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ScienceDirect
Topology and its Applications 153 (2006) 3308–3312
Topology and its Applications
www.elsevier.com/locate/topol

Minimax theorems for limits of parametrized functions having at most one local minimum lying in a certain set

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Received 30 December 2004; received in revised form 13 June 2005; accepted 13 June 2005

Abstract

In this paper, we establish some minimax theorems, of purely topological nature, that, through the variational methods, can be usefully applied to nonlinear differential equations. Here is a (simplified) sample: Let X be a Hausdorff topological space, $I \subseteq \mathbf{R}$ an interval and $\Psi : X \times I \to]-\infty, +\infty]$. Assume that the function $\Psi(x, \cdot)$ is lower semicontinuous and quasi-concave in I for all $x \in X$, while the function $\Psi(\cdot, q)$ has compact sublevel sets and one local minimum at most for each q in a dense subset of I. Then, one has

$$\sup_{q\in I}\inf_{x\in X}\Psi(x,q)=\inf_{x\in X}\sup_{q\in I}\Psi(x,q).$$

MSC: 49J35; 90C47; 54C30; 54D05

Keywords: Minimax theorems; Local minimum; Sublevel sets; Connectedness

If (X, τ) is a topological space, for any $\Psi : X \to]-\infty, +\infty]$, with $\mathrm{dom}(\Psi) \neq \emptyset$, we denote by τ_Ψ the weakest topology on X which contains both τ and the family of sets $\{\Psi^{-1}(]-\infty, r[)\}_{r\in\mathbf{R}}$.

The aim of this very short paper is to point out the following purely topological minimax result:

Theorem 1. *Let (X, τ) be a Hausdorff topological space, $I \subseteq \mathbf{R}$ an interval and $\Psi : X \times I \to]-\infty, +\infty]$ a function such that $\Psi(x, \cdot)$ is lower semicontinuous for all $x \in X$. Assume that there exist a number $\rho^* > \sup_I \inf_X \Psi$ and a set $D \subseteq I$, dense in I, such that, for each $\rho \in]-\infty, \rho^*[$ and each $q \in D$, the following conditions hold*:

(i) *the set $\{s \in I\colon\ \Psi(x, s) > \rho\}$ is an interval for all $x \in X$*;
(ii) *the set $\{x \in X\colon\ \Psi(x, q) \leqslant \rho\}$ is compact and sequentially compact*;
(iii) *there exist a function $\Phi_q : X \to \mathbf{R}$, bounded below on the set $\{x \in X\colon\ \Psi(x, q) \leqslant \rho^*\}$, and a sequence $\{\mu_n\}$ in $\mathbf{R}^+$ converging to 0 such that, for each $\lambda > 0$ small enough, the function $\Psi(\cdot, q) + \lambda\Phi_q(\cdot)$ is sequentially lower semicontinuous, and, for each $n \in \mathbf{N}$, the function $\Psi(\cdot, q) + \mu_n\Phi_q(\cdot)$ has at most one $\tau_{\Psi(\cdot,q)}$-local minimum lying in the set $\{x \in X\colon\ \Psi(x, q) < \rho^*\}$.*

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doi:10.1016/j.topol.2005.06.020

Then, one has

$$\sup_{q\in I}\inf_{x\in X}\Psi(x,q)=\inf_{x\in X}\sup_{q\in I}\Psi(x,q).$$

The most remarkable feature of Theorem 1 is its applicability, via the variational methods, to nonlinear differential equations. Indeed, in a successive paper, we will apply Theorem 1 in the setting where X is a Sobolev space endowed with the weak topology and Ψ is the energy functional of a Dirichlet problem of the type

$$\begin{cases}-\Delta u=f(x,u,q) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

We will derive Theorem 1 from the following more general result:

Theorem 2. *Let X be a topological space, $I\subseteq\mathbf{R}$ an interval and $\Psi: X\times I\to\,]-\infty,+\infty]$ a function such that $\Psi(x,\cdot)$ is lower semicontinuous for all $x\in X$. Assume that there exist a number $\rho^*>\sup_I\inf_X\Psi$, a point $\hat{q}\in I$ and two sets $D_1, D_2\subseteq I$, both dense in I, such that for each $\rho\in\,]-\infty,\rho^*[$, the following conditions hold*:

(α) *the set $\{q\in I\colon\ \Psi(x,q)>\rho\}$ is an interval for all $x\in X$;*
(β) *the set $\{x\in X\colon\ \Psi(x,q)\leqslant\rho\}$ is closed for all $q\in D_1$ and compact for $q=\hat{q}$, while the set $\{x\in X\colon\ \Psi(x,q)<\rho\}$ is connected for all $q\in D_2$.*

Then, one has

$$\sup_{q\in I}\inf_{x\in X}\Psi(x,q)=\inf_{x\in X}\sup_{q\in I}\Psi(x,q).$$

For other topological minimax theorems, we refer to [1–3] and to the references therein.

