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INTEGRALLY CLOSED IDEALS AND TYPE SEQUENCES IN ONE-DIMENSIONAL LOCAL RINGS

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0. Introduction. Let $(R, \mathfrak{m})$ be a one-dimensional, local, Noetherian domain. Let $\overline{R}$ be the integral closure of R in its quotient field. The conductor of R in $\overline{R}$ will be denoted by $\mathfrak{C}$, and the length function on R-modules by $\lambda(-)$. We also assume that R is analytically irreducible, that is, $\hat{R}$ is a domain, or equivalently $\overline{R}$ is a DVR and is a finite R-module. If $\mathfrak{n}$ is the maximal ideal of $\overline{R}$, we assume that $R/\mathfrak{m} \simeq \overline{R}/\mathfrak{n}$. To any such ring we can associate a numerical semigroup as follows. Let v denote the valuation of the quotient field K of R, $v(K) = \mathbf{Z} \cup \{\infty\}$, with valuation ring $\overline{R}$ and set $v(R) = \{v(x) \mid x \in R, x \neq 0\}$. As $\overline{R}$ is a DVR and a finite R-module, $\mathfrak{C} = r^{g+1}\overline{R}$, where $r\overline{R} = \mathfrak{n}$. Therefore, $v(R)$ is a numerical semigroup such that $|\mathbf{N} - v(R)| < \infty$. We have $v(R) = \{0 = s_0, s_1, \dots, S_{n-1}, s_n = g+1, \rightarrow\}$, where $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = g+1$, and the arrow indicates that any integer strictly greater than g is in $v(R)$. The integer g is the greatest integer not in $v(R)$ and is called the Frobenius number of R. Matsuoka [**7**] defines a chain of ideals $\mathfrak{U}\mathfrak{i}$ as follows

$$\mathfrak{U}\mathfrak{i} = \{x \in R \mid v(x) \geq s_i\} \quad \text{if } i \leq n.$$

Clearly $\mathfrak{C} = \mathfrak{U}_n \subset \mathfrak{U}_{n-1} \subset \cdots \subset \mathfrak{U}_1 = m \subset R \subset \mathfrak{U}_1^{-1} \subset \cdots \subset \mathfrak{U}_{n-1}^{-1} \subset \mathfrak{U}_n^{-1} = \overline{R}$. Since $\lambda(\mathfrak{U}_{i-1}/\mathfrak{U}_i) = |v(\mathfrak{U}_{i-1}) - v(\mathfrak{U}_i)| = 1$ for all i, cf. [**7**], $n = |v(R) \cap \{0, 1, \dots, g\}| = \lambda(R/\mathfrak{C})$. $\mathfrak{U}_i^{-1}$ is a ring for all i. Moreover, as $\overline{R}$ is local and finite over $\mathfrak{U}_i^{-1}$, $\mathfrak{U}_i^{-1}$ is a local ring. The sequence $t_i(R) = \lambda(\mathfrak{U}_i^{-1}/\mathfrak{U}_{i-1}^{-1})$ is called the *type sequence* of R (this terminology was first introduced in [**2**]). The name "type sequence" is related to the fact that, if $i = 1$, then $t_1(R) = \lambda(\mathfrak{m}^{-1}/R)$ is the Cohen-Macaulay type of R.

One can start with a numerical semigroup and define the analog of the notion of type sequence as follows. If $S = \{0 = s_0, s_1, \dots, s_n, \rightarrow\}$

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is a numerical semigroup, then we let $g(S)$ denote the Frobenius number of S, that is, the greatest integer not in S, and $n(S) = |S \cap \{0, 1, \dots, g(S)\}|$. We set

$$S_i = \{x \in S \mid x \geq s_i\},$$
$$S(i) = (S - S_i) = \{x \in \mathbf{Z} \mid x + S_i \subseteq S\}.$$

