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Positive, Negative, and Nodal Solutions to Elliptic Differential Inclusions Depending on a Parameter

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Abstract

The homogeneous Dirichlet problem for a partial differential inclusion involving the p-Laplace operator and depending on a parameter $\lambda > 0$ is investigated. The existence of three smooth solutions, a smallest positive, a biggest negative, and a nodal one, is obtained for any λ sufficiently large by combining variational methods with truncation techniques.

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1 Introduction and main result

In the present paper we deal with the following Dirichlet problem for a partial differential inclusion, depending on a parameter $\lambda > 0$:

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u \in \lambda \partial j(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with a C^2 boundary $\partial\Omega$, $p > 1$ and Δ_p denotes the p-Laplace operator

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \forall\, u \in W_0^{1,p}(\Omega).$$

Let p^* stand for the Sobolev critical exponent, i.e., $p^* = Np(N-p)^{-1}$ if $p < N$, and $p^* = +\infty$ if $p \geq N$. The reaction term $\partial j(x, s)$ is the generalized gradient of a non-smooth potential $s \mapsto j(x, s)$, which is subject to the following conditions.

$\mathbf{H}_j$ $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exist constants $a_1 > 0$, $1 < q < p^*$, $0 < a_2 \leq a_3$ such that:

- (i) $j(x, \cdot)$ is locally Lipschitz for almost every $x \in \Omega$ and $j(\cdot, 0) \in L^1(\Omega)$;
- (ii) $|\xi| \leq a_1 \left(1 + |s|^{q-1}\right)$ a.e. in Ω and for all $s \in \mathbb{R}$, $\xi \in \partial j(x, s)$;
- (iii) $\displaystyle\limsup_{|s| \to +\infty} \frac{j(x, s)}{|s|^p} \leq 0$ uniformly for almost every $x \in \Omega$;
- (iv) $\displaystyle a_2 \leq \liminf_{s \to 0^+} \frac{\min \partial j(x, s)}{s^{p-1}} \leq \limsup_{s \to 0^+} \frac{\max \partial j(x, s)}{s^{p-1}} \leq a_3$ uniformly a.e. in Ω;
- (v) $\displaystyle a_2 \leq \liminf_{s \to 0^-} \frac{\max \partial j(x, s)}{|s|^{p-2} s} \leq \limsup_{s \to 0^-} \frac{\min \partial j(x, s)}{|s|^{p-2} s} \leq a_3$ uniformly a.e. in Ω.

Let us remark that the functions $(x, s) \mapsto \min \partial j(x, s), \max \partial j(x, s)$ are well defined because $\partial j(x, s)$ is a compact interval. Moreover, they are measurable. Hypotheses $\mathbf{H}_j(iii), (iv), (v)$ define our potential $j(x, \cdot)$ as p-sub-linear at infinity and p-linear at zero, respectively. From $\mathbf{H}_j(iv)$ it follows that (P_λ) admits the zero solution for all $\lambda > 0$. We are interested in finding nontrivial solutions. More precisely, we shall establish the existence, for $\lambda > 0$ big enough, of at least three nontrivial solutions to (P_λ): a smallest positive, a greatest negative, and a nodal (that is, sign-changing) solution.

Many authors proved multiplicity results for boundary value problems driven by the p-Laplacian, with or without parameters, based on convenient assumptions about the behavior of the nonlinearities at infinity and/or at zero. We mention the classical work of Ambrosetti, Brézis & Cerami [1], which treats the case $p = 2$ and (single-valued) nonlinearities of the type

$$s \mapsto \lambda |s|^{q-2} s + |s|^{r-2} s \; (1 < q < 2 < r < 2^*).$$

A positive solution, for $\lambda > 0$ sufficiently small, is obtained via sub- and super-solution arguments. This result has been extended by Ambrosetti, Garcia Azorero & Peral [2] to the general case $p > 1$. Finally, the paper of Carl & Perera [6] exploits sub- and super-solution techniques to get both extremal constant sign and nodal solutions for a problem without parameters.

Here, we consider a *set-valued* reaction term, thus embracing also possibly discontinuous single-valued nonlinearities (see Chang [7]). In this new setting, variational methods and sub- and super-solutions can be still employed to achieve multiplicity results. For instance, Averna, Marano & Motreanu [3] proved the existence of positive, negative, and nodal solutions for a partial differential inclusion depending on a parameter λ. Carl & Motreanu [5] extended the ideas of [6] to the set-valued framework. Iannizzotto & Papageorgiou [12] obtained both constant sign and nodal solutions for an inclusion with Neumann boundary conditions. All these works are based on the critical point theory for locally Lipschitz functions.

Our main result reads as follows. By λ_2 we denote the second positive eigenvalue of the operator $-\Delta_p$ on $W_0^{1,p}(\Omega)$; cf. Section 2.

Theorem 1.1 *If hypotheses $\mathbf{H}_j$ hold then, for every $\lambda > \lambda_2/a_2 + 1$, problem (P_λ) possesses at least three nontrivial solutions, a smallest positive, a biggest negative, and a nodal one, lying in $C_0^1(\overline{\Omega})$.*

The approach taken is based on critical point theory for locally Lipschitz functions and truncation techniques. By minimizing a suitable truncated energy functional we first find a positive solution to

(P_λ). Sub- and super-solution arguments then provide a smallest positive solution $\hat{u}_\lambda \in C_0^1(\overline{\Omega})$. The construction of a greatest negative solution $\check{u}_\lambda \in C_0^1(\overline{\Omega})$ is analogous. A third solution $\tilde{u}_\lambda \in C_0^1(\overline{\Omega})$ is next obtained through a mountain pass-like theorem while a non-smooth deformation lemma ensures that it is not zero. Finally, due to the extremality of $\hat{u}_\lambda$ and $\check{u}_\lambda$, the solution $\tilde{u}_\lambda$ must be nodal. The C^1-case has been very recently investigated by Marano, Motreanu & Puglisi [15]. Our framework presents new nontrivial difficulties. In particular, the presence of a set-valued reaction term $\partial j(x, s)$ requires completely different devices in order to deal with truncations and verify the appropriate Palais-Smale condition.

As an example, the non-smooth potential $j : \mathbb{R} \to \mathbb{R}$ defined by

$$j(s) := \begin{cases} \ln(1 + |s|)|s|^{p-1} & \text{if } |s| \le 1, \\ \ln(2)|s|^q & \text{if } |s| > 1, \end{cases}$$

for every $s \in \mathbb{R}$, where $1 < q < p$, is locally Lipschitz and satisfies hypotheses $\mathbf{H}_j$ with $a_1 = \ln(2)$ and $a_2 = a_3 = p$.

The rest of the paper is organized as follows. Section 2 contains the necessary prerequisites. In Section 3 we prove the existence of extremal constant sign solutions (Theorems 3.1 and 3.2). Section 4 focuses on the existence of a nodal solution (Theorem 4.1).

2 Mathematical background

We start by recalling some basic facts from non-smooth analysis and refer the reader to the book of Gasiński & Papageorgiou [10] for a recent account of the theory.