Proof of Theorem 2. First, fix a non-decreasing sequence $\{I_n\}$ of compact sub-intervals of I, with $\hat{q}\in I_1$, such that $\bigcup_{n\in\mathbf{N}}I_n=I$. Now, fix $n\in\mathbf{N}$. We claim that

$$\sup_{q\in I_n}\inf_{x\in X}\Psi(x,q)=\inf_{x\in X}\sup_{q\in I_n}\Psi(x,q). \tag{1}$$

Arguing by contradiction, suppose that

$$\sup_{q\in I_n}\inf_{x\in X}\Psi(x,q)<\inf_{x\in X}\sup_{q\in I_n}\Psi(x,q).$$

Fix ρ satisfying

$$\sup_{q\in I_n}\inf_{x\in X}\Psi(x,q)<\rho<\min\Big\{\rho^*,\inf_{x\in X}\sup_{q\in I_n}\Psi(x,q)\Big\}.$$

Set

$$S=\big\{(x,q)\in X\times I_n\colon\ \Psi(x,q)<\rho\big\}$$

as well as, for each $q\in I_n$,

$$S^q=\big\{x\in X\colon\ (x,q)\in S\big\}.$$

Since $\sup_{I_n}\inf_X\Psi<\rho$, one has $S^q\neq\emptyset$ for all $q\in I_n$. Let $I_n=[a_n,b_n]$. Put

$$A=\Big\{(x,q)\in S\colon\ q<b_n,\ \sup_{s\in]q,b_n]}\Psi(x,s)>\rho\Big\}$$

and

$$B=\Big\{(x,q)\in S\colon\ q>a_n,\ \sup_{s\in[a_n,q[}\Psi(x,s)>\rho\Big\}.$$

Observe that A, B are non-empty. Indeed, let $x_1 \in S^{a_n}$ and $x_2 \in S^{b_n}$. Since $\rho < \inf_X \sup_{I_n} \Psi$, there are $t, s \in I_n$ such that $\min\{\Psi(x_1, t), \Psi(x_2, s)\} > \rho$. Since $\sup\{\Psi(x_1, a_n), \Psi(x_2, b_n)\} < \rho$, it follows that $t > a_n$ and $s < b_n$. Consequently, $(x_1, a_n) \in A$ and $(x_2, b_n) \in B$. Furthermore, observe that A, B are open in S. Let us see this for A, the other case being analogous. So, let $(x_0, q_0) \in A$. Since the function $\Psi(x_0, \cdot)$ is lower semicontinuous, the set $\{q \in]q_0, b_n]\colon \Psi(x_0, q) > \rho\}$ is non-empty and open in I_n and hence it contains a $q^* \in D_1$, by density. At this point, by (β), the set

$$\left(\left\{x \in X\colon \Psi(x, q^*) > \rho\right\} \times [a_n, q^*[\right) \cap S$$

is clearly a neighbourhood of (x_0, q_0) in S which is contained in A. We now prove that $S = A \cup B$. Indeed, let $(x, q) \in S \setminus A$. We have seen above that $S^{a_n} \times \{a_n\} \subseteq A$, and so $q > a_n$. If $q = b_n$, the fact that $(x, q) \in B$ has been likewise proved above. Suppose $q < b_n$. Thus, we have $\sup_{s \in]q, b_n]} \Psi(x, s) \leqslant \rho$. From this, it clearly follows that $\sup_{s \in [a_n, q[} \Psi(x, s) > \rho$ (note that $\Psi(x, q) < \rho$), and so $(x, q) \in B$. Furthermore, we have $A \cap B = \emptyset$. Indeed, if $(x_1, q_1) \in A \cap B$, there would be $t, s \in I_n$, with $t < q_1 < s$, such that $\min\{\Psi(x_1, t), \Psi(x_1, s)\} > \rho$. By (α), the set $\{s \in I\colon \Psi(x_1, s) > \rho\}$ is an interval, and so we would have $\Psi(x_1, q_1) > \rho$, against the fact that $(x_1, q_1) \in S$. Let $p_{\mathbf{R}}$ be the projection from $X \times \mathbf{R}$ onto $\mathbf{R}$. Now, consider the sets $p_{\mathbf{R}}(A)$ and $p_{\mathbf{R}}(B)$. Since $p_{\mathbf{R}}(S) = I_n$, thanks to the properties of A, B seen above, they are non-empty, open in I_n, and cover I_n. So, by the connectedness of I_n, we have $p_{\mathbf{R}}(A) \cap p_{\mathbf{R}}(B) \neq \emptyset$. Since D_2 is dense in I, there exists some $q' \in D_2 \cap p_{\mathbf{R}}(A) \cap p_{\mathbf{R}}(B)$. By (β), the set $S^{q'}$ (and hence $S^{q'} \times \{q'\}$ too) is connected. But $S^{q'} \times \{q'\}$ meets both A and B, and this just contradicts its being connected.