We also set $t_i(S) = |S(i) - S(i-1)|$. The sequence $\{t_1(S), \dots, t_n(S)\}$ is called the *type sequence* of S, and $t_1(S)$ is called the *type* of S. The properties of type sequences for numerical semigroups have been investigated by D'Anna in [**3**]. The type sequence of a ring need not be the same as the type sequence of the associated numerical semigroup. An example is given by the ring $k[[x^4, x^6+x^7, x^{10}]]$, where k is a field, cf. [**2**, Example II, 1.19]: the type sequence of the ring is $\{2, 2, 1, 1\}$, while the type sequence of the associated numerical semigroup is $\{3, 1, 1, 1\}$. In Section 1 we characterize the integrally closed ideals of R as the ideals of the form $I = \{x \in R \mid v(x) \geq r\}$ for some $r \in S$, cf. Proposition 1.1 and Corollary 1.3, and we give a criterion to check when the ideals $\mathfrak{U}_i$ are stable, cf. Proposition 1.13. In Section 2 we give an upper bound for $l^*(R) \leq (t-1)[\lambda(R/\mathfrak{C}) - \mathbf{1}]$, cf. Proposition 2.1, and we characterize the rings for which $l^*(R) = a \in \mathbf{N}$ and $t = e - 1$ in terms of the type sequence of the ring.

1. Integrally closed ideals and Arf rings.

Proposition 1.1. *Let I be an ideal of R. Then there exists an integer $g(I) \in \mathbf{N} - \gtrapprox (\mathbf{I})$ such that $I \supseteq \{x \in R \mid v(x) \geq g(I) + 1\}$.*

Proof. Let $e_1 = \min\{l \mid l \in v(I)\}$, and write $v(R) = \{0 = s_0, s_1, \dots, s_n, \rightarrow\}$. Since $v(R)$ contains all integers greater than s_n, we have that $v(I) \supseteq \{e_1 + s_n, \rightarrow\}$. Let $g(I) = \max\{l \in \mathbf{N} \mid \lessdot \notin \gtrapprox (\mathbf{I})\}$. Clearly $g(I) \geq g$. Let $x \in R$ be such that $v(x) \geq g(I) + 1$. Assume first that $v(x) \geq g(I) + 1 + e_1$. Let $y \in I$ be such that $v(y) = e_1$. Then $v(x/y) \geq g(I) + 1 \geq g + 1$, therefore $x/y \in R$ and $x \in I$. Assume now that $v(x) \geq g(I) + 1$. Let $z_1 \in I$ be such that $v(z_1) = v(x)$. Then there exists an invertible element u_1 in R such that $v(x - u_1 z_1) > v(z_1)$. If $v(x - u_1 z_1) \geq g(I) + 1 + e_1$, then we are done. Otherwise there exists $z_2 \in I$ such that $v(z_2) = v(x - u_1 z_1)$. Iterating the argument we get $x = u_1 z_1 + \dots + u_h z_h + i$, $i \in I$, $z_j \in I$ for all j, therefore $x \in I$. $\square$

Remark 1.2. $\mathfrak{C} = \{\mathfrak{x} \in \mathfrak{R} \mid \mathfrak{v}(\mathfrak{x}) \geq \mathfrak{g} + \mathbf{1}\}$, where $g = g(S) = g(\mathfrak{C})$.

Corollary 1.3. *An ideal I is integrally closed if and only if $I = \{x \in R \mid v(x) \geq r\}$ for a fixed $r \in S$.*

Corollary 1.3 has been proved independently by Barucci, Dobbs and Fontana in [**2**].

Remark 1.4. Let I be a nonzero ideal of R and $e_1 = \min\{l, l \in v(I)\}$. Then, for any $x \in I$ with $v(x) = e_1$, xR is a minimal reduction of I.

Proof. $x\overline{R} = I\overline{R}$ (since $\overline{R}$ is a DVR) so xR is a minimal reduction of I. □

Definition 1.5. Let $(R, \mathfrak{m})$ be a one-dimensional, reduced ring. The *reduction number* of an m-primary ideal I, $r(I)$, is defined to be $\min\{l \geq 0 \mid \text{there exists } x \in I \text{ such that } xI^l = I^{l+1}\}$.

