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Fat lines in $\mathbb{P}^3$: Powers versus symbolic powers

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ABSTRACT

We study the symbolic and regular powers of ideals I for a family of special configurations of lines in $\mathbb{P}^3$. For this family, we show that $I^{(m)} = I^m$ for all integers m if and only if $I^{(3)} = I^3$. We use these configurations to answer a question of Huneke that asks whether $I^{(m)} = I^m$ for all m if equality holds when m equals the big height of the ideal I.

1. Introduction

Let $R = k[x_0, \ldots, x_N] = k[\mathbb{P}^N]$ be a polynomial ring over an algebraically closed field of characteristic zero. Starting with the work of [6,18], and further refined by [1–3,5,7,10,11,17,19–21], the following containment question has been of interest: given a homogeneous ideal $(0) \neq I \subsetneq R$, for what integers m and r do we have $I^{(m)} \subseteq I^r$? Here, $I^{(m)}$ denotes the *m-th symbolic power* of the ideal I whose definition we now recall. If $I^m = \bigcap_j Q_j$ is the homogeneous primary decomposition of I^m, then $I^{(m)} = \bigcap_i Q_i'$, where the intersection is over those primary components Q_i' which have $\sqrt{Q_i'}$

contained in an associated prime ideal of I. From the definition, we always have $I^m \subseteq I^{(m)}$, but we do not always have $I^{(m)} \subseteq I^m$. For non-trivial ideals we never have $I^{(m)} \subseteq I^r$ when $m < r$, but by [6,18] we always have $I^{(m)} \subseteq I^r$ when $m \geqslant rN$; in fact, $I^{(me)} \subseteq I^m$ where e is the big height of I (i.e., the height of the associated prime ideal of I of biggest height). Consequently, for each fixed r, it is of interest to find the smallest $m \geqslant r$ with $I^{(m)} \subseteq I^r$. Apart from some sporadic examples [10,20], most of the cases for which this smallest m are known either are ideals of complete intersections (i.e., ideals I generated by a regular sequence, in which case $I^{(m)} = I^m$ for all $m \geqslant 1$ [23]) or are ideals defining zero-dimensional schemes [3].

Thus constructing families of ideals I of positive dimensional schemes which are not complete intersections but for which we can determine the least m for each r such that $I^{(m)} \subseteq I^r$ is of particular interest. Our focus will be on ideals with extremal behavior, in the sense that they satisfy $I^{(m)} = I^m$ for all $m \geqslant 1$. Adding to the interest of our results is that as an application we answer a question raised by C. Huneke.[1] In particular, for a homogeneous ideal I of big height c, Huneke asked whether it is true that $I^{(m)} = I^m$ for all $m \geqslant 1$ if $I^{(c)} = I^c$. Our results show that the answer in general is no. The ideals that we look at here are ideals I of special configurations of lines in $\mathbb{P}^3$. For these configurations, we completely characterize when I satisfies $I^{(m)} = I^m$ for all $m \geqslant 1$.

The key idea behind our constructions is to build finite configurations of lines in $\mathbb{P}^3$ so that the associated ideals have a bigraded structure with respect to which the ideals define finite sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Previous work on points in $\mathbb{P}^1 \times \mathbb{P}^1$, such as, for example, [8,13,14,22], then provides us with tools which we can exploit in our study of ideals of lines in $\mathbb{P}^3$. We now explain these two points of view. Let $R = k[\mathbb{P}^3] = k[x_0, x_1, y_0, y_1]$ and consider the two skew lines L_1 and L_2 in $\mathbb{P}^3$ defined by $I(L_1) = (x_0, x_1)$ and $I(L_2) = (y_0, y_1)$. If $B = [0 : 0 : b_0 : b_1]$ is a point on L_1 and $A = [a_0 : a_1 : 0 : 0]$ is a point on L_2, then the line L through A and B has defining ideal $I(L) = (a_1 x_0 - a_0 x_1, b_1 y_0 - b_0 y_1)$. We can regard R as being $k[\mathbb{P}^1 \times \mathbb{P}^1]$, and thus endowed with an $\mathbb{N}^2$-graded structure by setting $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$. With respect to this grading, $I(L)$ is a bigraded ideal which defines the point $(P, Q) \in \mathbb{P}^1 \times \mathbb{P}^1$ where $P = [a_0 : a_1]$ and $Q = [b_0 : b_1]$ and thus $I(L) = I((P, Q))$. Our configurations will be finite unions of such lines, i.e., each line L in our configuration meets both the lines L_1 and L_2. This allows us to reinterpret our union of lines in $\mathbb{P}^3$ as a finite set of points X in $\mathbb{P}^1 \times \mathbb{P}^1$. Conversely, the ideal of every finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ defines a finite union of lines in $\mathbb{P}^3$ where every line intersects L_1 and L_2. In addition, we will require that our configurations be arithmetically Cohen–Macaulay (ACM), i.e., that their associated coordinate rings be Cohen–Macaulay. Expressed in the language of points in $\mathbb{P}^1 \times \mathbb{P}^1$, our main result is:

