this minimization problem depends on the base space S where the collection of points A is located. In most of the cases considered in the literature, S is the k-dimensional space $\mathbb{R}^k$ endowed with the standard Euclidean distance; other base spaces that have been examined are directed trees [4] and metric spaces with Hamming distance [6]. In some cases considered in the literature, the set \$A\$ is infinite, e.g., it is a subset of an Euclidean space with continuously many points [1]. More recently, due to applications in bio-informatics, some attention has been devoted to an abstract version of this problem, where the base space S is an arbitrary metric or pseudo-metric space [2, 6].

If the base space S is the Euclidean space $\mathbb{R}^k$, the problem of finding a centroid (or the Steiner center) of a finite collection A of points is known as the *Fermat-Weber location problem* [3]. (A centroid is also called a 1-*median* of A.) More general versions of the Fermat-Weber problem have been considered in the literature. For example, given an integer number $d \geq 1$ and a finite collection A of points in $\mathbb{R}^k$, the *d-median problem* consists of finding d points (called *medians*) in a way such that the sum of the distances of each point of A from the closest median point is minimized. (For an even more general version of this problem, see [7].) Note that the classical Fermat-Weber location problem is the 1-median problem.

From a computational point of view, d-median problems in $\mathbb{R}^k$ with the Euclidean distance are rather lengthy and difficult to solve. On the other hand, if we endow $\mathbb{R}^k$ with the Manhattan distance, then the same problems can be solved more effectively by means of a linear algorithm. In this paper, we deal with a version of the problem that takes place in the space $\mathbb{R}^k$ endowed with the Manhattan distance.

To give a more detailed account of our result, we introduce basic notation and definitions. A *configuration* in $\mathbb{R}^k$ is a finite multi-set *A*, i.e., a finite set of points in $\mathbb{R}^k$ such that some of the points can be repeated more than once. Let *A* be a configuration in $\mathbb{R}^k$. For each $\mathbf{a} \in A$, we denote by $L_A(\mathbf{a})$ the total length of the *A-star centered* at **a**, i.e., the sum of the distances of **a** from all other points in *A*. Thus, a *centroid* of *A* is a point $\mathbf{c} \in A$ such that the length of the *A*-star centered at **c** is minimum among all points in *A*; we denote by $\mathrm{Centr}(A)$ the set of all centroids in *A*. Since by definition, we have $L_A(\mathbf{c}) = L_A(\mathbf{c}')$ for all $\mathbf{c}, \mathbf{c}' \in \mathrm{Centr}(A)$, we simplify notation and write C_A for the length of the *A*-star centered at any of the centroids of *A*. The *Steiner center* of *A* is the unique point $\mathbf{s} \in \mathbb{R}^k$ (not necessarily in *A*) that minimizes the total length of the *A*-star centered at it; we denote the length of its star by S_A.

We aim at showing that the worst-case ratio of C_A/S_A is independent of the number of points in the configuration *A*, being only a function of the dimension of the base space $\mathbb{R}^k$. Specifically, we prove the following:

Theorem 1.1. *For each configuration A in* $\mathbb{R}^k$, *we have*:

$$\frac{C_A}{S_A} \leq 1 + \frac{k-1}{k}.$$

In [5], the two authors provide separate proofs of this inequality in the 2-dimensional and the 3-dimensional case, showing that the worst-case ratio of C_A/S_A is $3/2$ in $\mathbb{R}^2$ and $5/3$ in $\mathbb{R}^3$. In the next section, we extend their technique to prove Theorem 1.1.

2. The Result

First, we introduce some notation.