Corollary 1.6. *If I is an integrally closed ideal, then $r(I) \leq \max\{r(\mathfrak{U}_i), i = 1, \ldots, n\}$.*

Proof. By Corollary 1.3 either $I = \mathfrak{U}_i$ or $I \subseteq \mathfrak{C}$ and $I = \{x \in R \mid v(x) \geq r\}$ for a fixed $r \in S$. In the second case we have $I^2 = xI$, where x is a minimal reduction of I. □

Definition 1.7 [**6**]. Let R be a one-dimensional Cohen-Macaulay semi-local ring.

(i) An ideal I is said to be *open* if it contains a regular element of R.

(ii) An element $x \in I$ is *I-transversal* if $I^{n+1} = xI^n$ for some integer $n \geq 1$.

(iii) R is an *Arf ring* if any integrally closed, open ideal has a transversal element and if the following condition is satisfied: $x, y, z \in R$, x regular, y, z integral over $xR \Rightarrow yz \in xR$.

Definition 1.8 [**6**, Definition 1.3]. Set $R^I = \cup(I^n : I^n)$. An open ideal I of R is *stable* if $R^I = (I : I)$.

Lemma 1.9 [**6**, Lemma 1.11]. *An open ideal of R is stable if and only if one of the following equivalent conditions is satisfied*:

(i) $I^2 = xI$ *for some* $x \in I$;

(ii) *there exists* $x \in I$ *such that* x *is regular and* Ix^{-1} *is a ring.*

Moreover, if I is stable and x is any transversal element of I, then $I^2 = xI$.

Proposition 1.10 [**6**, Lemma 2.2]. *Let R be a one-dimensional, semi-local, Cohen-Macaulay Noetherian ring. The following are equivalent*:

(i) *R is Arf*;

(ii) *every integrally closed open ideal is stable.*

The following proposition shows that to see if a ring is Arf we only need to check if the ideals $\mathfrak{U}_i$ are stable.

Proposition 1.11. *The following are equivalent*:

(i) *R is Arf*;

(ii) $r(\mathfrak{U}_i) = 1$ *for all* i.

Proof. We only need to prove (ii) $\Rightarrow$ (i). By Proposition 1.10 and Lemma 1.9, it suffices to show that if I is integrally closed, then $r(I) = 1$. By Corollary 1.3 either $I = \mathfrak{U}_i$ or $I \subseteq \mathfrak{C}$ and $I = \{x \in R \mid v(x) \geq r \text{ for some } r \in S\}$. In both cases $r(I) = 1$. $\square$

We have remarked earlier that $\mathfrak{U}_i^{-1}$ is a local ring for all i. Let $\mathfrak{C}_i$ be its conductor, $g(\mathfrak{U}_i^{-1})$ the Frobenius number and $e(\mathfrak{U}_i^{-1})$ the multiplicity. Let x_iR be a minimal reduction of $\mathfrak{U}_i$. Then $v(x_i) = s_i = \min\{v(x), x \in \mathfrak{U}_i\}$.

Remark 1.12. $\mathfrak{C} = x_i\mathfrak{C}_i$ for all i.

Proof. We first show that $x_i\mathfrak{C}_i \subseteq \mathfrak{C}$. Let $u \in \mathfrak{C}_i$ and α in the integral closure $\overline{R}$ of R. We need to show that $\alpha x_i u \in R$. Since $u \in \mathfrak{C}_i$ and $\alpha \in \overline{R}$, $\alpha u \in \mathfrak{U}_i^{-1}$. As $x_i \in \mathfrak{U}$, $\alpha x_i u \in R$. Conversely, let $z \in \mathfrak{C}$. We need to show that $z/x_i \in \mathfrak{C}_i$. Let $u \in \overline{R}$. We will show that $uz/x_i \in \mathfrak{U}_i^{-1}$. Let $w \in \mathfrak{U}_i$. Then $v(zuw/x_i) = v(z)+v(u)+v(w)-v(x) \geq (g+1)+s_i-s_i = g+1$, therefore $zuw/x_i \in R$. □

Proposition 1.13. *The following are equivalent*:

(i) $\mathfrak{U}_i$ *is stable*;

(ii) $\mathfrak{U}_i = x_i\mathfrak{U}_i^{-1}$;

(iii) $\lambda(\mathfrak{U}_i^{-1}/\mathfrak{C}_i) = \lambda(R/\mathfrak{C}) - i$.