Theorem 1.1. *Let $I = I(X)$ be the ideal of a finite reduced ACM subscheme X in $\mathbb{P}^1 \times \mathbb{P}^1$. Then $I^{(m)} = I^m$ for all $m \geqslant 1$ if and only if $I^{(3)} = I^3$.*

To prove Theorem 1.1, we use a result of Morey [21] to first show that $I^{(m)} = I^m$ for all $m \geqslant 1$ if and only if equality holds for $m = 2$ and 3. We use results of the first and third author [14] to show that $I^{(2)} = I^2$ always holds for ideals of a finite reduced ACM subscheme in $\mathbb{P}^1 \times \mathbb{P}^1$, and thus $I^{(m)} = I^m$ for all $m \geqslant 1$ if and only if $I^{(3)} = I^3$. We note that $I^{(3)} \neq I^3$ and $I^{(3)} = I^3$ both occur; see Example 3.1 and Theorem 4.6 for examples of the former, and Example 3.2 and Theorem 4.5 for examples of the latter. Note that the examples with $I^{(3)} \neq I^3$ give a negative answer to Huneke's question. To see why, note that the ideal I is an unmixed ideal of height two (in particular, its big height is two) that has $I^{(2)} = I^2$, but fails to have $I^{(m)} \neq I^m$ for all $m \geqslant 1$ since it fails for $m = 3$.

In Section 2 we present the background needed for the proof of Theorem 1.1 in Section 3. The last section presents a conjecture, along with some evidence, for a geometric description of all finite reduced ACM subschemes X in $\mathbb{P}^1 \times \mathbb{P}^1$ with $I(X)^{(3)} = I(X)^3$.

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[1] This question was posed by Huneke in a talk "Comparing Powers and Symbolic Powers of Ideals" that he gave at the University of Nebraska, Lincoln in May 2008.
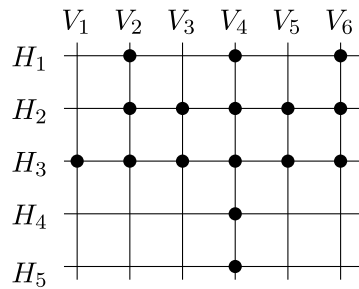
Fig. 1. An ACM point configuration.

2. Background

To any non-empty finite set $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ of points we associate a set S_X of integer lattice points indicating which points lie on the same horizontal or vertical rule. The idea is to enumerate the horizontal and vertical rules whose intersection with X is non-empty. We thus obtain, say, $H_1, \ldots, H_h$ and $V_1, \ldots, V_v$ where $X \subset \bigcup H_i$ and $X \subset \bigcup V_j$, and S_X consists of all pairs (i, j) such that $X \cap H_i \cap V_j \neq \emptyset$. We also associate to X its bi-homogeneous ideal $I(X) = \bigcap_{(P,Q) \in X} I((P, Q)) \subset R = k[x_0, x_1, y_0, y_1]$ and its coordinate ring $R/I(X)$.

As is usual, we say that a subscheme $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is arithmetically Cohen–Macaulay (ACM) if its coordinate ring $R/I(X)$ is Cohen–Macaulay. It is important to emphasize that while the coordinate rings of zero-dimensional subschemes in $\mathbb{P}^n$ are always Cohen–Macaulay, coordinate rings of zero-dimensional subschemes in a multiprojective space need not be Cohen–Macaulay. As an example of this, take two distinct points P and Q in $\mathbb{P}^1$. Then the coordinate ring $R/I(X)$ of $X = \{(P, P), (Q, Q)\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is not Cohen–Macaulay. This can be seen from the fact that if we consider only the graded structure of $R/I(X)$, then X is the union of two skew lines in $\mathbb{P}^3$ which is well known not to be Cohen–Macaulay. Reduced zero-dimensional ACM subschemes of $\mathbb{P}^1 \times \mathbb{P}^1$ can be characterized in terms of their Hilbert functions [8]. We recall an alternative geometric characterization found in [15, Theorem 4.3]:

Theorem 2.1. *Let X be the reduced subscheme consisting of a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then X is ACM if and only if whenever (P_1, Q_1) and (P_2, Q_2) are points in X with $P_1 \neq P_2$ and $Q_1 \neq Q_2$, then either (P_1, Q_2) or (P_2, Q_1) (or both) also belongs to X.*

When a finite reduced subscheme $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is ACM, Theorem 2.1 implies that we can relabel the H_i's and V_j's so that S_X resembles the Ferrers diagram of a partition $\lambda = (\lambda_1, \ldots, \lambda_h)$ with $\sum \lambda_i = |X|$ and $\lambda_i \geqslant \lambda_{i+1}$ for all $1 \leqslant i < h$, where λ_i equals the number of points on the rule H_i for each i.