Notation 2.1. The set of all configurations (i.e., finite multi-sets) in $\mathbb{R}^k$ is denoted by $\mathcal{P}^{(k)}$. In particular, the set of all configurations with *n* points in $\mathbb{R}^k$ is denoted by $\mathcal{P}_n^{(k)}$; thus, $\mathcal{P}^{(k)} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n^{(k)}$. Without loss of generality, we work with configurations whose Steiner center is the origin $\mathbf{0} = (0, 0, ..., 0) \in \mathbb{R}^k$. Let

$$\rho_n^{(k)} := \sup\{L_A(\mathbf{c})/L_A(\mathbf{0}) : \mathbf{c} \in \text{Centr}(A) \text{ and } A \in \mathcal{P}_n^{(k)}\}.$$

We may also assume without loss of generality that the length of the *A*-star centered at the Steiner center **0** of *A* is 1, i.e., $L_A(\mathbf{0}) = S_A = 1$. Thus, denoted by C_A the length of the *A*-star centered at any of the centroids of *A*, we have

$$\rho_n^{(k)} = \sup\{C_A/S_A : A \in \mathcal{P}_n^{(k)}\} = \sup\{C_A : A \in \mathcal{P}_n^{(k)}\}.$$

Finally, we denote by $\rho^{(k)}$ the worst-case value of the ratio C_A/S_A for configurations of points in $\mathbb{R}^k$, i.e.,

$$\rho^{(k)} := \sup\{C_A/S_A : A \in \mathcal{P}^{(k)}\} = \sup\{\rho_n^{(k)} : n \in \mathbb{N}\}.$$

Remark 2.2. The assumptions that the Steiner center **s** of a configuration *A* in $\mathbb{R}^k$ is **0** and the length of its *A*-star is $S_A = 1$ causes no loss of generality, because we are indeed working with equivalence classes of configurations in $\mathbb{R}^k$. Specifically, given a configuration *A* in $\mathbb{R}^k$, the equivalence class of *A* comprises all configurations in $\mathbb{R}^k$ obtained from *A* by re-scaling all points by a constant factor $\alpha \in \mathbb{R}$ and/or translating all *k* coordinates of each point by the same constants β_i, $i = 1, 2, ..., k$. To keep notation simple, henceforth, we avoid any explicit mention to equivalence classes of configurations, implicitly selecting in each class the representative such that its Steiner center is **0** and the length of the *A*-star centered at **0** is equal to 1.

Our goal is to prove the following result, which immediately yields Theorem 1.1 as a corollary:

Theorem 2.3. *For each* $k \geq 2,$ *the value* $\rho^{(k)}$ *is attained and is equal to* $1 + (k-1)/k.$

Note that the equalities $\rho^{(2)} = 3/2$ and $\rho^{(3)} = 5/3$ yield, respectively, Theorems 6 and 8 in [5]. The proof of Theorem 2.3 is similar - *mutatis mutandis* - to the proofs given for the cases $k = 2$ and $k = 3;$ therefore, we shall only point out the necessary modifications and sketch the rest. Before stating all results needed to prove Theorem 2.3, we introduce some new definitions.

Definition 2.4. A configuration $A \in \mathcal{P}_n^{(k)}$ is called an *extremal configuration* if it attains the value $\rho_n^{(k)},$ i.e., $C_A/S_A = \rho_n^{(k)}.$

Definition 2.5. Let $\mathbf{a} = (a_1, ..., a_k)$ and $\mathbf{b} = (b_1, ..., b_k)$ be two points of a configuration $A \in \mathcal{P}_n^{(k)}.$ We say that **a** *star-dominates* **b** if for each $i \in \{1, ..., k\},$ we have $0 \leq |b_i| \leq |a_i|;$ furthermore, if at least one of the inequalities is strict, then **a** *strictly star-dominates* **b**.