Proof. (i) $\Rightarrow$ (ii). We have $x_i\mathfrak{U}_i^{-1} \subseteq \mathfrak{U}_i$ by definition of $\mathfrak{U}_i^{-1}$. Let $y \in \mathfrak{U}_i$. We need to show that $y/x_i \in \mathfrak{U}_i^{-1}$. Let $z \in \mathfrak{U}_i$. We have $yz/x_i = x_iw/x_i = w \in \mathfrak{U}_i$, therefore $y/x_i \in \mathfrak{U}_i^{-1}$.

(ii) $\Rightarrow$ (i). We only need to show that $\mathfrak{U}_i^2 \subseteq x_i\mathfrak{U}_i$. Let $x \in \mathfrak{U}_i^2$. We want to show that $w/x_i \in \mathfrak{U}_i$. It suffices to assume $w = uz$ with $u, z \in \mathfrak{U}_i$. $w/x_i = uz/x_i$ and $u/x_i \in \mathfrak{U}_i^{-1}$, so $uz/x_i \in \mathfrak{U}_i$.

(ii) $\Leftrightarrow$ (iii). Computing lengths in the short exact sequence

$$0 \longrightarrow x_i\mathfrak{U}_i^{-1}/x_i\mathfrak{C}_i \longrightarrow \mathfrak{U}_i/\mathfrak{C} \longrightarrow \mathfrak{U}_i/x_i\mathfrak{U}_i^{-1} \longrightarrow 0,$$

we get: $\lambda(\mathfrak{U}_i^{-1}/\mathfrak{C}_i) + \lambda(\mathfrak{U}_i/x_i\mathfrak{U}_i^{-1}) = \lambda(x_i\mathfrak{U}_i^{-1}/x_i\mathfrak{C}_i) + \lambda(\mathfrak{U}_i/x_i\mathfrak{U}_i^{-1}) = \lambda(\mathfrak{U}_i/\mathfrak{C}) = \lambda(R/\mathfrak{C}) - i$, where the last equality follows from the definition of $\mathfrak{U}_i$. □

Proposition 1.14 [**1**, Theorem 22]. *The following are equivalent*:

(i) R *is Arf*;

(ii) $\lambda(\mathfrak{U}_i^{-1}/\mathfrak{C}_i) = \lambda(R/\mathfrak{C}) - i$ *and* $g(\mathfrak{U}_i^{-1}) = g(R) - \sum_{k=0}^{i-1} e(\mathfrak{U}_i^{-1})$ *for all* i.

We now show that the second condition in (ii) of Proposition 1.14 is redundant.

Proposition 1.15. *The following are equivalent*:

(i) *R is Arf*;

(ii) $\lambda(\mathfrak{U}_i^{-1}/\mathfrak{C}_i) = \lambda(R/\mathfrak{C}) - i$ *for all* i.

Proof. Apply Proposition 1.13. □

2. Type sequences. In [**4**] it is shown that if R is a one-dimensional, Noetherian, local, reduced, excellent ring with infinite residue field, then the inequality $\lambda(\overline{R}/R) \leq t\lambda(R/\mathfrak{C})$ always holds. The main ingredients of the proof are the fact that R has a canonical module which is isomorphic to an m-primary ideal of R, and the existence of a minimal reduction of the canonical module which is generated by one element. If we assume R to be analytically irreducible, then it has a canonical module which is isomorphic to an m-primary ideal, since $\hat{R}$ is reduced. By Remark 1.4 the canonical module has a minimal reduction generated by one element, so the same proof as in [**4**, Proposition 2.1] allows us to conclude that the inequality $\lambda(\overline{R}/R) \leq t\lambda(R/\mathfrak{C})$ holds. We set $l^*(R) = t\lambda(R/\mathfrak{C}) - \lambda(\overline{R}/R)$, cf. [**4**].