Example 2.2. Any finite reduced subscheme $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ whose diagram S_X is as in Fig. 1 is ACM. This is because after relabeling the H_i's and V_j's, as in Fig. 2, S_X becomes the Ferrers diagram for the partition $\lambda = (6, 5, 3, 1, 1)$ (where the entries of λ count the number of points on each H_i, arranged to be non-increasing).

When $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a finite reduced ACM subscheme, then the generators of $I(Z)$ can be described in terms of the partition λ. Note that H_i and V_j are divisors on $\mathbb{P}^1 \times \mathbb{P}^1$, defined by forms in $k[x_0, x_1, y_0, y_1]$ of degrees $(1, 0)$ and $(0, 1)$, respectively. In the sequel, we shall abuse notation and let H_i and V_j denote both the divisor and the form that defines it.

Lemma 2.3. *(See [22, Theorem 5.1].) Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a finite reduced ACM subscheme with associated partition $\lambda = (\lambda_1, \ldots, \lambda_h)$. Let $H_1, \ldots, H_h$ denote the associated horizontal rules and $V_1, \ldots, V_v$ denote the associated vertical rules which minimally contain X (i.e., $H_i \cap X \neq \emptyset$ and $V_j \cap X \neq \emptyset$ for all i and j). Then a minimal homogeneous set of generators of $I(X)$ is given by*
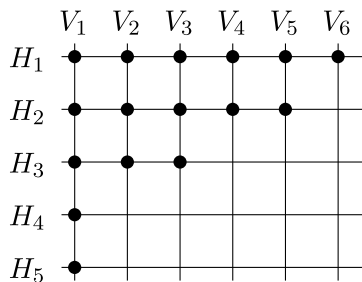
Fig. 2. Fig. 1, relabeled.

$$\{H_1 \cdots H_h, V_1 \cdots V_v\} \cup \{H_1 \cdots H_i V_1 \cdots V_{\lambda_{i+1}} \mid \lambda_{i+1} - \lambda_i < 0\}.$$

Example 2.4. In Example 2.2, we have $\lambda = (6, 5, 3, 1, 1)$, so $I(X)$ has generators

$$\{H_1 H_2 H_3 H_4 H_5, V_1 V_2 V_3 V_4 V_5 V_6, H_1 V_1 V_2 V_3 V_4 V_5, H_1 H_2 V_1 V_2 V_3, H_1 H_2 H_3 V_1\}.$$

In light of Lemma 2.3, we can write down the generators of $I(X)^2$ for any finite reduced ACM sub-scheme X in $\mathbb{P}^1 \times \mathbb{P}^1$. We end this section by showing that $I(X)^2 = I(X)^{(2)}$. Note that the ideal $I(X)^{(2)}$ defines a subscheme whose support is X and whose points all have multiplicity two (alternatively, when viewed as a graded ideal, $I(X)^{(2)}$ defines a union of "fat lines" in $\mathbb{P}^3$). We first recall a relevant fact. For our purposes, it is sufficient to know that the algorithm described in [14] always produces a set of generators for $I(X)^{(2)}$ of the following form.

Lemma 2.5. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a finite reduced ACM subscheme. Let $H_1, \ldots, H_h$ denote the horizontal rules and $V_1, \ldots, V_v$ denote the vertical rules which minimally contain X. There is a minimal set of generators of $I(X)^{(2)}$ such that every generator F has one of the following forms*:

1) $H_1^2 \cdots H_h^2$;
2) $H_1 \cdots H_h V_1 \cdots V_v$;
3) $V_1^2 \cdots V_v^2$;
4) *or there exist $1 \leqslant a \leqslant b \leqslant h$ and $1 \leqslant c \leqslant d \leqslant v$ such that*

$$F = H_1^2 H_2^2 \cdots H_a^2 H_{a+1}^1 \cdots H_b^1 V_1^2 V_2^2 \cdots V_c^2 V_{c+1}^1 \cdots V_d^1.$$

(If $a = b$ in case 4), F has the form $H_1^2 H_2^2 \cdots H_a^2 V_1^2 V_2^2 \cdots V_c^2 V_{c+1}^1 \cdots V_d^1$, and similarly for $c = d$.)