Let $(X, \|\cdot\|)$ be a real Banach space. The symbols $(X^*, \|\cdot\|_*)$ denote its topological dual while $\langle\cdot,\cdot\rangle$ is the duality pairing between X^* and X. A functional $\varphi : X \to \mathbb{R}$ is called *locally Lipschitz* when for every $u \in X$ there correspond a neighborhood U of u and a constant $L > 0$ such that

$$|\varphi(v) - \varphi(w)| \le L\|v - w\| \quad \forall\, v, w \in U.$$

The *generalized directional derivative* of φ at $u \in X$ along the direction $v \in X$ is defined as

$$\varphi^\circ(u; v) := \limsup_{w \to u,\ \tau \to 0^+} \frac{\varphi(w + \tau v) - \varphi(w)}{\tau}.$$

Proposition 2.1 ([10, Proposition 1.3.7]) *If φ, $\psi : X \to \mathbb{R}$ are locally Lipschitz, then*

(*i*) $\varphi^\circ(u; \cdot)$ *is positively homogeneous, sub-additive, and continuous for all $u \in X$;*

(*ii*) $\varphi^\circ(u; -v) = (-\varphi)^\circ(u; v)$ *for all $u, v \in X$;*

(*iii*) *if $\varphi \in C^1(X)$, then $\varphi^\circ(u; v) = \langle\varphi'(u), v\rangle$ for all $u, v \in X$;*

(*iv*) $(\varphi + \psi)^\circ(u; v) \le \varphi^\circ(u; v) + \psi^\circ(u; v)$ *for all $u, v \in X$.*

The *generalized gradient* of φ at $u \in X$ is the set

$$\partial\varphi(u) := \{u^* \in X^* \ :\ \langle u^*, v\rangle \le \varphi^\circ(u; v) \quad \forall\, v \in X\}.$$

Proposition 2.2 ([10, Propositions 1.3.8 and 1.3.9]) *If φ, $\psi : X \to \mathbb{R}$ are locally Lipschitz, then*

(*i*) $\partial\varphi(u)$ *is convex, closed, and weakly* compact for all $u \in X$;*

(*ii*) *the multifunction* $\partial\varphi : X \to 2^{X^*}$ *is upper semicontinuous with respect to the weak** *topology on* X^**;*

(*iii*) *if* $\varphi \in C^1(X)$*, then* $\partial\varphi(u) = \{\varphi'(u)\}$ *for all* $u \in X$*;*

(*iv*) $\partial(\varphi + \psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u)$ *for all* $u \in X$.

Let us also recall Lebourg's mean value theorem.

Proposition 2.3 ([10, Proposition 1.3.14]) *If* $\varphi : X \to \mathbb{R}$ *is locally Lipschitz and* $u, v \in X$*, then there exist* $\tau \in (0, 1)$ *and* $w^* \in \partial\varphi(\tau u + (1 - \tau)v)$ *such that*

$$\varphi(v) - \varphi(u) = \langle w^*, v - u\rangle.$$

We say $u \in X$ is a *critical point* of φ provided $0 \in \partial\varphi(u)$. Write, as usual,

$$K(\varphi) := \{u \in X \ : \ u \text{ is a critical point of } \varphi\}$$

and, for any $c \in \mathbb{R}$,

$$K_c(\varphi) := \{u \in K(\varphi) \ : \ \varphi(u) = c\}\,.$$

Moreover, set

$$\varphi^c := \{u \in X \ : \ \varphi(u) \leq c\}\,.$$

The functional φ is said to satisfy the *(nonsmooth) Palais-Smale condition* (shortly, (PS)) when every sequence (u_n) in X such that $(\varphi(u_n))$ is bounded and

$$\varphi^\circ(u_n; v - u_n) + \varepsilon_n\|v - u_n\| \geq 0 \quad \forall\, v \in X,\ n \in \mathbb{N}$$

with a sequence $(\varepsilon_n) \subset \mathbb{R}^+$ such that $\varepsilon_n \to 0$, has a convergent subsequence.

The following nonsmooth versions of two classical results in critical point theory, namely the *deformation theorem* and the *mountain pass theorem*, hold.

Theorem 2.1 (special case of [10, Theorem 2.1.1]; see also [12, Theorem 2]) *If the locally Lipschitz function* $\varphi : X \to \mathbb{R}$ *satisfies* (PS) *and there are real numbers* $a < b$ *such that* $K_a(\varphi)$ *is a finite set consisting only of local minimizers of* φ *while* $K_c(\varphi) = \emptyset$ *for all* $c \in (a, b]$*, then there exists a continuous function* h *from* $[0, 1] \times \varphi^b$ *into* φ^b *complying with*

(*i*) $h(0, u) = u$ *and* $h(1, u) \in K_a(\varphi)$ *for all* $u \in \varphi^b$*;*

(*ii*) $\varphi(h(t, u)) \leq \varphi(u)$ *for all* $(t, u) \in [0, 1] \times \varphi^b$.

Theorem 2.2 ([10, Theorem 2.1.3]) *If the locally Lipschitz function* $\varphi : X \to \mathbb{R}$ *satisfies* (PS) *and there exist* $\hat{u}, \check{u} \in X$ *and* $0 < r < \|\hat{u} - \check{u}\|$ *such that*

$$\max\{\varphi(\hat{u}), \varphi(\check{u})\} \leq \eta_r := \inf_{u\in\partial B_r(\hat{u})} \varphi(u),$$

then, for

$$\Gamma := \{\gamma \in C([-1, 1], X) \ : \ \gamma(-1) = \check{u},\ \gamma(1) = \hat{u}\}\,, \quad c := \inf_{\gamma\in\Gamma} \max_{t\in[-1,1]} \varphi(\gamma(t)),$$

one has $c \geq \eta_r$ *and* $K_c(\varphi) \neq \emptyset$.

Our study will involve two real Banach spaces. The first is the Sobolev space $W_0^{1,p}(\Omega)$, equipped with the norm $\|u\| = \|\nabla u\|_p$. It is known that it is separable, reflexive, and that $W^{-1,p'}(\Omega)$ $(1/p + 1/p' = 1)$ denotes its dual. Moreover, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^\nu(\Omega)$ is compact for any $\nu \in [1, p^*)$. Define, provided $u, v \in W_0^{1,p}(\Omega)$,

$$\langle A(u), v\rangle := \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx.$$

The nonlinear operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is strictly monotone and enjoys the $(S)_+$-property; see e.g. Motreanu, Motreanu & Papageorgiou [17]. The other space is $C_0^1(\overline{\Omega})$, which turns out to be an ordered Banach space with order cone

$$C_+ := \left\{u \in C_0^1(\overline{\Omega}) \ : \ u(x) \geq 0 \quad \forall\, x \in \Omega\right\},$$

whose interior is

$$\mathrm{int}(C_+) = \left\{u \in C_+ \ : \ u(x) > 0 \ \forall\, x \in \Omega \text{ and } \frac{\partial u}{\partial n}(x) < 0 \ \forall\, x \in \partial\Omega\right\}.$$

Here, as usual, $n(x)$ denotes the outward unit normal to $\partial\Omega$ at any point x; cf. Gasiński & Papageorgiou [11, Remark 6.2.10].

