PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 3, March 1984

ON INDUCTIVELY OPEN REAL FUNCTIONS

BIAGIO RICCERI[1]

ABSTRACT. In this note, given a locally connected topological space X, we characterize those continuous and locally nonconstant real functions on X which are inductively open there.

Throughout this note, X denotes a locally connected topological space. Let f be a real function on X. We recall that f is said to be *inductively open in* X (see [**1**]) if there exists a set $X^* \subseteq X$ such that $f(X^*) = f(X)$ and the function $f_{|X^*}: X^* \to f(X)$ is open.

Recently, in [**2**], as a consequence of a general lower semicontinuity theorem for certain multifunctions, we have

THEOREM 1 [**2**, THÉORÈME 2.4]. *Let X also be connected. Then any continuous real function f on X, such that $\operatorname{int}(f^{-1}(t)) = \emptyset$ for every $t \in]\inf f(X), \sup f(X)[$, is inductively open in X.*

It is easy to show by means of simple examples that none of the hypotheses of Theorem 1 can be dropped. In particular, this theorem is no longer true if X is disconnected. Indeed, it suffices to take $X = [0,1] \cup]2,3]$ and $f: X \to \mathbf{R}$ defined as follows:

$$f(x) = \begin{cases} x - 1 & \text{if } x \in [0,1], \\ x - 2 & \text{if } x \in]2,3]. \end{cases}$$

f cannot be inductively open in X, since, otherwise, it would be open there, being one-to-one. But f is not open in X (for instance, $[0,1]$ is open in X but $f([0,1])$ is not open in $f(X)$).

The aim of this note is to characterize those continuous and locally nonconstant real functions on X which are inductively open there.

We first recall a lemma established in [**3**].

LEMMA 1 [**3**, LEMMA 3.1]. *Let S be a topological space, Y a connected subset of S, s_0, s_1 two points of Y, g a real function on S. Moreover, assume:*

(1) *s_0 is a local maximum (resp. minimum) point for g;*

(2) $g(s_0) < g(s_1)$ (*resp.* $g(s_0) > g(s_1)$);

(3) *g is continuous at every point of Y.*

Then there exists $s^* \in Y$ with the following properties:

(i) $g(s^*) = g(s_0)$;

(ii) *s^* is not a local maximum (resp. minimum) point for g;*

(iii) *s^* is not a local minimum (resp. maximum) point for g, provided that for every open set $\Omega \subseteq S$, with $\Omega \cap Y \neq \emptyset$, there exists $\bar{s} \in \Omega$ such that $g(\bar{s}) \neq g(s_0)$.*

Received by the editors April 19, 1983.
1980 *Mathematics Subject Classification*. Primary 54C10, 54C30; Secondary 54D05.
[1]Supported by M.P.I.

Now, we can prove

THEOREM 2. *Let f be a continuous real function on X such that for every connected component Γ of $f(X)$ and every $t \in \mathrm{int}(\Gamma)$ the set $\mathrm{int}(f^{-1}(t))$ is empty.*

Then the following are equivalent:

(1) *The function f is inductively open in X.*

(2) *For every $t \in f(X)$ there exists a connected set $X_t \subseteq X$ such that t belongs to the interior of $f(X_t)$ in $f(X)$.*

PROOF. Let us show that (1)$\Rightarrow$(2). As f is inductively open in X, there exists $X^* \subseteq X$ such that $f(X^*) = f(X)$ and $f_{|X^*}\colon X^* \to f(X)$ is open. Let $t \in f(X)$. Choose $x \in X^*$ such that $f(x) = t$. Since X is locally connected at x, there is, in particular, a connected neighbourhood X_t of x. Thus, with obvious meaning of the symbols, we have

$$t \in \mathrm{int}_{f(X)}(f(X_t \cap X^*)) \subseteq \mathrm{int}_{f(X)}(f(X_t)),$$

so (2) follows.

Now let us show that (2)$\Rightarrow$(1). Put $E = \{x \in X\colon x \text{ is a local extremum point for } f\}$, $\tilde{X} = \{x \in X\colon f(x)$ is an extreme of a connected component of $f(X)$ and, for every neighbourhood V of x, $f(x) \in \mathrm{int}_{f(X)}(f(V))\}$, and $X^* = (X \setminus E) \cup \tilde{X}$. We claim $f(X^*) = f(X)$. Indeed, let $t \in f(X)$. By (2) there is a connected set $X_t \subseteq X$ such that $t \in \mathrm{int}_{f(X)}(f(X_t))$. Let $x \in X_t$ be such that $f(x) = t$. Suppose $x \notin X^*$, that is, $x \in E \setminus \tilde{X}$. Let Γ_t be the connected component of $f(X)$ containing t and, first, assume t is not an extreme of Γ_t. Then there exist $x_1, x_2 \in X_t$ such that $f(x_1) < f(x) < f(x_2)$. By hypothesis the interior of $f^{-1}(t)$ is empty. Therefore, since x is a local extremum point for f and f is continuous, by Lemma 1, there is a point $x^* \in X_t \cap (f^{-1}(t) \setminus E)$, so $t \in f(X \setminus E) \subseteq f(X^*)$.