Sketch of the proof. For a minimal set of generators for $I(X)^{(2)}$ in terms of the partition λ, see [12, Theorem 3.15] and [14, Algorithm 5.1]; in particular, explicit values of a, b, c, and d are given for the elements described in 4). $\quad\square$

Theorem 2.6. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a finite reduced ACM subscheme. Then $I(X)^2 = I(X)^{(2)}$.*

Proof. Let $I = I(X)$. It suffices to prove $I^{(2)} \subseteq I^2$. Let F be any generator of $I^{(2)}$. By Lemma 2.5, F must have one of four forms. Since $H_1 \cdots H_h$ and $V_1 \cdots V_v$ are generators of I, the generators $H_1^2 \cdots H_h^2$, $H_1 \cdots H_h V_1 \cdots V_v$, and $V_1^2 \cdots V_v^2$ all belong to I^2.

We therefore take a generator of $I^{(2)}$ of the form

$$F = H_1^2 H_2^2 \cdots H_a^2 H_{a+1}^1 \cdots H_b^1 V_1^2 V_2^2 \cdots V_c^2 V_{c+1}^1 \cdots V_d^1$$

for some $1 \leqslant a \leqslant b \leqslant h$ and $1 \leqslant c \leqslant d \leqslant v$. Factor F as
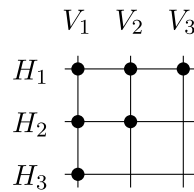
$$V_1 \quad V_2 \quad V_3$$

Fig. 3. A point configuration, S_X.

$$F_1 = H_1 H_2 \cdots H_a V_1 \cdots V_c V_{c+1} \cdots V_d \quad \text{and} \quad F_2 = H_1 H_2 \cdots H_a H_{a+1} \cdots H_b V_1 \cdots V_c.$$

We claim that both F_1 and F_2 are elements of I (and hence $F = F_1 F_2 \in I^2$). We show that $F_1 \in I$, since the other case is similar. Let $(P, Q) \in X$. Then (P, Q) is either on one of the rulings $H_1, \ldots, H_a$ or it is not. If it is on one of these rulings, say H_i, then $F_1((P, Q)) = 0$ because $H_i((P, Q)) = 0$. On the other hand, suppose that (P, Q) is not on any of these rulings. Since F vanishes with multiplicity two at (P, Q), and because (P, Q) can lie on at most one of the rulings $H_{a+1}, \ldots, H_b$, there must be at least one vertical ruling V_j among $V_1, \ldots, V_d$ such that $V_j((P, Q)) = 0$. But this means $F_1((P, Q)) = 0$. Hence $F_1 \in I$. □

3. Main result

We now present the proof of the main result of this paper, Theorem 1.1.

Proof of Theorem 1.1. Since I is homogeneous, we have $I^{(m)} = I^m$ if and only if $J^{(m)} = J^m$, where $J = I R_M$, R_M being the localization of $R = k[\mathbb{P}^1 \times \mathbb{P}^1]$ at the ideal M generated by the variables. Note that J is a perfect ideal (i.e., $\mathrm{pd}_{R_M}(R_M/J) = \mathrm{depth}(J, R_M)$; we have $\mathrm{depth}(J, R_M) = \mathrm{codim}(J) = 2$ since R_M is Cohen–Macaulay, and we obtain $\mathrm{pd}_{R_M}(R_M/J) = 2$ from the Auslander–Buchsbaum formula), it has codimension 2, R_M/J is Cohen–Macaulay, and J is generically a complete intersection (i.e., the localizations of J at its minimal associated primes are complete intersections). Now [21, Theorem 3.2] asserts that $J^{(m)} = J^m$ for all $m \geqslant 1$ if and only if $J^{(m)} = J^m$ for $1 \leqslant m \leqslant \dim(R_M) - 1$. Because $\dim(R_M) = 4$, it follows that $J^{(2)} = J^2$ and $J^{(3)} = J^3$ implies $J^{(m)} = J^m$ for all $m \geqslant 1$, and thus $I^{(2)} = I^2$ and $I^{(3)} = I^3$ implies $I^{(m)} = I^m$ for all $m \geqslant 1$. But we always have $I^{(2)} = I^2$ by Theorem 2.6, so the conclusion follows. □

The next example shows that $I^{(3)} \neq I^3$ can occur for an ideal I of a finite reduced ACM subscheme in $\mathbb{P}^1 \times \mathbb{P}^1$, while the example after that shows that $I^{(3)} = I^3$ can occur for a finite reduced ACM subscheme, even if it is not a complete intersection.