Now, consider the following nonlinear weighted eigenvalue problem driven by the p-Laplacian:

$$\begin{cases} -\Delta_p u = \lambda m(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $m \in L^\infty(\Omega)_+$, $m \neq 0$. Problem (2.1) admits a sequence of eigenvalues $0 < \lambda_1(m) < \lambda_2(m) < \ldots$ which are known as *Ljusternik-Schnirelman eigenvalues*. A complete description of the spectrum of the p-Laplacian is yet an open question. However, we only need the first two eigenvalues, whose well-known properties are summarized below; see [11, Section 6.2], Cuesta, de Figueiredo & Gossez [8], besides Lê [13].

Proposition 2.4 *If $m \in L^\infty(\Omega)_+ \setminus \{0\}$ then $\lambda_1(m)$ is simple, isolated, and the corresponding eigenfunctions have constant sign in Ω. The eigenfunctions associated with any other eigenvalue $\lambda > \lambda_1(m)$ are nodal. If $m = 1$ then $\lambda_1 := \lambda_1(1)$ admits the following variational characterization:*

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p}{\|u\|_p^p},$$

with the infimum being attained at an eigenfunction $\hat{u}_1 \in \mathrm{int}(C_+)$ that fulfils $\|\hat{u}_1\|_p = 1$. Finally, if $\hat{m}, \check{m} \in L^\infty(\Omega)_+ \setminus \{0\}$, $\hat{m} \geq \check{m}$, and $\hat{m} \neq \check{m}$, then $\lambda_1(\hat{m}) < \lambda_1(\check{m})$.

We will also use the following min-max characterization of the second eigenvalue.

Proposition 2.5 *If $m = 1$ then $\lambda_2 := \lambda_2(1)$ possesses the variational characterization*

$$\lambda_2 = \inf_{\gamma \in \Gamma_0} \max_{t \in [-1,1]} \|\gamma(t)\|^p,$$

where

$$\Gamma_0 := \{\gamma \in C([-1, 1], S) \ : \ \gamma(\pm 1) = \pm\hat{u}_1\}$$

and

$$S := \left\{u \in W_0^{1,p}(\Omega) \ : \ \|u\|_p = 1\right\}$$

is endowed with the induced $W_0^{1,p}(\Omega)$-topology. Moreover, there are no eigenvalues of (2.1) *between λ_1 and λ_2.*

3 Extremal constant-sign solutions

To simplify notation, write $X := W_0^{1,p}(\Omega)$. Given $\lambda > 0$, we say that $u \in X$ is a *(weak) solution* of problem (P_λ) if $\Delta_p u \in L^{q'}(\Omega)$, where $1/q + 1/q' = 1$, and

$$-\Delta_p u(x) \in \lambda \partial j(x, u(x)) \text{ for almost every } x \in \Omega.$$

Hypothesis $\mathbf{H}_j(iv)$ forces $0 \in \partial j(x, 0)$. Thus, as pointed out in the Introduction, (P_λ) admits the zero solution. The nonsmooth energy functional $\varphi_\lambda : X \to \mathbb{R}$ associated with (P_λ) is defined by setting

$$\varphi_\lambda(u) := \frac{\|u\|^p}{p} - \lambda \int_\Omega j(x, u)dx \quad \forall\, u \in X.$$

The next simple result will be helpful. For general regularity information we refer to Lieberman [14].

Proposition 3.1 *The functional $\varphi_\lambda : X \to \mathbb{R}$ is locally Lipschitz. Moreover, if $u \in K(\varphi_\lambda)$ then $u \in C_0^1(\overline{\Omega})$ and u solves (P_λ).*

Proof. Obviously, $u \mapsto \|u\|^p/p$ is a C^1-functional whose derivative is the operator A. Aubin-Clarke's theorem [10, Theorem 1.3.10] ensures that the function

$$u \mapsto \int_\Omega j(x, u)dx$$

is Lipschitz continuous on any bounded subset of $L^q(\Omega)$ and its generalized gradient is included in the set

$$N(u) := \left\{ w \in L^{q'}(\Omega) \ : \ w(x) \in \partial j(x, u(x)) \text{ for almost every } x \in \Omega \right\}.$$

Since X continuously embeds in $L^q(\Omega)$, the functional φ_λ turns out to be locally Lipschitz on X. By Proposition 2.2 we have

$$\partial\varphi_\lambda(u) \subseteq A(u) - \lambda N(u).$$

Now, if $u \in X$ complies with $0 \in \partial\varphi_\lambda(u)$ then

$$A(u) = \lambda w \text{ in } X^* \tag{3.1}$$

for some $w \in N(u)$. Hence, $\Delta_p u \in L^{q'}(\Omega)$ and u solves (P_λ). Combining $\mathbf{H}_j(ii)$ with (3.1) yields the estimate

$$-u\Delta_p u \le a_1(|u| + |u|^q) \text{ a.e. in } \Omega,$$

which, on account of [10, Theorem 1.5.5], implies $u \in L^\infty(\Omega)$. From $\mathbf{H}_j(ii)$ and (3.1) it follows $\Delta_p u \in L^\infty(\Omega)$. So, by [10, Theorem 1.5.6], the function u belongs to $C_0^1(\overline{\Omega})$. □

The next maximum principle-type result for problem (P_λ) holds.

Proposition 3.2 *If $u \in C_+$ is a solution of (P_λ) and $u \neq 0$, then $u \in \mathrm{int}(C_+)$.*

Proof. Fix $0 < \theta < a_2$. Through $\mathbf{H}_j(iv)$ one can find a $\delta > 0$ such that

$$\min \partial j(x, s) \ge \theta s^{p-1}$$

for almost every $x \in \Omega$ and all $0 < s < \delta$. Since in (3.1) we now have $w \in L^\infty(\Omega)$, there exists a constant $c > 0$ such that

$$\Delta_p u \le c u^{p-1} \text{ a.e. in } \Omega.$$

The Vázquez maximum principle [19, Theorem 5] directly gives $u \in \text{int}(C_+)$. □

A comparison principle for differential inclusions, due to Carl & Motreanu [5, Corollary 4.1] (see also Carl, Le & Motreanu [4, Corollary 4.24]), will be employed. Recall that $\underline{u} \in W^{1,p}(\Omega)$ is called a *sub-solution* of (P_λ) provided $\underline{u} \le 0$ on $\partial\Omega$ and there exists $w \in N(\underline{u})$ such that

$$\langle A(\underline{u}), v\rangle \le \lambda \int_\Omega wv\, dx \text{ for all } v \in X \text{ with } v(x) \ge 0 \text{ a.e. in } \Omega.$$

The definition of a *super-solution* is analogous.