Now suppose t is an extreme of Γ_t, for instance, the maximum of Γ_t. As $x \notin \tilde{X}$, there exists a neighbourhood V of x such that $t \notin \mathrm{int}_{f(X)}(f(V))$. Hence, as $t \in \mathrm{int}_{f(X)}(f(X_t))$, it follows that $f(X_t)$ is a nondegenerate interval contained in Γ_t. Applying Lemma 1 again, we then get a point $\tilde{x} \in f^{-1}(t) \cap X_t$ that is not a local minimum point for f. Thus, $\tilde{x} \in \tilde{X}$ so $t \in f(\tilde{X}) \subseteq f(X^*)$. A similar argument holds if t is the minimum of Γ_t. Now we prove that the function $f_{|X^*}\colon X^* \to f(X)$ is open. Let Ω be any open subset of X. We must show that $f(\Omega \cap X^*)$ is open in $f(X)$. To this end, let $\bar{t} \in f(\Omega \cap X^*)$. Choose a point $\bar{x} \in \Omega \cap X^*$ such that $f(\bar{x}) = \bar{t}$. Furthermore, let U be an open connected neighbourhood of $\bar{x}$ contained in Ω. Let $\Gamma_{\bar{t}}$ be the connected component of $f(X)$ containing $\bar{t}$. First suppose $\bar{t} \in \mathrm{int}(\Gamma_{\bar{t}})$, so that, by hypothesis, we have $\mathrm{int}(f^{-1}(\bar{t})) = \emptyset$. As $\bar{x} \neq E$, the set $f(U)$ is a neighbourhood of $\bar{t}$. But, by Lemma 1, we have $\mathrm{int}(f(U)) \subseteq f(U \setminus E) \subseteq f(\Omega \cap X^*)$. Hence, $\bar{t}$ is an interior point of $f(\Omega \cap X^*)$. Now suppose $\bar{t}$ is an extreme of $\Gamma_{\bar{t}}$, for instance, the minimum of $\Gamma_{\bar{t}}$. In this case $\bar{x} \in \tilde{X}$, so $\bar{t} \in \mathrm{int}_{f(X)}(f(U))$. If $\Gamma_{\bar{t}} = \{\bar{t}\}$, then, since $f(U) \subseteq \Gamma_{\bar{t}}$, we have $U \subseteq \Omega \cap \tilde{X}$ so $\bar{t} \in \mathrm{int}_{f(X)}(f(\Omega \cap X^*))$. If, on the contrary, $\Gamma_{\bar{t}} \neq \{\bar{t}\}$, then there exists $\epsilon > 0$ such that $f(U) \supseteq [\bar{t}, \bar{t} + \epsilon[$. Always by Lemma 1, we have $]\bar{t}, \bar{t} + \epsilon[\subseteq f(U \setminus E)$ so $[\bar{t}, \bar{t} + \epsilon[\subseteq f(\Omega \cap X^*)$. Hence $\bar{t} \in \mathrm{int}_{f(X)}(f(\Omega \cap X^*))$. A similar argument holds if $\bar{t}$ is the maximum of $\Gamma_{\bar{t}}$.

We conclude, observing that by means of Theorem 2 it is possible to extend Theorems 2.8 and 2.10 of [**4**] as well as Theorem 2.2 of [**5**]. The details are left to the reader.

References

1. A. Arhangel'skii, *Open and almost open mappings of topological spaces*, Soviet Math. Dokl. **3** (1962), 1738–1741.
2. B. Ricceri, *Sur la semi-continuité inférieure de certaines multifonctions*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), 265–267.
3. B. Ricceri and A. Villani, *Openness properties of continuous real functions on connected spaces*, Rend. Mat. (7) (to appear).
4. B. Ricceri, *Applications de théorèmes de semi-continuité inférieure*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), 75–78.
5. ——, *Solutions lipschitziennes d'équations différentielles sous forme implicite*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), 245–248.

Seminario Matematico, Citta' Universitaria, Viale A. Doria 6, 95125 Catania, Italy