Example 3.1. Let X be the reduced subscheme consisting of six points (unique up to choice of bi-homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$) having diagram S_X as in Fig. 3. Let $I = I(X)$ be the ideal of X and let $\alpha(I) = \min\{d \mid I_d \neq (0)\}$, where I_d is the homogeneous component of I of degree d (with respect to the usual grading on $R = k[\mathbb{P}^3]$). It is easy to check that $\alpha(I) = 3$ and hence $\alpha(I^3) = 3\alpha(I) = 9$, and so the bi-homogeneous component $(I^3)_{(4,4)}$ of bidegree $(4, 4)$ is (0) (since $(I^3)_{(4,4)} \subseteq (I^3)_8 = (0)$). But there is a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(4, 4)$ vanishing on X to order 3. It is, as shown in Fig. 4, the zero locus of $H_1^2 V_1^2 H_2 V_2 F$, where $\deg F = (1, 1)$, with the zero locus of F represented by the diagonal line. (One can check by Bézout's Theorem that in fact C is unique.)

So, $(I^{(3)})_{(4,4)} \neq (0)$, whence $I^{(3)} \neq I^3$. As pointed out in the introduction, this example gives a negative answer to the question of Huneke discussed in the opening section.

In the next example we consider the case (unique up to choice of bi-homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$) of a finite reduced ACM subscheme consisting of 3 points which is not a complete intersection, but whose ideal I nevertheless has $I^{(3)} = I^3$.
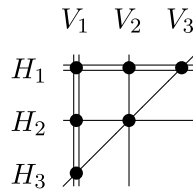
Fig. 4. The divisor $H_1^2 V_1^2 H_2 V_2 F$ through S_X.

Example 3.2. Consider a reduced ACM subscheme X consisting of 3 points, $Q_1, Q_2, Q_3 \in \mathbb{P}^1 \times \mathbb{P}^1$, not all on a single rule. Up to choice of coordinates, we may assume $P_1 = [1 : 0]$, $P_2 = [0 : 1] \in \mathbb{P}^1$ and that $Q_1 = (P_1, P_1)$, $Q_2 = (P_1, P_2)$, $Q_3 = (P_2, P_1)$. In this case, the ideal $I = I(X)$ of X is the monomial ideal $I = (x_1, y_1) \cap (x_1, y_0) \cap (x_0, y_1) = (x_0 x_1, y_0 y_1, x_0 y_0)$. Then

$$I^3 = \left(y_0^3 y_1^3, x_0 y_0^3 y_1^2, x_0 x_1 y_0^2 y_1^2, x_0^2 y_0^3 y_1, x_0^2 x_1 y_0^2 y_1, x_0^2 x_1^2 y_0 y_1, x_0^3 y_0^3, x_0^3 x_1 y_0^2, x_0^3 x_1^2 y_0, x_0^3 x_1^3\right).$$

A tedious but elementary computation shows that

$$I^{(3)} = (x_1, y_1)^3 \cap (x_1, y_0)^3 \cap (x_0, y_1)^3$$
$$= \left(y_0^3 y_1^3, x_0 y_0^3 y_1^2, x_0 x_1 y_0^2 y_1^2, x_0^2 y_0^3 y_1, x_0^2 x_1 y_0^2 y_1, x_0^2 x_1^2 y_0 y_1, x_0^3 y_0^3, x_0^3 x_1 y_0^2, x_0^3 x_1^2 y_0, x_0^3 x_1^3\right).$$

Thus $I^3 = I^{(3)}$, and hence $I^m = I^{(m)}$ for all $m \geqslant 1$ by Theorem 1.1.

4. Numerical conditions that imply $I^{(3)} = I^3$

Let I be the ideal of a reduced finite ACM subscheme X in $\mathbb{P}^1 \times \mathbb{P}^1$. Theorem 1.1 reduces the problem of determining whether $I^{(m)} = I^m$ for all $m \geqslant 1$ to simply checking if equality holds when $m = 3$. It has been shown that many of the algebraic invariants of I (e.g., the graded Betti numbers of its bigraded minimal free resolution [22]) are encoded into its associated partition λ. This motivates us to ask if knowing λ is enough to determine whether $I^{(3)} = I^3$, and consequently, whether $I^{(m)} = I^m$ for all $m \geqslant 1$. Computer experimentation using [4,9] has suggested the following specific characterization.

Conjecture 4.1. *Let X be a finite reduced ACM subscheme in $\mathbb{P}^1 \times \mathbb{P}^1$ with associated partition $\lambda = (\lambda_1, \ldots, \lambda_h)$. Let $I = I(X)$. Then $I^{(3)} = I^3$ if and only if λ has one of the following two forms:*

(i) *$\lambda = (a, a, \ldots, a)$ for some integer $a \geqslant 1$.*
(ii) *$\lambda = (a, \ldots, a, b, \ldots, b)$ for some integers $a > b \geqslant 1$.*

In other words, the conjecture is that $I^{(3)} \neq I^3$ if and only if λ contains at least three distinct entries. Notice that this is the case in Example 3.1 because the associated tuple is $\lambda = (3, 2, 1)$. We round out this paper by giving supporting evidence for this conjecture.