Proposition 3.3 *If $\underline{u}$, $\overline{u} \in X$ are a sub-solution and a super-solution, respectively, of (P_λ) and $\underline{u}(x) \le \overline{u}(x)$ for almost all $x \in \Omega$ then the set*

$$H := \left\{ u \in X \ :\ u \textit{ solves } (P_\lambda),\ \underline{u}(x) \le u(x) \le \overline{u}(x) \textit{ a.e. in } \Omega \right\}$$

is nonempty, compact, and directed with respect to the point-wise order. Moreover, H has a smallest and a biggest element.

Let us now come to our first result.

Theorem 3.1 *If $\lambda > \lambda_1/a_2$ then (P_λ) possesses a smallest positive solution $\hat{u}_\lambda \in \text{int}(C_+)$.*

Proof. Fix $\lambda > \lambda_1/a_2$. We split the proof in several steps.

Claim 1. Problem (P_λ) has a positive solution $\hat{u} \in \text{int}(C_+)$.

Consider the truncation $j_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$j_+(x, s) := j(x, s^+) \quad \forall\, (x, s) \in \Omega \times \mathbb{R}.$$

Here, as usual, $s^+ := \max\{0, s\}$. A simple computation shows that $j(x, \cdot)$ is locally Lipschitz and its generalized gradient with respect to s is

$$\partial j_+(x, s) = \begin{cases} \{0\} & \text{if } s < 0, \\ \partial j(x, s) & \text{if } s > 0, \end{cases}$$

$$\partial j_+(x, 0) \subseteq \{\tau\xi \ :\ \xi \in \partial j(x, 0),\ \tau \in [0, 1]\}.$$

Moreover, the functional

$$\varphi_\lambda^+(u) := \frac{\|u\|^p}{p} - \lambda \int_\Omega j_+(x, u)dx \quad \forall\, u \in X$$

is weakly sequentially lower semi-continuous and locally Lipschitz, which can be verified through the same arguments introduced in the proof of Proposition 3.1. We shall prove that φ_λ^+ turns out to be coercive. Let $0 < \varepsilon < \lambda_1/(p\lambda)$. By $\mathbf{H}_j(iii)$ there exists $M > 0$ such that

$$j_+(x, s) < \varepsilon s^p \text{ for almost all } x \in \Omega \text{ and all } s > M,$$

while $\mathbf{H}_j(ii)$ and Proposition 2.3 give

$$j_+(x, s) \le j(x, 0) + a_1(M + M^q) \text{ a.e. in } \Omega \text{ and for every } 0 \le s \le M.$$

Using Proposition 2.4 we obtain

$$\begin{aligned}\varphi_\lambda^+(u) &\geq \frac{\|u\|^p}{p} - \lambda \int_{\{u \leq M\}} (j(x,0) + a_1(M + M^q))\, dx - \lambda\varepsilon \|u\|_p^p \\ &\geq \left(\frac{1}{p} - \frac{\varepsilon\lambda}{\lambda_1}\right)\|u\|^p - M',\end{aligned}$$

with a constant $M' > 0$. Since $p > 1$, the choice of ε allows to conclude that φ_λ^+ is coercive.

Now, the Weierstrass theorem yields $\hat{u} \in X$ fulfilling

$$\varphi_\lambda^+(\hat{u}) = \inf_{u \in X} \varphi_\lambda^+(u). \tag{3.2}$$

Like in the proof of Proposition 3.1, we see that $\hat{u} \in C_0^1(\overline{\Omega})$ and there exists $\hat{w} \in L^{q'}(\Omega)$ having the following properties:

$$\hat{w}(x) \in \partial j_+(x, \hat{u}(x)) \text{ for almost every } x \in \Omega;$$

$$A(\hat{u}) = \lambda \hat{w} \text{ in } X^*. \tag{3.3}$$

Exploiting (3.3) with the test function $\hat{u}^- := -\min\{0, \hat{u}\}$ yields

$$-\int_\Omega |\nabla \hat{u}^-|^p dx = \lambda \int_\Omega \hat{w}\, \hat{u}^- dx = 0$$

because $\hat{w}(x) = 0$ whenever $\hat{u}(x) < 0$. Thus, $\hat{u} \in C_+$.

We shall next prove that $\hat{u} \neq 0$. Let $\theta < a_2$ satisfy $\lambda > \lambda_1/\theta$. By $\mathbf{H}_j(iv)$ there exists $\delta > 0$ such that

$$\min \partial j_+(x, s) > \theta s^{p-1} \text{ a.e. in } \Omega \text{ and for every } 0 < s < \delta.$$

Proposition 2.3 leads to

$$j_+(x, s) \geq j(x, 0) + \frac{\theta s^p}{p}.$$

Since the function $\hat{u}_1$ introduced in Proposition 2.4 is bounded, we can find $\tau > 0$ such that $\tau\hat{u}_1(x) < \delta$ for all $x \in \Omega$, which enables us to write

$$\begin{aligned}\varphi_\lambda^+(\tau\hat{u}_1) &\leq \frac{\tau^p \|\hat{u}_1\|^p}{p} - \lambda \int_\Omega \left(j(x,0) + \frac{\theta\tau^p \hat{u}_1^p}{p}\right) dx \\ &\leq \frac{\tau^p}{p}(\lambda_1 - \lambda\theta) + \varphi_\lambda^+(0) \\ &< \varphi_\lambda^+(0).\end{aligned}$$

On account of (3.2) this implies $\hat{u} \neq 0$. Hence, by Proposition 3.2, $\hat{u} \in \mathrm{int}(C_+)$. Since $\partial j_+(x, s) = \partial j(x, s)$, for almost all $x \in \Omega$ and all $s > 0$, the function $\hat{u}$ solves (P_λ).

Claim 2. The set of positive solutions of (P_λ) has a smallest element $\hat{u}_\lambda$ in $\mathrm{int}(C_+)$.

Let $\varepsilon_0 > 0$ be such that $\varepsilon\hat{u}_1(x) < \hat{u}(x)$ in Ω for all $\varepsilon \in (0, \varepsilon_0)$. Such a constant ε_0 exists because $\hat{u} \in \mathrm{int}(C_+)$. Thanks to $\mathbf{H}_j(iv)$ and the inequality $\lambda > \lambda_1/a_2$ one has, provided ε is sufficiently small,

$$-\Delta_p(\varepsilon\hat{u}_1) = \lambda_1 \varepsilon^{p-1} \hat{u}_1^{p-1} \leq \lambda \min \partial j_+(x, \varepsilon\hat{u}_1) \text{ a.e. in } \Omega.$$

Therefore, $\varepsilon\hat{u}_1$ turns out to be a sub-solution of (P_λ). Obviously, $\hat{u}$ can be regarded as a super-solution. Proposition 3.3 guarantees that the ordered set

$$H_\varepsilon := \{u \in X \ : \ \varepsilon\hat{u}_1(x) \leq u(x) \leq \hat{u}(x)\ \forall\, x \in \Omega \text{ and } u \text{ solves } (P_\lambda)\}$$

is nonempty and contains a smallest solution of (P_λ) with respect to the point-wise order. If u_n, with $n \in \mathbb{N}$ big enough, denotes the smallest solution of (P_λ) lying in $H_{1/n}$ then

$$A(u_n) = \lambda w_n \text{ in } X^*, \tag{3.4}$$

for some $w_n \in N(u_n)$. By the minimality property we get $u_n \geq u_{n+1}$. So, there exists a function $\hat{u}_\lambda : \Omega \to \mathbb{R}^+$ such that

$$\lim_{n\to+\infty} u_n(x) = \hat{u}_\lambda(x) \text{ pointwise in } \Omega. \tag{3.5}$$