So, we have proved (1). Finally, let us prove the theorem. Again arguing by contradiction, suppose that

$$\sup_{q \in I} \inf_{x \in X} \Psi(x, q) < \inf_{x \in X} \sup_{q \in I} \Psi(x, q).$$

Choose r satisfying

$$\sup_{q \in I} \inf_{x \in X} \Psi(x, q) < r < \min\left\{\rho^*, \inf_{x \in X} \sup_{q \in I} \Psi(x, q)\right\}.$$

For each $n \in \mathbf{N}$, put

$$C_n = \left\{x \in X\colon \sup_{q \in I_n} \Psi(x, q) \leqslant r\right\}.$$

Note that $C_n \neq \emptyset$. Indeed, otherwise, we would have

$$r \leqslant \inf_{x \in X} \sup_{q \in I_n} \Psi(x, q) = \sup_{q \in I_n} \inf_{x \in X} \Psi(x, q) \leqslant \sup_{q \in I} \inf_{x \in X} \Psi(x, q).$$

Furthermore, for each $x \in X$, we have

$$\sup_{q \in I_n} \Psi(x, q) = \sup_{q \in D_1 \cap I_n} \Psi(x, q)$$

since $\Psi(x, \cdot)$ is lower semicontinuous and D_1 is dense in I. So, we have

$$C_n = \bigcap_{q \in D_1 \cap I_n} \left\{x \in X\colon \Psi(x, q) \leqslant r\right\}.$$

Consequently, $\{C_n\}$ is a non-increasing sequence of non-empty closed subsets of the compact set $\{x \in X\colon \Psi(x, \hat{q}) \leqslant \rho^*\}$. Therefore, one has $\bigcap_{n \in \mathbf{N}} C_n \neq \emptyset$. Let $x^* \in \bigcap_{n \in \mathbf{N}} C_n$. Then, one has

$$\sup_{q \in I} \Psi(x^*, q) = \sup_{n \in \mathbf{N}} \sup_{q \in I_n} \Psi(x^*, q) \leqslant r$$

and so

$$\inf_{x \in X} \sup_{q \in I} \Psi(x, q) \leqslant r,$$

a contradiction. The proof is complete. □

Remark 1. It is clear from the proof that when I is compact, Theorem 2 holds without requiring the existence of the point $\hat{q}$ with the indicated property. Likewise, when $D_1 = I$, the assumption that $\Psi(x, \cdot)$ is lower semicontinuous for all $x \in X$ becomes superfluous.

Before deriving Theorem 1 from Theorem 2, for the reader's convenience, we recall the basic result of [4].

Theorem A. ([4], Theorem 1) *Let (X,τ) be a Hausdorff topological space and $\Phi, \Psi : X \to]-\infty,+\infty]$ two functions. Assume that there is $\sigma > \inf_X \Psi$ such that the set $\overline{\Psi^{-1}(]-\infty,\sigma[)}$ is compact and sequentially compact, has at least k connected components and each of them intersects the interior of* $\mathrm{dom}(\Phi)$. *Moreover, suppose that the function Φ is bounded below in $\overline{\Psi^{-1}(]-\infty,\sigma[)}$ and that the function $\Psi+\lambda\Phi$ is sequentially lower semicontinuous for each $\lambda>0$ small enough.*

Then, there exists $\lambda^*>0$ such that, for each $\lambda\in]0,\lambda^*[$, the function $\Psi+\lambda\Phi$ has at least k τ_Ψ-local minima lying in $\mathrm{dom}(\Phi)\cap\Psi^{-1}(]-\infty,\sigma[)$.