4.1. λ has at most two distinct entries

We first focus on the case that $\lambda = (a, \ldots, a)$. If X is a finite reduced ACM subscheme in $\mathbb{P}^1 \times \mathbb{P}^1$ with associated tuple $\lambda = (a, \ldots, a)$, then by Lemma 2.3, we have $I = I(X) = (H_1 \cdots H_{|\lambda|}, V_1 \cdots V_a)$, and so I is a complete intersection. Because I is a complete intersection, we have $I^{(m)} = I^m$ for all m, and in particular, for $m = 3$. Thus, the λ's of the form (i) in Conjecture 4.1 have the desired property.

We therefore turn our attention to the case that $\lambda = (a, \ldots, a, b, \ldots, b)$, and give evidence for the conjecture by proving it when $\lambda = (\underbrace{a, \ldots, a}_{t}, a - 1)$. In this case

$$I(X) = (H_1 \cdots H_{t+1}, H_1 \cdots H_t V_1 \cdots V_{a-1}, V_1 \cdots V_a).$$

We first construct two new zero-dimensional subschemes Y and W such that $Z \subseteq W \subseteq Y$ where Z is the subscheme defined by $I(X)^{(3)}$. Let C denote the complete intersection defined by $I(C) = (H_1 \cdots H_{t_1+1}, V_1 \cdots V_a)$. Note that $X \subseteq C$. We then define Y, respectively W, to be the schemes defined by the ideals

$$I(Y) = I(X)^{(3)} \cap \bigcap_{P \in C \setminus X} I(P)^2 \quad \text{and} \quad I(W) = I(X)^{(3)} \cap \bigcap_{P \in C \setminus X} I(P). \tag{4.1}$$

Note that in this case, $C \setminus X$ consists of a single point, namely the point $P = H_{t+1} \cap V_a$. Our strategy is to first find the generators of $I(Y)$, then find the generators of $I(W)$ in terms of $I(Y)$, and then find the generators of $I(X)^{(3)}$ in terms of the generators of $I(W)$.

As in [12], we call Y the *completion* of Z. Moreover, by [13], we have:

Lemma 4.2. *Let X be a finite reduced ACM subscheme in $\mathbb{P}^1 \times \mathbb{P}^1$ with partition $\lambda = (\underbrace{a, \ldots, a}_{t}, a - 1)$. Let Z denote the subscheme defined by $I(X)^{(3)}$ (i.e., Z is the scheme whose points all have multiplicity three and whose support is X), and let Y denote the completion of Z as described above. Then a minimal set of generators of $I(Y)$ is given by*

$$
\begin{aligned}
G_1 &= V_1^3 \cdots V_a^3, \\
G_2 &= H_1 \cdots H_t V_1^3 \cdots V_{a-1}^3 V_a^2, \\
G_3 &= H_1 \cdots H_{t+1} V_1^2 \cdots V_a^2, \\
G_4 &= H_1^2 \cdots H_t^2 H_{t+1} V_1^2 \cdots V_{a-1}^2 V_a, \\
G_5 &= H_1^2 \cdots H_t^2 H_{t+1}^2 V_1 \cdots V_a, \\
G_6 &= H_1^3 \cdots H_t^3 H_{t+1}^2 V_1 \cdots V_{a-1}, \quad \text{and} \\
G_7 &= H_1^3 \cdots H_{t_1+1}^3.
\end{aligned}
$$

Proof. For the reader's convenience, we sketch the main ideas. As described at the beginning of Section 3 of [13], we can associate to Y two tuples α_Y and β_Y. In our case, the tuples α_Y and β_Y will satisfy the condition of [13, Theorem 4.8] for Y to be Cohen–Macaulay. As a consequence, [13, Theorem 4.11] gives us the degrees of the minimal generators of $I(Y)$ directly from α_Y. Each of the forms listed in the lemma belong to $I(Y)$ since they vanish at each point with the correct multiplicity. Furthermore, their degrees correspond to the degrees of the minimal generators of $I(Y)$, so they form our minimal set of generators. $\square$

