J. Math. Anal. Appl. 394 (2012) 225–230

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Three non-zero solutions for a nonlinear eigenvalue problem

Francesca Faraci [a,*], Alexandru Kristály [b]

[a] Department of Mathematics, Università di Catania, Catania, Italy
[b] Department of Economics, Babeş-Bolyai University, Cluj-Napoca, Romania

ARTICLE INFO

Article history:
Received 16 August 2011
Available online 27 April 2012
Submitted by Manuel del Pino

Keywords:
Nonlinear eigenvalue problem
Dirichlet boundary conditions
Multiple solutions

ABSTRACT

In the present paper we prove a novel multiplicity result for a model quasilinear Dirichlet problem (P_λ) depending on a positive parameter λ. By a variational method, we prove that for every $\lambda > 1$ problem (P_λ) has at least two non-zero solutions, while there exists $\hat{\lambda} > 1$ such that problem $(P_{\hat{\lambda}})$ has at least three non-zero solutions.

1. Introduction

In the present paper we deal with the problem of multiplicity results for the following quasilinear equation coupled with the Dirichlet boundary condition

$$\begin{cases} -\Delta_p u = \lambda\alpha(x) f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where Ω is a bounded open connected set in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $p > n$, Δ_p is the p-Laplacian operator, λ is a positive parameter, $\alpha \in L^1(\Omega)$ is a non-zero potential, and $f : [0, +\infty[\to \mathbb{R}$ is a continuous function with $f(0) = 0$.

Problems of the type (P_λ) have been the object of intensive investigations in the recent years, see [1–8], and references therein. Many of the aforementioned contributions guarantee the existence of *at least two* non-trivial weak solutions of (P_λ) for $\lambda > 0$ large enough where the key geometric assumptions on the nonlinear term F, where $F : [0, +\infty[\to \mathbb{R}$ is the primitive of f, that is $F(s) = \int_0^s f(t)dt$ for every $s \geq 0$, can be summarized as

$$\begin{cases} \sup\limits_{[0,+\infty[} F > 0; \\ \limsup\limits_{s\to 0^+} \dfrac{F(s)}{s^p} \leq 0 \quad \text{and} \quad \limsup\limits_{s\to +\infty} \dfrac{F(s)}{s^p} \leq 0. \end{cases} \tag{1.1}$$

In order to obtain the aforementioned multiplicity results, various variational approaches are exploited; for instance, Morse theory [5,6], the mountain pass theorem and Ricceri-type three critical points results [1–4,7,9].

Notice that under (1.1) one can have even an exact multiplicity result for (P_λ). To see this, let $p = 2$, $n = 1$, $\Omega = I \subset \mathbb{R}$ be a large interval, $\alpha = 1$, and $f : [0, +\infty[\to \mathbb{R}$ defined by $f(s) = s(s-a)(1-s)_+$ with $0 < a < 1/2$; here, $t_+ = \max(0, t)$. It is clear that F verifies (1.1). Moreover, via a bifurcation argument, Wei [10] proved that there exists $\lambda_0 > 0$ such that for

* Corresponding author.
E-mail addresses: ffaraci@dmi.unict.it (F. Faraci), alexandrukristaly@yahoo.com (A. Kristály).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.04.045

all $0 < \lambda < \lambda_0$ problem (P_λ) has no positive solution, it has exactly one positive solution for $\lambda = \lambda_0$, and *exactly two* positive solutions for $\lambda > \lambda_0$; see also [11].

The main purpose of the present paper is to guarantee the existence of *at least three* non-zero, non-negative weak solutions for (P_λ) for certain values of $\lambda > 0$ when (1.1) holds. According to the above exact multiplicity result, our aim requires more specific assumptions both on f (or F) and α. In order to state our main result, we introduce the notation

$$k_\infty := \frac{n^{\frac{-1}{p}}}{\sqrt{\pi}} \left[\Gamma\left(1+\frac{n}{2}\right)\right]^{\frac{1}{n}} \left(\frac{p-1}{p-n}\right)^{1-\frac{1}{p}} m(\Omega)^{\frac{1}{n}-\frac{1}{p}}, \tag{1.2}$$

where Γ denotes the Euler Gamma-function.

Our main result reads as follows:

Theorem 1.1. *Let $p > n$, $\alpha \in L^1(\Omega)$ be a non-negative, non-zero function with compact support K. Assume that*

(i) $S_F := \sup_{[0,+\infty[} F < +\infty$;
(ii) $\limsup_{s\to 0^+} \frac{F(s)}{s^p} \le 0$.

Moreover, there exists $c > 0$ such that

(iii) $F(c) = \max_{[0,k_\infty(pS_F\|\alpha\|_{L^1})^{\frac{1}{p}}]} F < S_F$;
(iv) $\frac{F(c)}{c^p} > \frac{m(\Omega\setminus K)}{p\,\mathrm{dist}(K,\partial\Omega)^p\|\alpha\|_{L^1}}$.