Proposition 2.1. *Let (R, m) be a one-dimensional, Noetherian, local ring. Assume that R is either reduced and excellent, with infinite residue field, or that it is an analytically irreducible domain with $R/\mathfrak{m} \simeq \overline{R}/\mathfrak{n}$, where n is the maximal ideal of $\overline{R}$. Then $l^*(R) \leq (t-1)[\lambda(R/\mathfrak{C}) - 1]$.*

Proof. Let zR be a minimal reduction of K_R, the canonical module of R. Then $z\mathfrak{C} = \mathfrak{C}K_R$, as K_R is integral over zR. Computing lengths in the short, exact sequence,

$$0 \longrightarrow \frac{zR}{z\mathfrak{C}} \longrightarrow K_R/\mathfrak{C}K_R \longrightarrow \frac{K_R}{zR} \longrightarrow 0,$$

we obtain: $\lambda(\overline{R}/R) = \lambda(K_R/\mathfrak{C}K_R) = \lambda(R/\mathfrak{C}) + \lambda(K_R/zR)$. We have

$$\begin{aligned} l^*(R) &= t\lambda(R/\mathfrak{C}) - \lambda(\overline{R}/R) = t\lambda(R/\mathfrak{C}) - \lambda(K_R/\mathfrak{C}K_R) \\ &= t[\lambda(R/\mathfrak{C}) - 1] - \lambda(K_R/\mathfrak{C}K_R) + t \\ &= t[\lambda(R/\mathfrak{C}) - 1] - \lambda(R/\mathfrak{C}) - \lambda(K_R/zR) + t. \end{aligned}$$

Now $\lambda(K_R/zR) \geq t-1$ as $\mu(K_R/zR) = t-1$. It follows that

$$\begin{aligned} l^*(R) &= t[2(R/\mathfrak{C}) - 1] - \lambda(R/\mathfrak{C}) - \lambda(K_R/zR) + t \\ &\leq [\lambda(R/\mathfrak{C}) - 1] - \lambda(R/\mathfrak{C}) - t + 1 + t \\ &= (t-1)[\lambda(R/\mathfrak{C}) - 1]. \qquad \square \end{aligned}$$

Remark 2.2. Assume R is an analytically irreducible domain with $R/\mathfrak{m} \simeq \overline{R}/\mathfrak{n}$, where n is the maximal ideal of $\overline{R}$. The equality $l^*(R) = (t-1)[\lambda(R/\mathfrak{C}) - 1]$ holds if and only if the type sequence is $\{t, 1, \dots, 1\}$. Indeed, $l^*(R) = t\lambda(R/\mathfrak{C}) - \sum_{i=1}^n t_i(R) = t(\lambda(R/\mathfrak{C}) - 1) - \sum_{i=2}^n (t_i(R))$. There is always a ring with a type sequence as follows. It suffices to take $R = k[[x^s, s \in S]]$, where k is an infinite field, $S = \{0, t+n-1, t+n, \dots, t+2n-3, t+2n-1, \rightarrow\}$, and n is the number of elements in the type sequence.

Proposition 2.3 [**4**, Theorem 2.10 and Corollary 2.14]. *Let a be a nonnegative integer, $t \geq a$, $t = e-1$ and $e \geq 3$. Then $l^*(R) = a$ if and only if*

(i) *if $a = 0$, then $v\{R\} = \{0, e, 2e, \dots, ne, \rightarrow\}$ with $n \geq 1$;*

(ii) *if $a > 0$, then $v\{R\} = \{0, e, 2e, \dots, ne - a, \rightarrow\}$ with $n \geq 2$.*

Remark 2.4 cf. [**3**]. Numerical semigroups of the form $S = \{0, e, 2e, 3e, \dots, (n-1)e, ne-a, \rightarrow\}$ have type sequence $\{t = e-1, t, \dots, t, t-a\}$.

Proof. We have that $S(i) = \{0, e, 2e, \dots, (n-i-1)e, (n-i)e - a, \rightarrow\}$ for all i. Thus $t_1(S) = t_2(S) = \cdots = t_{n-1}(S) = e-1$ and $t_n(S) = e-1-a = t-a$. $\square$

Lemma 2.5. *We have*

(i) $v(\mathfrak{U}_i^{-1}) \subseteq S(i)$ *for all i, and* $v(\mathfrak{U}_{n-1}^{-1}) = S(n-1) = \{0, s_n - s_{n-1}, \rightarrow\}$;

(ii) $g(\mathfrak{U}_i^{-1}) = g - s_i$.