Lemma 4.3. *Let X be a finite reduced ACM subscheme in $\mathbb{P}^1 \times \mathbb{P}^1$ with partition $\lambda = (\underbrace{a, \ldots, a}_{t}, a - 1)$. Let Z denote the subscheme defined by $I(X)^{(3)}$, and let Y and W be the schemes defined as above. Then*

$$I(W) = I(Y) + \left(H_1^2 \cdots H_t^2 V_1^3 \cdots V_{a-1}^3 V_a, H_1^3 \cdots H_t^3 H_{t+1} V_1^2 \cdots V_{a-1}^2 \right),$$

and

$$I(X)^{(3)} = I(Z) = I(W) + \left(H_1^3 \cdots H_t^3 V_1^3 \cdots V_{a-1}^3 \right).$$

To prove this lemma, we introduce the notion of a separator:

Definition 4.4. Let $Z = m_1 P_1 + \cdots + m_i P_i + \cdots + m_s P_s$ be the subscheme in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the ideal $\bigcap_j I(P_j)^{m_j}$. We say that F is a *separator of the point P_i of multiplicity m_i* if $F \in I(P_i)^{m_i-1} \setminus I(P_i)^{m_i}$ and $F \in I(P_j)^{m_j}$ for all $j \neq i$. A set $\{F_1, \ldots, F_p\}$ is a set of *minimal separators of P_i of multiplicity m_i* if $I(Z')/I(Z) = (\overline{F}_1, \ldots, \overline{F}_p)$, and there does not exist a set $\{G_1, \ldots, G_q\}$ with $q < p$ such that $I(Z')/I(Z) = (\overline{G}_1, \ldots, \overline{G}_q)$. Here, $Z' = m_1 P_1 + \cdots + (m_i - 1)P_i + \cdots + m_s P_s$.

We now return to the proof of Lemma 4.3.

Proof. Let P denote the point given by $H_{t+1} \cap V_a$. In particular, the only point of Y which is a double point is the point whose support is P. Because Y is ACM, we can use [16] to compute the minimal separators P of multiplicity 2. Indeed, applying [16, Theorem 3.4], we find that $H_1^2 \cdots H_t^2 V_1^3 \cdots V_{a-1}^3 V_a$ and $H_1^3 \cdots H_t^3 H_{t+1} V_1^2 \cdots V_{a-1}^2$ are these minimal separators, thus proving the first statement about the generators of $I(W)$.

We now wish to show that $I(W) + (H_1^3 \cdots H_t^3 V_1^3 \cdots V_{a-1}^3) = I(X)^{(3)}$. Note that $F = H_1^3 \cdots H_t^3 \times V_1^3 \cdots V_{a-1}^3$ is a separator of P of multiplicity 1. To complete the proof, we need to show that this form is the only minimal separator of P of multiplicity 1. So, suppose that G is some other separator of P of multiplicity 1.

Let W_1 consist of the subscheme of W which contains all the points on the ruling H_{t+1}, and let W_2 consist of all the points on the ruling V_a. So, W_1 consists of $a - 1$ triple points and one reduced point, while W_2 contains t triple points and one reduce point. The separator G is also a separator of this point of multiplicity 1 in both these schemes.

Now, W_1, respectively W_2, is ACM, so by [16, Theorem 3.4], it will have a unique, up to scalar, minimal separator of degree $(0, 3(a-1))$, respectively, $(3t, 0)$. Because G is a separator of P of multiplicity 1 in both these schemes, we must have $\deg G \succeq (0, 3(a-1))$ and $\deg G \succeq (3t, 0)$. So, $\deg G \succeq (3t, 3(a-1))$. So, $\deg G \succeq \deg F$. But because $\deg Z = \deg W - 1$, we must have that $\dim_k(I(W)/I(Z))_{i,j} = 0$ or 1 for all (i, j). We have shown that if $G \in I(W) \setminus I(Z)$, we must have $\deg G \succeq \deg F = (3t, 3(a-1))$. Thus we have $(I(W)/I(Z))$ is principally generated by $\overline{F}$ in $R/I(Z)$, i.e., F is the only minimal separator of P of multiplicity 1. This gives the desired conclusion. $\square$

We can now prove the following special case of Conjecture 4.1.

Theorem 4.5. *Let X be a finite reduced ACM subscheme in $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose that the partition λ associated to X has the form $\lambda = (\underbrace{a, a, \ldots, a}_{t}, a - 1)$. Then $I(X)^{(3)} = I(X)^3$.*

Proof. The generators of $I(X)^{(3)}$ are given in Lemmas 4.2 and 4.3; an exercise shows that each generator belongs to $I(X)^3 = (H_1 \cdots H_{t+1}, H_1 \cdots H_t V_1 \cdots V_{a-1}, V_1 \cdots V_a)^3$. $\square$

4.2. λ has at least three distinct entries

One strategy to show that the converse of Conjecture 4.1 holds is to show that if λ has three or more distinct entries, then $I(X)^{(3)} \neq I(X)^3$. While we have not been able to prove this statement in general, we conclude with some infinite families that exhibit this behavior.