The sequence (u_n) is bounded in X because (3.4), $\mathbf{H}_j(ii)$, and Proposition 2.3 entail

$$\begin{aligned} \|u_n\|^p &= \langle A(u_n), u_n\rangle \\ &= \lambda \int_\Omega w_n\, u_n\, dx \\ &\leq \lambda a_1 \int_\Omega \left(u_n + u_n^q\right) dx \\ &\leq \lambda a_1 \int_\Omega (\hat{u} + \hat{u}^q)\, dx. \end{aligned}$$

From (3.5), passing to a subsequence when necessary, it follows $\hat{u}_\lambda \in X$, $u_n \rightharpoonup \hat{u}_\lambda$ in X, as well as $u_n \to \hat{u}_\lambda$ in $L^q(\Omega)$. Through (3.4) again, $\mathbf{H}_j(ii)$, and Proposition 2.3 we achieve

$$\begin{aligned} \langle A(u_n), u_n - \hat{u}_\lambda\rangle &= \lambda \int_\Omega w_n(u_n - \hat{u}_\lambda)dx \\ &\leq \lambda a_1 \int_\Omega (1 + u_n^{q-1})(u_n - \hat{u}_\lambda)dx \\ &\leq M\|u_n - \hat{u}_\lambda\|_q, \end{aligned}$$

for some $M > 0$, which implies

$$\limsup_{n\to+\infty} \langle A(u_n), u_n - \hat{u}_\lambda\rangle \leq 0.$$

Since A has the $(S)_+$ property, $u_n \to \hat{u}_\lambda$ in X.

By $\mathbf{H}_j(ii)$ the sequence (w_n) turns out to be bounded in X^*. Thus, possibly along a subsequence, $w_n \rightharpoonup \hat{w}_\lambda$ in X^*. At this point, Proposition 2.2, combined with (3.4), ensures that $\hat{w}_\lambda \in N(\hat{u}_\lambda)$ and

$$A(\hat{u}_\lambda) = \lambda \hat{w}_\lambda \text{ in } X^*, \tag{3.6}$$

namely $\hat{u}_\lambda$ solves (P_λ). The same reasoning exploited in the proof of Proposition 3.1 produces here $\hat{u}_\lambda \in C_+$.

Let us next verify that $\hat{u}_\lambda \neq 0$. Arguing by contradiction, suppose $\hat{u}_\lambda = 0$. The function $v_n := u_n/\|u_n\|$ is evidently a weak solution of the Dirichlet problem

$$\begin{cases} -\Delta_p v = \lambda \dfrac{w_n(x)}{u_n(x)^{p-1}}|v|^{p-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

whose weight function belongs to $L^\infty(\Omega)$. Indeed, if $\theta \in (\lambda_1/\lambda, a_2)$ then (3.5), with $\hat{u}_\lambda = 0$, and $\mathbf{H}_j(iv)$ give, for any sufficiently large n,

$$\frac{w_n(x)}{u_n(x)^{p-1}} \geq \frac{\min \partial j(x, u_n(x))}{u_n(x)^{p-1}} > \theta \tag{3.8}$$

as well as

$$\frac{w_n(x)}{u_n(x)^{p-1}} \leq \frac{\max \partial j(x, u_n(x))}{u_n(x)^{p-1}} < 2a_3$$

a.e. in Ω. Now, through (3.8) and Proposition 2.4 we obtain

$$\lambda_1\left(\frac{w_n}{u_n^{p-1}}\right) < \lambda_1(\theta) = \frac{\lambda_1}{\theta} < \lambda,$$

i.e., $\lambda > 0$ is an eigenvalue of the weighted problem (3.7) greater than the first one. Hence, by Proposition 2.4, the corresponding eigenfunction v_n should be nodal. However, this contradicts the definition of v_n. Consequently, $\hat{u}_\lambda \neq 0$, and the assertion immediately follows from Proposition 3.2.

Finally, we prove that $\hat{u}_\lambda$ turns out to be the smallest among positive solutions of (P_λ).

Suppose $u \in X$ is a positive solution to (P_λ). Because of Proposition 3.2 we have $u \in \mathrm{int}(C_+)$, whence

$$\frac{\hat{u}_1(x)}{n} \leq u(x) \quad \forall\, x \in \Omega$$

provided n is big enough. The function $\frac{1}{n}\hat{u}_1$ is a sub-solution of (P_λ), while it is easily seen that $\min\{u(x), \hat{u}(x)\}$ turns out to be a super-solution. By Proposition 3.3, there exists a solution $v \in X$ of (P_λ) such that

$$\frac{\hat{u}_1(x)}{n} \leq v(x) \leq \min\{u(x), \hat{u}(x)\} \text{ for almost every } x \in \Omega.$$

The minimality of u_n in $H_{1/n}$ entails

$$u_n(x) \leq v(x) \leq u(x) \text{ a.e. in } \Omega.$$

Letting $n \to \infty$ we obtain $\hat{u}_\lambda \leq u$. This completes the proof. □

The next result is achieved in the same way as Theorem 3.1.

Theorem 3.2 *If $\lambda > \lambda_1/a_2$ then problem (P_λ) possesses a biggest negative solution $\check{u}_\lambda \in -\mathrm{int}(C_+)$.*

Remark 3.1 The preceding results can be established with a more general differential operator in place of the p-Laplacian. Precisely, let $\mathcal{A} : X \to X^*$ be a continuous, $(p-1)$-homogeneous, odd, and uniformly positive potential operator. Assuming that $\mathcal{A}$ has the $(S)_+$ property, a variational characterization of the *first* eigenvalue of $\mathcal{A}$, analogous to that given by Proposition 2.4, holds true; see for instance Perera, Agarwal & O'Regan [18, Theorem 4.6]. So, a straightforward extension of our method leads to the existence of extremal constant-sign solutions for the problem

$$\begin{cases} -\mathcal{A}u \in \lambda\partial j(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We do not know whether a variational characterization of the *second* eigenvalue of $\mathcal{A}$, similar to the one in Proposition 2.5, is available. On the other hand, it will play an essential role for finding nodal solutions. Accordingly, in what follows we only treat the case $\mathcal{A} = \Delta_p$.