Proof of Theorem 1. We have only to check that Ψ satisfies the hypotheses of Theorem 2. So, let $q\in D$, and $r<\sigma<\rho^*$. By (ii), it clearly follows that the set $\{x\in X\colon \Psi(x,q)\leqslant\sigma\}$ is closed (since X is Hausdorff) and that the set $\overline{\{x\in X\colon \Psi(x,q)<\sigma\}}$ is compact and sequentially compact. From (iii), it follows that the functions $\Psi(\cdot,q)$, Φ_q do not satisfy the conclusion of Theorem A with $k=2$, and so, since function Φ_q is bounded below in $\overline{\{x\in X\colon \Psi(x,q)<\sigma\}}$ and the function $\Psi(\cdot,q)+\lambda\Phi_q(\cdot)$ is sequentially lower semicontinuous for each $\lambda>0$ small enough, it necessarily follows that the set $\overline{\{x\in X\colon \Psi(x,q)<\sigma\}}$ is connected. Now, observe that, since $\overline{\{x\in X\colon \Psi(x,q)<\sigma\}}\subseteq\{x\in X\colon \Psi(x,q)\leqslant\sigma\}$, one has

$$\{x\in X\colon \Psi(x,q)\leqslant r\}=\bigcap_{r<\sigma<\rho^*}\overline{\{x\in X\colon \Psi(x,q)<\sigma\}}.$$

Therefore, the closed set $\{x\in X\colon \Psi(x,q)\leqslant r\}$, as the intersection of a non-increasing sequence of compact and connected sets, is connected too. Finally, let $\rho\in]-\infty,\rho^*[$. Since

$$\{x\in X\colon \Psi(x,q)<\rho\}=\bigcup_{r<\rho}\{x\in X\colon \Psi(x,q)\leqslant r\},$$

it follows that the set $\{x\in X\colon \Psi(x,q)<\rho\}$ is connected. So, all the assumptions of Theorem 2 are satisfied, and the conclusion follows. □

Remark 2. We do not know whether, in Theorem 1, condition (iii) can be improved replacing $\tau_{\Psi(\cdot,q)}$ with τ. However, this is the case when we are allowed to take $\Phi_q=0$. To see this, we first establish the following.

Proposition 1. *Let X be a Hausdorff topological space and $\Psi : X\to]-\infty,+\infty]$ a function. Assume that, for some $r>\inf_X\Psi$, Ψ has at most one local minimum lying in $\Psi^{-1}(]-\infty,r])$ and that $\Psi^{-1}(]-\infty,\rho])$ is compact for all $\rho\in]-\infty,r]$.*

Then, the set $\Psi^{-1}(]-\infty,r])$ is connected.

Proof. Assume that the set $\Psi^{-1}(]-\infty,r])$ is disconnected. Then, since it is closed, there would be two non-empty, closed and disjoint sets A, B such that

$$\Psi^{-1}\big(]-\infty,r]\big)=A\cup B.$$

Since the restriction of Ψ to $\Psi^{-1}(]-\infty,r])$ is lower semicontinuous and A, B are compact, there are $x_1\in A$ and $x_2\in B$ such that $\Psi(x_1)=\inf_{x\in A}\Psi$ and $\Psi(x_2)=\inf_{x\in B}\Psi$. Now, choose two open and disjoint sets $\Omega_1,\Omega_2\in X$ such that $A\subseteq\Omega_1$ and $B\subseteq\Omega_2$. It is readily seen that $\Psi(x_1)\leqslant\Psi(x)$ for all $x\in\Omega_1$ and that $\Psi(x_2)\leqslant\Psi(x)$ for all $x\in\Omega_2$. Therefore, x_1 and x_2 would be two distinct local minima of Ψ lying in $\Psi^{-1}(]-\infty,r])$, against one of the hypotheses. □

Theorem 3. *Let X be a Hausdorff topological space, $I\subseteq\mathbf{R}$ an interval and $\Psi : X\times I\to]-\infty,+\infty]$ a function such that $\Psi(x,\cdot)$ is lower semicontinuous for all $x\in X$. Assume that there exist a number $\rho^*>\sup_I\inf_X\Psi$ and a set $D\subseteq I$, dense in I, such that, for each $\rho\in]-\infty,\rho^*[$ and each $q\in D$, the following conditions hold*:

(i′) *the set $\{s\in I\colon \Psi(x,s)>\rho\}$ is an interval for all $x\in X$;*

(ii′) *the set* $\{x \in X\colon\ \Psi(x,q) \leqslant \rho\}$ *is compact*;
(iii′) *the function* $\Psi(\cdot,q)$ *has at most one local minimum lying in the set* $\{x \in X\colon\ \Psi(x,q) < \rho^*\}$.