Theorem 4.6. *Let X be a finite reduced ACM subscheme in $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose that the partition λ associated to X has either of the two forms:*

  (i) $\lambda = (\lambda_1, \ldots, \lambda_{t-3}, 3, 2, 1)$ *with $\lambda_i \geq t - i + 1$ for $i = 1 \ldots, t - 3$; or*
  (ii) $\lambda = (\underbrace{t, t, \ldots, t}_{m}, t - 1, t - 2, \ldots, 5, 4, 3, 2, 1)$ *for some integer $t \geq 3$.*

Then $I(X)^{(3)} \neq I(X)^3$.

Proof. Suppose that λ has the form in (i). Consider the bi-homogeneous form

$$F = H_1^3 H_2^3 \cdots H_{t-3}^3 H_{t-2}^2 H_{t-1}^1 V_1^2 V_2 D$$

where H_i and V_j correspond to the horizontal and vertical rulings that minimally contain X, and D is the degree $(1,1)$-form that vanishes at the three points $H_{t-2} \cap V_3$, $H_{t-1} \cap V_2$, and $H_t \cap V_1$. Then $F \in I(X)^{(3)}$ since F vanishes at each of the points with multiplicity at least three. In addition, the bidegree of F is $(3t-5, 4)$, so the total degree of F is $3t-1$.

By Lemma 2.3, the bidegrees of the generators of $I(X)$ are $(t,0)$, $(0,\lambda_1)$ and (i, λ_{i+1}) whenever $\lambda_{i+1} - \lambda_i < 0$. But by our hypotheses, $\lambda_i \geqslant t - i + 1$ for all i, thus all generators of $I(X)$ have total degree at least t, so $I(X)^3$ has no nonzero elements of total degree less than $3t$, whence $F \in I(X)^{(3)} \setminus I(X)^3$.

Now assume (ii). The form $F = H_1^3 H_2^3 \cdots H_m^3 H_{m+1}^3 \cdots H_{m+t-4}^3 H_{m+t-3}^2 H_{m+t-2} V_1^2 V_2 D$ vanishes at each of the points with multiplicity at least three; again D is a $(1,1)$-form that vanishes at the three points $H_{m+t-3} \cap V_3$, $H_{m+t-2} \cap V_2$, and $H_{m+t-1} \cap V_1$. The form F has bidegree $(3m + 3t - 8, 4)$.

By Lemma 2.3, $I(X)$ has $t+1$ generators, say $G_0, \ldots, G_t$ with $\deg G_i = (t + m - i, i)$ for $i = 0, \ldots, t-1$, and $\deg G_t = (0, t)$. If $F \in I(X)^3$, then there exists some non-negative integer solution to $a_0 + a_1 + \cdots + a_t = 3$ such that $(3m + 3t - 8, 4) \succeq \deg G_0^{a_0} G_1^{a_1} \cdots G_t^{a_t}$. This expression implies that our non-negative integer solution to $a_0 + a_1 + \cdots + a_t = 3$ must also satisfy

$$0a_0 + 1a_1 + 2a_2 + \cdots + ta_t \leqslant 4.$$

We can write out all such solutions:

$$(3, 0, 0, 0, 0, 0, \ldots, 0), \quad (0, 3, 0, 0, 0, 0, \ldots, 0), \quad (2, 1, 0, 0, 0, 0, \ldots, 0),$$

$$(2, 0, 1, 0, 0, 0, \ldots, 0), \quad (2, 0, 0, 1, 0, 0, \ldots, 0), \quad (2, 0, 0, 0, 1, 0, \ldots, 0),$$

$$(1, 2, 0, 0, 0, 0, \ldots, 0), \quad (1, 0, 2, 0, 0, 0, \ldots, 0), \quad (1, 1, 1, 0, 0, 0, \ldots, 0),$$

$$(1, 1, 0, 1, 0, 0, \ldots, 0).$$

However, for any such solution, $(3m + 3t - 8, 4) \not\succeq \deg G_0^{a_0} G_1^{a_1} \cdots G_t^{a_t}$ because the first coordinate of $\deg G_0^{a_0} G_1^{a_1} \cdots G_t^{a_t}$ will be larger than $3m + 3t - 8$. So, $I(X)^3$ will be empty in bidegree $(3m + 3t - 8, 4)$, but $I(X)^{(3)}$ is not empty. This implies the desired conclusion. $\square$

Remark 4.7. Theorem 4.6 generalizes Example 3.1, which has $\lambda = (3, 2, 1)$.

Acknowledgments

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