4 Nodal solutions

Let $\lambda > \lambda_1/a_2$. The present section deals with the existence of solutions to problem (P_λ) that lie in the interval $[\check{u}_\lambda, \hat{u}_\lambda]$. To this end, we introduce the following truncation-perturbation of j:

$$\tilde{j}(x, s) := \begin{cases} j(x, \check{u}_\lambda(x)) + \max \partial j(x, \check{u}_\lambda(x))(s - \check{u}_\lambda(x)) & \text{if } s < \check{u}_\lambda(x), \\ j(x, s) & \text{if } \check{u}_\lambda(x) \leq s \leq \hat{u}_\lambda(x), \\ j(x, \hat{u}_\lambda(x)) + \min \partial j(x, \hat{u}_\lambda(x))(s - \hat{u}_\lambda(x)) & \text{if } s > \hat{u}_\lambda(x). \end{cases}$$

A simple argument ensures that $\tilde{j} : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory's conditions, $\tilde{j}(x,\cdot)$ is locally Lipschitz, and hypotheses like $\mathbf{H}_j$ hold true. So, due to Proposition 3.1, the functional $\tilde{\varphi}_\lambda : X \to \mathbb{R}$ defined by

$$\tilde{\varphi}_\lambda(u) = \frac{\|u\|^p}{p} - \lambda \int_\Omega \tilde{j}(x,u)dx$$

for all $u \in X$ is locally Lipschitz.

Proposition 4.1 *If $u \in K(\tilde{\varphi}_\lambda)$ then $u \in C_0^1(\overline{\Omega})$, u solves (P_λ), and $\check{u}_\lambda(x) \le u(x) \le \hat{u}_\lambda(x)$ in Ω.*

Proof. Since $0 \in \partial\tilde{\varphi}_\lambda(u)$, the same argument exploited in the proof of Proposition 3.1 ensures here that $u \in C_0^1(\overline{\Omega})$ and there exists $w \in \tilde{N}(u)$, where

$$\tilde{N}(u) := \left\{ w \in L^{q'}(\Omega) \ : \ w(x) \in \partial\tilde{j}(x,u(x)) \text{ a.e. in } \Omega \right\},$$

such that

$$A(u) = \lambda w \text{ in } X^*. \tag{4.1}$$

Using (4.1) and (3.6) with the test function $(u - \hat{u}_\lambda)^+ \in X$ yields

$$\begin{aligned} \langle A(u) - A(\hat{u}_\lambda), (u-\hat{u}_\lambda)^+\rangle &= \lambda \int_\Omega (w - \hat{w}_\lambda)(u-\hat{u}_\lambda)^+ dx \\ &= \lambda \int_\Omega (\min \partial j(x,\hat{u}_\lambda) - \hat{w}_\lambda)\,(u-\hat{u}_\lambda)^+ dx \\ &\le 0, \end{aligned}$$

because

$$\partial\tilde{j}(x,s) = \{\min \partial j(x,\hat{u}_\lambda(x))\} \text{ a.e. in } \Omega \text{ and for all } s > \hat{u}_\lambda(x)$$

while $\hat{w}_\lambda \in N(\hat{u}_\lambda)$. The strict monotonicity of A now leads to $u \le \hat{u}_\lambda$. A similar argument provides $u \ge \check{u}_\lambda$. Thus, $\check{u}(x) \le u(x) \le \hat{u}(x)$ in Ω. Since

$$\partial\tilde{j}(x,\hat{u}_\lambda(x)) \subseteq \partial j(x,\hat{u}_\lambda(x)) \quad \text{and} \quad \partial\tilde{j}(x,\check{u}_\lambda(x)) \subseteq \partial j(x,\check{u}_\lambda(x))$$

we clearly have $w \in N(u)$. So, by (4.1), the function u solves (P_λ). □

The next result deals with the existence of sign-changing solutions.

Theorem 4.1 *If $\lambda > \lambda_2/a_2 + 1$ then (P_λ) possesses a nodal solution $\tilde{u}_\lambda \in C_0^1(\overline{\Omega})$ such that $\check{u}_\lambda(x) \le \tilde{u}_\lambda(x) \le \hat{u}_\lambda(x)$ for every $x \in \Omega$.*

Proof. Since $\lambda_2/a_2 + 1 > \lambda_1/a_2$, Theorems 3.1–3.2 provide a smallest positive solution $\hat{u}_\lambda \in \mathrm{int}(C_+)$ and a biggest negative solution $\check{u}_\lambda \in -\mathrm{int}(C_+)$ of (P_λ). Write $\tilde{j}_+(x,s) := \tilde{j}(x,s^+)$. The functional $\tilde{\varphi}_\lambda^+ : X \to \mathbb{R}$ given by

$$\tilde{\varphi}_\lambda^+(u) := \frac{\|u\|^p}{p} - \lambda \int_\Omega \tilde{j}_+(x,u)dx \quad \forall u \in X$$

turns out to be locally Lipschitz, weakly sequentially lower semi-continuous, and coercive. So, as in the proof of Theorem 3.1, there exists $\tilde{u} \in X$ fulfilling

$$\tilde{\varphi}_\lambda^+(\tilde{u}) = \inf_{u\in X} \tilde{\varphi}_\lambda^+(u) < \tilde{\varphi}_\lambda^+(0). \tag{4.2}$$

We claim that $\tilde{u} = \hat{u}_\lambda$. In fact, Proposition 3.2 yields $\tilde{u} \in \mathrm{int}(C_+)$ while the restrictions to C_+ of the functionals $\tilde{\varphi}_\lambda$ and $\tilde{\varphi}_\lambda^+$ coincide. Hence, $\tilde{u}$ is a $C_0^1(\overline{\Omega})$-local minimizer for the functional

$\tilde{\varphi}_\lambda$. Thanks to [10, Proposition 4.6.10], namely a non-smooth extension of Theorem 1.1 in Garcia Azorero, Manfredi & Peral Alonso [9], this entails that $\tilde{u}$ is also a X-local minimizer for $\tilde{\varphi}_\lambda$, whence $0 \in \partial\tilde{\varphi}_\lambda(\tilde{u})$. Through Proposition 4.1 we get

$$0 < \tilde{u}(x) \leq \hat{u}_\lambda(x) \quad \forall\, x \in \Omega.$$

The minimality of $\hat{u}_\lambda$ forces $\tilde{u} = \hat{u}_\lambda$, and the assertion follows.

Accordingly, (4.2) rephrases as

$$\tilde{\varphi}_\lambda(\hat{u}_\lambda) = \inf_{u\in X} \tilde{\varphi}^+_\lambda(u) < \tilde{\varphi}_\lambda(0). \tag{4.3}$$

Observe that $\hat{u}_\lambda$ turns out to be a local minimizer of $\tilde{\varphi}_\lambda$ because $\hat{u}_\lambda \in \mathrm{int}(C_+)$. The same holds true for $\check{u}_\lambda \in -\mathrm{int}(C_+)$, with

$$\tilde{\varphi}_\lambda(\check{u}_\lambda) < \tilde{\varphi}_\lambda(0). \tag{4.4}$$

Two situations may now occur.