Definition 2.6. The *extremal boundary* of a configuration $A \in \mathcal{P}_n^{(k)}$ is the subset of $\mathbb{R}^k,$

$$\mathcal{E}(A) := \{\mathbf{e}_1, ..., \mathbf{e}_k, \mathbf{e}_{-1}, ..., \mathbf{e}_{-k}\}$$

whose $2k$ points are defined as follows. For each $i \in \{1, ..., k\},$ let $x_i :=$ $\max\{a_i : \mathbf{a} = (a_1, ..., a_n) \in A\}$ and $y_i := \min\{a_i : \mathbf{a} = (a_1, ..., a_n) \in A\}.$ (Note that $x_i \geq 0$ and $y_i \leq 0,$ because **0** is assumed to be the Steiner center of A.) For each $i \in \{1, ..., k\},$ set $\mathbf{e}_i := (0, ..., 0, x_i, 0, ..., 0)$ and $\mathbf{e}_{-i} :=$ $(0, ..., 0, y_i, 0, ..., 0),$ where x_i and y_i are the ith coordinates of, respectively, $\mathbf{e}_i$ and $\mathbf{e}_{-i}.$ We say that A is *e-closed* if $\mathcal{E}(A) \subseteq A.$

Next, we state some preliminary results; their proofs are similar to those given in [5], and are omitted.

Lemma 2.7. *For each* $k, n \in \mathbb{N} \setminus \{0, 1\}$, *there exists an extremal configuration* $A \in \mathcal{P}_n^{(k)}$.

Lemma 2.8. *For each extremal configuration* $A \in \mathcal{P}_n^{(k)}$, *we have* $L_A = L_A(\mathbf{a}) = \rho_n^{(k)}$ *for any* $\mathbf{a} = A$.

Corollary 2.9. *For each extremal configuration* $A \in \mathcal{P}_n^{(k)}$, *we cannot move a point such that* S_A *decreases by* $\varepsilon > 0$ *and* L_A *decreases by no more than* ε.

Corollary 2.10. *For each extremal configuration* $A \in \mathcal{P}_n^{(k)}$, *we cannot move a point such that* S_A *remains the same, one or more of the* $L_A(\mathbf{a})$ *increase, and none of them decreases.*

Corollary 2.11. *For each extremal configuration* $A \in \mathcal{P}_n^{(k)}$, *there cannot be two points* $\mathbf{a}, \mathbf{b} \in A$ *such that* **a** *strictly star-dominates* **b**.

Lemma 2.12. *Each extremal configuration* $A \in \mathcal{P}_n^{(k)}$ *is e-closed.*

Lemma 2.13. *For each* $n, k \in \mathbb{N} \setminus \{0, 1\}$, *we have* $\rho_n^{(k)} \leq \rho_{rn}^{k}$ *for all* $r \in \mathbb{N} \setminus \{0\}$.

In [5], in order to determine the limit of the sequence $\rho_n^{(k)}$ (for a fixed $k \in \{2, 3\}$), the authors define a subsequence $\beta_n^{(k)}$, which is obtained as the supremum of the values C_A / S_A for all configurations $A \in \mathcal{P}_n^{(k)}$ such that each point has *at least* one zero coordinate. They show that the following result holds:

Lemma 2.14. *For each extremal configuration* $A \in \mathcal{P}_n^{(k)}$, *where* $k \in \{2, 3\}$, *there can be at most* 2^k *points such that all of their coordinates are nonzero.*

Then, with the help of Corollary 2.11, Lemma 2.13 and Lemma 2.14, the authors show that for configurations with a sufficiently large number of points, the bounded number of points having all nonzero coordinates becomes negligible for the worst-case ratio, namely:

Lemma 2.15. *For* $k \in \{2, 3\}$, *we have*

$$\limsup_{n\to\infty} \rho_n^{(k)} = \limsup_{n\to\infty} \beta_n^{(k)}.$$

To generalize their approach, we need to define a slightly different subsequence. Fix $k \geq 2$. For each $n \geq 2$, let $\gamma_n^{(k)}$ be the supremum of the values C_A/S_A for all configurations $A \in \mathcal{P}_n^{(k)}$ such that each point of A has *exactly* one nonzero coordinate. Note that for each n, $k \geq 2$, we have $\gamma_n^{(k)} \leq \beta_n^{(k)}$. In order to give a general version of Lemma 2.14 (with $\gamma_n^{(k)}$ in place of $\beta_n^{(k)}$), which holds for all $k \geq 2$, we need a definition and a lemma.