Then, the following statements hold:

(a) *For every $\lambda > 1$, problem (P_λ) has at least two non-zero, non-negative weak solutions.*
(b) *There exists $\hat\lambda > 1$ such that problem $(P_{\hat\lambda})$ has at least three non-zero, non-negative weak solutions.*

Before proving Theorem 1.1 some remarks are in order.

Remark 1.1. (a) Under the assumptions of Theorem 1.1, one can prove the existence of two non-zero weak solutions for (P_λ) for enough large values of $\lambda > 0$; the first one is the global minimum of the energy functional associated with (P_λ) with negative energy-level, while the second one is a mountain-pass type solution with positive energy-level. A much precise conclusion can be deduced as follows. Since (i),(ii) and (iv) imply (1.1), a suitable choice in [9] guarantees the existence of at least two non-zero weak solutions for (P_λ) for every $\lambda > \lambda_0$, where

$$\lambda_0 = \inf\left\{\frac{\int_\Omega |\nabla u|^p}{p\int_K \alpha(x)F(u(x))dx} : u \in W_0^{1,p}(\Omega),\ \int_K \alpha(x)F(u(x))dx > 0\right\}. \tag{1.3}$$

A simple estimate by means of a suitable truncation function and assumption (iv) show that

$$\lambda_0 < \frac{c^p m(\Omega\setminus K)}{pF(c)\mathrm{dist}(K,\partial\Omega)^p\|\alpha\|_{L^1}} < 1,$$

which concludes the proof of (a) in Theorem 1.1; for details see (3.6). Even more, under these assumptions, Ricceri's result (see [9]) provides a *stability* of problem (P_λ) with respect to any small nonlinear perturbation whenever $\lambda > \lambda_0$. However, for $\lambda > 0$ *small* enough, problem (P_λ) has usually only the trivial solution. Example 3.1 supports this fact as well.

(b) Assumption (iii) requires that the function F has a local maximum $c > 0$ on a quite large set whose size depends on the function F itself, namely, on the interval $I_F := [0, k_\infty(pS_F\|\alpha\|_{L^1})^{\frac{1}{p}}]$. Note that a simple estimate together with hypothesis (iv) shows that c belongs to the interval I_F. In view of the above discussion, the technical assumption (iii) is behind on the existence of a third non-zero weak solution for (P_λ).

Remark 1.2. Note that in Theorem 1.1 we are able to prove the existence of a single value of $\hat\lambda > 1$ such that problem $(P_{\hat\lambda})$ has at least three non-zero, non-negative weak solutions. A challenging problem is to know if this phenomenon is stable/unstable with respect to the parameter λ; namely, to confirm/infirm the existence of certain functions f satisfying all the assumptions of Theorem 1.1 such that problem (P_λ) has exactly two non-zero weak solutions for $\lambda \in]1,+\infty[\setminus\{\hat\lambda\}$ and at least three solutions for $\lambda = \hat\lambda$.

Remark 1.3. Taking into account the special character of the function α (i.e., α has a compact support K in Ω), we could expect to construct in a trivial way some weak solutions for (P_λ) via p-harmonic functions. The reason is the following; for simplicity, let us consider the case when $\Omega = B(0,R)$ and $K = \overline{B}(0,r)$ for some $0 < r < R$. Due to (iii), the nonlinearity f attains the zero value at least in two points (c being one of them since it is a local maximum for F). Let us denote such an element by $c > 0$. A simple calculation shows that the function $\tilde u_c \in W_0^{1,p}(B(0,R))$ defined by

$$\tilde u_c(x) = \begin{cases} c & \text{if } x \in K = \overline{B}(0,r), \\ c\dfrac{|x|^{\frac{p-n}{p-1}} - R^{\frac{p-n}{p-1}}}{r^{\frac{p-n}{p-1}} - R^{\frac{p-n}{p-1}}} & \text{if } x \in B(0,R)\setminus K, \end{cases}$$

verifies almost everywhere problem (P_λ) for every $\lambda \in \mathbb{R}$. Thus, one can construct in a direct way two *almost pointwise* solutions of (P_λ) for every $\lambda \in \mathbb{R}$. However, it is worth noticing that $\tilde{u}_c$ is *not* a weak solution for (P_λ). Indeed, according to a standard regularity result (see for instance Lieberman [12, Theorem 1]), a bounded weak solution of (P_λ) is a function of class $C^{1,\beta}$ for some $\beta > 0$ which is not our case. Note that without α having compact support K in Ω we are not able to state a similar multiplicity result as Theorem 1.1; we postpone this discussion after giving the whole proof (see Remark 3.1).

Remark 1.4. We are able to state a similar result also in the case when $S_F := \sup_{[0,+\infty[} F = +\infty$. Note that in this case the assumptions/proof are more involved, see Theorem 3.1.