Case 1. $\tilde{\varphi}_\lambda$ has not a strict local minimum at $\hat{u}_\lambda$ or $\check{u}_\lambda$. Then there exists another local minimizer $\tilde{u}_\lambda \in X \setminus \{\hat{u}_\lambda, \check{u}_\lambda\}$ satisfying

$$\tilde{\varphi}_\lambda(\tilde{u}_\lambda) = \tilde{\varphi}_\lambda(\hat{u}_\lambda) \text{ or } \tilde{\varphi}_\lambda(\tilde{u}_\lambda) = \tilde{\varphi}_\lambda(\check{u}_\lambda).$$

Proposition 4.1 ensures that $\tilde{u}_\lambda \in C^1_0(\overline{\Omega})$, $\tilde{u}_\lambda$ solves (P_λ), and

$$\check{u}_\lambda(x) \leq \tilde{u}_\lambda(x) \leq \hat{u}_\lambda(x) \quad \forall\, x \in \Omega. \tag{4.5}$$

From (4.3) or (4.4) it clearly follows $\tilde{u}_\lambda \neq 0$. Let us show that $\tilde{u}_\lambda$ has to be nodal. If the assertion were false, as there is no loss of generality in assuming $\tilde{u}_\lambda \in C_+$, then we would get $\tilde{u}_\lambda \in \mathrm{int}(C_+)$ thanks to Proposition 3.2. However, by (4.5) and the minimality of $\hat{u}_\lambda$, this is impossible.

Case 2. $\tilde{\varphi}_\lambda$ has a strict local minimum both at $\hat{u}_\lambda$ and at $\check{u}_\lambda$. Then we can find $r \in (0, \|\hat{u}_\lambda - \check{u}_\lambda\|)$ such that

$$\max\{\tilde{\varphi}_\lambda(\hat{u}_\lambda), \tilde{\varphi}_\lambda(\check{u}_\lambda)\} < \eta_r := \inf_{u\in\partial B_r(\hat{u}_\lambda)} \tilde{\varphi}_\lambda(u). \tag{4.6}$$

The same reasoning made in the proof of Theorem 3.1 guarantees here that $\tilde{\varphi}_\lambda$ is coercive. Moreover, it fulfils condition (PS). To see this, pick a sequence (u_n) in X such that $(\tilde{\varphi}_\lambda(u_n))$ is bounded and

$$(\tilde{\varphi}_\lambda)^\circ(u_n; v - u_n) + \varepsilon_n\|v - u_n\| \geq 0 \quad \forall\, v \in X,\ n \in \mathbb{N}, \tag{4.7}$$

where $\varepsilon_n \to 0^+$. By coercivity of $\tilde{\varphi}_\lambda$, the sequence (u_n) must be bounded. So, passing to a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^q(\Omega)$. Through (4.7), $\mathbf{H}_j(ii)$, Propositions 2.1 and 2.3 one has

$$\begin{aligned} 0 &\leq \langle A(u_n), u - u_n\rangle + \lambda \int_\Omega a_1(1 + |u_n|^{q-1})|u - u_n|dx + \varepsilon_n\|u - u_n\| \\ &\leq \langle A(u_n), u - u_n\rangle + \lambda M\|u - u_n\|_q + \varepsilon_n\|u - u_n\|, \end{aligned}$$

with a constant $M > 0$. Therefore,

$$\limsup_{n\to+\infty} \langle A(u_n), u_n - u\rangle \leq 0.$$

Since A has the $(S)_+$ property, $u_n \to u$ in X, as desired.

We are now in a position to apply Theorem 2.2. Set

$$\Gamma := \{\gamma \in C([-1, 1], X) \ :\ \gamma(-1) = \check{u}_\lambda,\ \gamma(1) = \hat{u}_\lambda\}$$

and

$$\tilde{c} := \inf_{\gamma\in\Gamma} \max_{t\in[-1,1]} \tilde{\varphi}_\lambda(\gamma(t)).$$

Then there exists a critical point $\tilde{u}_\lambda \in X$ of $\tilde{\varphi}_\lambda$ such that

$$\tilde{\varphi}_\lambda(\tilde{u}_\lambda) = \tilde{c} \geq \eta_r.$$

By (4.6) this implies $\tilde{u}_\lambda \neq \hat{u}_\lambda, \check{u}_\lambda$. Proposition 4.1 ensures that $\tilde{u}_\lambda \in C^1_0(\overline{\Omega})$, $\tilde{u}_\lambda$ solves (P_λ), and

$$\check{u}_\lambda(x) \leq \tilde{u}_\lambda(x) \leq \hat{u}_\lambda(x) \quad \forall\, x \in \Omega.$$

Claim. $\tilde{u}_\lambda \neq 0$.

Choose $\theta \in (0, a_2)$ such that $\lambda > \lambda_2/\theta + 1$. Proposition 2.5 provides $\gamma_0 \in \Gamma_0$ satisfying

$$\|\gamma_0(t)\|^p < \lambda_2 + \theta \quad \forall\, t \in [-1,1]. \tag{4.8}$$

Without loss of generality, we may suppose $\gamma_0 \in C([-1,1], C^1_0(\Omega))$, because $S \cap C^1_0(\Omega)$ turns out to be dense in S (endowed with the induced X-topology) while $\gamma_0([-1,1])$ is compact, Through $\mathbf{H}_j(i)$, (iv), (v), and Proposition 2.3, we obtain $\delta > 0$ such that

$$j(x,s) > j(x,0) + \frac{\theta|s|^p}{p} \text{ a.e. in } \Omega \text{ and for every } |s| \leq \delta\,. \tag{4.9}$$

Finally, since $\gamma_0([-1,1])$ is compact in $C^1_0(\overline{\Omega})$ while $\hat{u}_\lambda, -\check{u}_\lambda \in \operatorname{int}(C_+)$, there exists $\varepsilon > 0$ such that

$$\varepsilon|u(x)| < \min\{\delta,\ \hat{u}_\lambda(x),\ -\check{u}_\lambda(x)\} \quad \forall\, x \in \Omega,\ u \in \gamma_0([-1,1]). \tag{4.10}$$

Gathering (4.8)–(4.10) together and recalling that $\|u\|_p = 1$ leads to

$$\begin{aligned}
\tilde{\varphi}_\lambda(\varepsilon u) &= \frac{\varepsilon^p}{p}\|u\|^p - \lambda \int_\Omega \tilde{j}(x,\varepsilon u)dx \\
&\leq \frac{\varepsilon^p}{p}(\lambda_2 + \theta) - \lambda \int_\Omega \left(j(x,0) + \frac{\theta\varepsilon^p|u|^p}{p} \right) dx \\
&\leq \frac{\varepsilon^p}{p}(\lambda_2 + \theta - \lambda\theta) + \tilde{\varphi}_\lambda(0) \quad \forall\, u \in \gamma_0([-1,1]).
\end{aligned}$$

By the choice of λ this entails

$$\tilde{\varphi}_\lambda(\varepsilon\gamma_0(t)) < \tilde{\varphi}_\lambda(0) \quad \forall\, t \in [-1,1]. \tag{4.11}$$

Let us now apply Theorem 2.1 to the functional $\tilde{\varphi}^+_\lambda$, with $a := \tilde{\varphi}^+_\lambda(\hat{u}_\lambda)$ and $b := \tilde{\varphi}^+_\lambda(\varepsilon\hat{u}_1)$. The same arguments exploited in the proof of Theorem 3.1 yield here

$$a < b < \tilde{\varphi}^+_\lambda(0).$$

We may assume that $K_c(\tilde{\varphi}^+_\lambda) \subseteq \{\hat{u}_\lambda\}$ for any $c < \tilde{\varphi}^+_\lambda(0)$. Indeed, if $u \in K_c(\tilde{\varphi}^+_\lambda) \setminus \{\hat{u}_\lambda\}$ then, thanks to Propositions 3.2 and 4.1, $u \in \operatorname{int}(C_+)$, u is a positive solution of (P_λ), $u \leq \hat{u}_\lambda$, and $u \neq \hat{u}_\lambda$, against the minimality of $\hat{u}_\lambda$.