Proof. i) Take any $a \in \mathfrak{U}_i^{-1}$ and $r \in \mathfrak{U}_i$ (so that $v(r) = l \geq s_i$). Then $ar \in R$, so $v(a) + l \in S$. By definition of $S(i)$, $v(\mathfrak{U}_i^{-1}) \subseteq S(i)$

for all i. We now show that $v(\mathfrak{U}_{n-1}^{-1}) \supseteq S(n-1)$. Take $r \in \overline{R}$ such that $v(r) \geq s_n - s_{n-1}$ (so that $v(r) \in S(n-1)$). If $x \in \mathfrak{U}_{n-1}$, then $v(rx) = v(r) + v(x) \geq s_n - s_{n-1} + s_{n-1} = s_n$, which implies that $rx \in R$. It follows that $r \in \mathfrak{U}_{n-1}^{-1}$. Moreover, $0 \in \mathfrak{U}_{n-1}^{-1}$.

(ii) As $v(\mathfrak{U}_i^{-1}) \subseteq S(i)$, $g(\mathfrak{U}_i^{-1}) \geq g(S(i)) = g - s_i$ (the last equality is proved in [**3**, Proposition 1.1]). We only need to show that $\{g-s_i+1, \rightarrow\} \subseteq v(\mathfrak{U}_i^{-1})$. Let y be an element of the quotient field of R such that $v(y) \geq g - s_i + 1$. Set $z \in \mathfrak{U}_i$. We have $v(yz) = v(y) + v(z) \geq g - s_i + 1 + s_i = g + 1$. It follows that $yz \in R$, therefore $y \in \mathfrak{U}_i^{-1}$. □

Lemma 2.6. *We have* $t_n(R) = t_n(S) = g - s_{n-1}$.

Proof. We have $t_n(R) = \lambda(\overline{R}/\mathfrak{U}_{n-1}^{-1}) = |v(\overline{R}) - v(\mathfrak{U}_{n-1}^{-1})|$, where the second equality follows from [**5**] (here we need the fact that the residue fields of R and $\overline{R}$ are isomorphic). By Lemma 2.5, $v(\mathfrak{U}_{n-1}^{-1}) = S(n-1)$, therefore $t_n(R) = |\mathbf{N} - S(n-1)| = t_n(S)$. □

The following proposition generalizes [**2**, Theorem II5.3] and [**4**, Theorem 2.10].

Proposition 2.7. *Let a be a nonnegative integer and assume that $t \geq a$ and $e \geq 3$. The following conditions are equivalent*:

1) $l^*(R) = a$;

2) *for all reductions xR of m, $m = \mathfrak{C} + xR$ and $\lambda(\mathfrak{C}/x^p\overline{R}) = a$, (here $p = \min\{i \mid x^i \in \mathfrak{C}\}$)*;

3) *there exists a reduction xR of m such that $m = \mathfrak{C} + xR$ and $\lambda(\mathfrak{C}/x^p\overline{R}) = a$, where $p = \min\{i \mid x^i \in \mathfrak{C}\}$.*

4) *$t = e-1$ and the type sequence of R is $t_1(R) = \cdots = t_{n-1}(R) = t$, $t_n(R) = t - a$.*

Proof. We only need to prove 1) ⇔ 4). Assume $l^*(R) = a$. We have

$$a = l^*(R) = tn - \lambda(\overline{R}/R) = tn - \sum_{i=1}^{n} t_i(R),$$

therefore $\sum_{i=1}^{n} t_i(R) = tn - a$. By Lemma 2.6, $t_n(R) = t_n(S)$. Finally,

$t_n(R) = t_n(S) = t - a$ by Proposition 2.3 and the above remark. It follows that $t - a + \sum_{i=1}^{n-1} t_i(R) = t(n-1) + (t-a)$, so $t_i(R) = t$ for all $i \leq n - 1 (t_i(R) \leq t$ for all i by [**7**]). Conversely, assume that the type sequence of R is $t, \dots, t, t-a$. Then $l^*(R) = tn - \lambda(\overline{R}/R) = tn - (tn - a) = a$. □

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