Then, one has

$$\sup_{q\in I}\inf_{x\in X}\Psi(x,q) = \inf_{x\in X}\sup_{q\in I}\Psi(x,q).$$

Proof. The proof is exactly the same as that of Theorem 1, with the only change of using Proposition 1 instead of Theorem 1 of [4]. □

It is worth noticing the following consequence of Theorem 1 that we will likewise apply to nonlinear differential equations:

Theorem 4. *Let* (X,τ) *be a Hausdorff topological space,* $I \subseteq \mathbf{R}$ *an interval, and* $\Psi : X \times I \to]-\infty,+\infty]$ *a function such that* $\Psi(x,\cdot)$ *is lower semicontinuous for all* $x \in X$ *and*

$$\sup_{q\in I}\inf_{x\in X}\Psi(x,q) < \inf_{x\in X}\sup_{q\in I}\Psi(x,q).$$

Assume that there exist a number $\rho^* > \sup_I \inf_X \Psi$ *and an open set* $D \subseteq I$*, dense in* I*, such that, for each* $\rho \in]-\infty,\rho^*[$*, the set* $\{q \in I\colon\ \Psi(x,q) > \rho\}$ *is an interval for all* $x \in X$*, while the set* $\{x \in X\colon\ \Psi(x,q) \leqslant \rho\}$ *is compact and sequentially compact for all* $q \in D$.

Then, there exist a non-empty open set $A \subset I$ *such that, for every* $q \in A$ *and for every function* $\Phi : X \to \mathbf{R}$*, bounded below on the set* $\{x \in X\colon\ \Psi(x,q) \leqslant \rho^*\}$ *and such that, for each* $\lambda > 0$ *small enough, the function* $\Psi(\cdot,q) + \lambda\Phi(\cdot)$ *is sequentially lower semicontinuous, there exists* $\delta > 0$ *such that, for each* $\mu \in]0,\delta]$*, the function* $\Psi(\cdot,q) + \mu\Phi(\cdot)$ *has at least two* $\tau_{\Psi(\cdot,q)}$*-local minima lying in the set* $\{x \in X\colon\ \Psi(x,q) < \rho^*\}$.

Proof. Denote by D' the set of all $q \in I$ such that there exist a function $\Phi_q : X \to \mathbf{R}$, bounded below on the set $\{x \in X\colon\ \Psi(x,q) \leqslant \rho^*\}$, and a sequence $\{\mu_n\}$ in $\mathbf{R}^+$ converging to 0 such that, for each $\lambda > 0$ small enough, the function $\Psi(\cdot,q) + \lambda\Phi_q(\cdot)$ is sequentially lower semicontinuous, and, for each $n \in \mathbf{N}$, the function $\Psi(\cdot,q) + \mu_n\Phi_q(\cdot)$ has at most one $\tau_{\Psi(\cdot,q)}$-local minimum lying in the set $\{x \in X\colon\ \Psi(x,q) < \rho^*\}$. By Theorem 1, the set $D \cap D'$ is not dense in I. Consequently, since D is open and dense in I, the set D' is not dense in I, and so the set $A = \operatorname{int}(I \setminus D')$ satisfies the conclusion. □

Analogously, from Theorem 3 we get

Theorem 5. *Let* X *be a Hausdorff topological space,* $I \subseteq \mathbf{R}$ *an interval, and* $\Psi : X \times I \to]-\infty,+\infty]$ *a function such that* $\Psi(x,\cdot)$ *is lower semicontinuous for all* $x \in X$ *and*

$$\sup_{q\in I}\inf_{x\in X}\Psi(x,q) < \inf_{x\in X}\sup_{q\in I}\Psi(x,q).$$

Assume that there exist a number $\rho^* > \sup_I \inf_X \Psi$ *and an open set* $D \subseteq I$*, dense in* I*, such that, for each* $\rho \in]-\infty,\rho^*[$*, the set* $\{q \in I\colon\ \Psi(x,q) > \rho\}$ *is an interval for all* $x \in X$*, while the set* $\{x \in X\colon\ \Psi(x,q) \leqslant \rho\}$ *is compact for all* $q \in D$.

Then, there exist a non-empty open set $A \subset I$ *such that, for every* $q \in A$*, the function* $\Psi(\cdot,q)$ *has at least two local minima lying in the set* $\{x \in X\colon\ \Psi(x,q) < \rho^*\}$.

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