Consequently, $K_a(\tilde{\varphi}^+_\lambda) = \{\hat{u}_\lambda\}$ while $K_c(\tilde{\varphi}^+_\lambda) = \emptyset$ for each $c \in (a,b]$. Due to Theorem 2.1, there exists a continuous deformation $h : [0,1] \times Z \to Z$, where

$$Z := \{u \in X \ :\ \tilde{\varphi}^+_\lambda(u) \leq b\},$$

such that $h(0,u) = u$, $h(1,u) = \hat{u}_\lambda$ for all $u \in Z$, and

$$\tilde{\varphi}_\lambda^+(h(t,u)) \leq \tilde{\varphi}_\lambda^+(u) \quad \forall\,(t,u) \in [0,1] \times Z.$$

The path $\gamma_+ : [0,1] \to X$ defined by setting

$$\gamma_+(t) := (h(t, \varepsilon\hat{u}_1))^+ \quad \forall t \in [0,1]$$

enjoys the following properties: $\gamma_+(0) = \varepsilon\hat{u}_1$, $\gamma_+(1) = \hat{u}_\lambda$, $\tilde{\varphi}_\lambda = \tilde{\varphi}_\lambda^+$ on the set $\gamma_+([0,1])$, and

$$\tilde{\varphi}_\lambda(\gamma_+(t)) < \tilde{\varphi}_\lambda(0) \quad \forall\, t \in [0,1]. \tag{4.12}$$

Similarly, we construct a continuous function $\gamma_- : [0,1] \to X$ such that $\gamma_-(0) = \check{u}_\lambda$, $\gamma_-(1) = -\varepsilon\hat{u}_1$, as well as

$$\tilde{\varphi}_\lambda(\gamma_-(t)) < \tilde{\varphi}_\lambda(0) \quad \forall\, t \in [0,1]. \tag{4.13}$$

Concatenating γ_-, $\varepsilon\gamma_0$, and γ_+ produces a path $\gamma \in \Gamma$, which, in view of (4.11)–(4.13), fulfils

$$\tilde{\varphi}_\lambda(\gamma(t)) < \tilde{\varphi}_\lambda(0) \quad \forall\, t \in [-1,1].$$

Thus, $\tilde{c} < \tilde{\varphi}_\lambda(0)$ and, a fortiori, $\tilde{u}_\lambda \neq 0$.

Let us finally verify that $\tilde{u}_\lambda$ is nodal. If, on the contrary, $\tilde{u}_\lambda \in C_+$ then, by Proposition 3.2,

$$\tilde{u}_\lambda \in \mathrm{int}(C_+), \quad 0 < \tilde{u}_\lambda \leq \hat{u}_\lambda.$$

Since $\hat{u}_\lambda$ was minimal, this actually means $\tilde{u}_\lambda = \hat{u}_\lambda$, which is impossible according to (4.6). A similar reasoning shows that $\tilde{u}_\lambda \notin -C_+$. Consequently, $\tilde{u}_\lambda$ has to be nodal. □

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References

[1] A. Ambrosetti, H. Brézis, and G. Cerami, *Combined effects of concave-convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), 519–543.

[2] A. Ambrosetti, J. Garcia Azorero, and I. Peral, *Multiplicity results for some nonlinear elliptic equations*, J. Funct. Anal. **137** (1996), 219–242.

[3] D. Averna, S.A. Marano, and D. Motreanu, *Multiple solutions for a Dirichlet problem with p-Laplacian and set-valued nonlinearity*, Bull. Austral. Math. Soc. **77** (2008), 285–303.

[4] S. Carl, V.K. Le, and D. Motreanu, Nonsmooth Variational Problems and Their Inequalities, Springer Monogr. Math., Springer, New York, 2007.

[5] S. Carl and D. Motreanu, *General comparison principle for quasilinear elliptic inclusions*, Nonlinear Anal. **70** (2009), 1105–1112.

[6] S. Carl and K. Perera, *Sign-changing and multiple solutions for the p-Laplacian*, Abstr. Appl. Anal. **7** (2002), 613–625.

[7] K.C. Chang, *Variational methods for non-differentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl. **80** (1981), 102–129.

[8] M. Cuesta, D. de Figueiredo, and J.P. Gossez, *The beginning of the Fučik spectrum for the p-Laplacian*, J. Differential Equations **159** (1999), 212–238.

[9] J.P. Garcia Azorero, J. Manfredi, and I. Peral Alonso, *Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations*, Commun. Contemp. Math. **2** (2000), 385–404.

[10] L. Gasiński and N.S. Papageorgiou, Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems, Ser. Math. Anal. Appl. **8**, Chapman & Hall/CRC Press, Boca Raton, 2005.

[11] L. Gasiński and N.S. Papageorgiou, Nonlinear Analysis, Ser. Math. Anal. Appl. **9**, Chapman & Hall/CRC Press, Boca Raton, 2006.

[12] A. Iannizzotto and N.S. Papageorgiou, *Existence of three nontrivial solutions for nonlinear Neumann hemivariational inequalities*, Nonlinear Anal. **70** (2009), 3285–3297.

[13] A. Lê, *Eigenvalue problems for the p-Laplacian*, Nonlinear Anal. **64** (2006), 1057-1099.

[14] G.M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), 1203–1219.

[15] S.A. Marano, D. Motreanu, and D. Puglisi, *Multiple solutions to a Dirichlet eigenvalue problem with p-Laplacian*, Topol. Methods Nonlinear Anal., in press.

[16] S.A. Marano and N.S. Papageorgiou, *Positive solutions to a Dirichlet problem with p-Laplacian and concave-convex nonlinearity depending on a parameter*, Comm. Pure Appl. Anal. **12** (2013), 815–829.

[17] D. Motreanu, V. V. Motreanu, and N. S. Papageorgiou, *Multiple constant sign and nodal solutions for Nonlinear Neumann eigenvalue problems*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **10** (2011), 729–755.

[18] K. Perera, R.P. Agarwal, and D. O'Regan, Morse Theoretic Aspects of p-Laplacian Type Operators, Math. Surveys Monogr. **161**, Amer. Math. Soc., Providence, 2010.

[19] J.L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191–202.