Definition 2.16. Denote by $S = \{+, 0, -\}$ the set of signs. Let $\sigma : \mathbb{R} \to S$ be the *sign-map* in $\mathbb{R}$, defined by $\sigma(x) := +$ if $x > 0$, $\sigma(x) := 0$ if $x = 0$, and $\sigma(x) := -$ if $x < 0$. More generally, for each $k \geq 1$, define the *k-dimensional sign-map* $\sigma^k : \mathbb{R}^k \to S^k$ as follows: for all $\mathbf{x} = (x_1, ..., x_k) \in \mathbb{R}^k$, let $\sigma^k(\mathbf{x}) := (\sigma(x_1), ..., \sigma(x_k))$; the vector $\sigma^k(\mathbf{x})$ is called the *sign-string* of $\mathbf{x}$.

Lemma 2.17 *For each extremal configuration* $A \in \mathcal{P}_n^{(k)}$, *the sign-map* σ^k *restricted to* $A \setminus \mathcal{E}(A)$ *is injective.*

Proof. We prove the result by contradiction. Assume that $A \in \mathcal{P}_n^{(k)}$ is an extremal configuration such that there are two different points $\mathbf{a} = (a_1, ..., a_k)$ and $\mathbf{b} = (b_1, ..., b_k)$ in $A \setminus \mathcal{E}(A)$ having the same sign-string. Without loss of generality, assume that all coordinates of $\mathbf{a}$ and $\mathbf{b}$ are non-negative. Since $\mathbf{a}$ and $\mathbf{b}$ do not belong to $\mathcal{E}(A)$, Corollary 2.11 implies that both $\mathbf{a}$ and $\mathbf{b}$ have at least two nonzero coordinates and that there exist $i, j \in \{1, ..., k\}$ such that $0 < b_i < a_i$ and $0 < a_j < b_j$. Now proceed as in the proof of Lemma 10 in [5], and create a new configuration $A' := A \setminus \{\mathbf{a}, \mathbf{b}\} \cup \{\mathbf{a}', \mathbf{b}'\} \in \mathcal{P}_n^{(k)}$, where the points $\mathbf{a}', \mathbf{b}'$ are chosen in a way such that $S_A = S_{A'}$, $L_A(\mathbf{x}) = L_{A'}(\mathbf{x})$ for each $\mathbf{x} \in A \setminus \{\mathbf{a}, \mathbf{b}\}$, but $L_A(\mathbf{a}) < L_{A'}(\mathbf{a}')$ and $L_A(\mathbf{b}) < L_{A'}(\mathbf{b}')$. The configuration A' contradicts Corollary 2.10. □

In our setting, we obtain the following generalization of Lemma 2.14:

Corollary 2.18. *For each extremal configuration* $A \in \mathcal{P}_n^{(k)}$, *there can be at most* $3^k - 2k - 1$ *points in* $A \setminus \mathcal{E}(A)$.

Proof. Let $A \in \mathcal{P}_n^{(k)}$ be an extremal configuration. The total number of distinct *k*-sequences of symbols chosen from the set $S = \{+, 0, -\}$ is 3^k. If **a** is a point in $A \setminus \mathcal{E}(A)$, then Lemma 2.8 and Corollary 2.11 yield that its sign-string $\sigma^k(\mathbf{a})$ has at least two nonzero occurrences. Thus, the claim follows from Lemma 2.17. □

The same technique used in [5] allows us to generalize and sharpen Lemma 2.15 as follows:

Lemma 2.19. *For each* $k \geq 2$, *we have*

$$\limsup_{n \to \infty} \rho_n^{(k)} = \limsup_{n \to \infty} \gamma_n^{(k)}.$$

At this point, we can proceed as in [5] to complete the proof of Theorem 2.3.

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