We conclude this section with a simple consequence of Theorem 1.1 when we are dealing with a highly quasilinear problem, i.e., the value of p in the p-Laplacian is large enough. Namely, we have

Corollary 1.1. *Let $\alpha \in L^1(\Omega)$ be a non-negative, non-zero function with compact support K, and besides of* (i) *from Theorem 1.1, we assume that 0 is a local maximum of F, and that there exist constants c, δ, with $0 < c < k_\infty < \delta$ and $c \leq \operatorname{dist}(K, \partial\Omega)$ such that $0 < F(c) = \max_{[0,\delta]} F < S_F$. Then, there exists $p_0 > n$ such that for each $p \geq p_0$, both conclusions of Theorem 1.1 hold.*

The proof of Corollary 1.1 is based on the observations that $\lim_{p\to\infty}(pS_F\|\alpha\|_{L^1})^{\frac{1}{p}} = 1$ and $\lim_{p\to\infty}(c/\operatorname{dist}(K, \partial\Omega))^p/p = 0$; these limits are used to prove items (iii) and (iv) from Theorem 1.1, respectively.

2. Preliminaries

Our main tool is the following theorem by Ricceri which is a consequence of a more general result [13, Theorem 1]:

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a reflexive real Banach space, $p > 1, J : X \to \mathbb{R}$ a C^1 functional, with compact derivative such that $J(0) = 0$ and*

$$\max\left\{\limsup_{u\to 0}\frac{J(u)}{\|u\|^p},\ \limsup_{\|u\|\to\infty}\frac{J(u)}{\|u\|^p}\right\} \leq 0.$$

Moreover, assume that there exist $\gamma > 0$, $\rho \in \mathbb{R}$, $\nu \in [0, \frac{1}{p}[$, $u_1 \in X \setminus \{0\}$, $u_2 \in X$ such that

(j) $J(u) \leq \gamma + \nu\|u\|^p$ *for all* $u \in X$;
(jj) $\|u_1\|^p \leq pJ(u_1)$ *and* $\|u_2\|^p < p(J(u_2) + \rho)$;
(jjj) $J(u_2) = \sup_{\|u\|^p \leq \frac{p(\rho+\gamma)}{1-p\nu}} J(u) < \sup_X J$.

Then, there exists $\hat{\lambda} > 1$ such that the functional $\Phi : X \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{p}\|u\|^p - \hat{\lambda}J(u)$$

has at least three non-zero critical points.

Remark 2.1. Notice that, from the assumptions of Theorem 2.1 one has also that 0, being a strict local minimum, belongs to the set of the critical points of Φ. Also, one has that u_2 is a local maximum of J, hence a critical point of J.

In the sequel we will denote by X the space $W_0^{1,p}(\Omega)$ endowed with the usual norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p dx\right)^{\frac{1}{p}}.$$

Since $p > n$, the embedding of X into $C^0(\bar{\Omega})$ is continuous and compact. Let c_∞ be the embedding constant, that is

$$c_\infty = \sup_{u\in X\setminus\{0\}} \frac{\max_{\bar{\Omega}} |u|}{\|u\|}.$$

We have the following estimate for c_∞, see [14]:

$$c_\infty \leq \frac{n^{\frac{-1}{p}}}{\sqrt{\pi}}\left[\Gamma\left(1+\frac{n}{2}\right)\right]^{\frac{1}{n}}\left(\frac{p-1}{p-n}\right)^{1-\frac{1}{p}} m(\Omega)^{\frac{1}{n}-\frac{1}{p}},$$

where Γ denotes the Euler Gamma-function, and equality occurs when Ω is a ball. From (1.2), we clearly have that $c_\infty \leq k_\infty$.

Recall that by a weak solution of $(P_{\hat{\lambda}})$ we mean a function $u \in X$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v - \hat{\lambda}\int_\Omega \alpha(x)f(u)v = 0,$$

for every $v \in X$. It is well known that critical points of Φ are weak solutions of $(P_{\hat{\lambda}})$.

3. Proof of Theorem 1.1

Without loss of generality we can assume that f is defined on the whole real axis, putting $f(s) = 0$ for all $s \le 0$. If we still denote by F the primitive of f, then $F(s) = 0$ for all $s \le 0$. Define $J : X \to \mathbb{R}$ by

$$J(u) = \int_\Omega \alpha(x)F(u) \quad \text{for all } u \in X.$$

The functional J is of class C^1, has compact derivative and $J(0) = 0$. Assumption (i) implies that J is bounded and

$$\sup_X J \le \sup_{[0,+\infty[} F\|\alpha\|_{L^1} = S_F\|\alpha\|_{L^1}. \tag{3.1}$$

In particular,

$$\limsup_{\|u\|\to\infty} \frac{J(u)}{\|u\|^p} \le 0. \tag{3.2}$$

Moreover, from (ii) it follows that for fixed $\varepsilon > 0$, there exists $s_0 > 0$ such that $F(s) \le \varepsilon s^p$ for all $s \in [0, s_0]$ and so, $F(s) \le \varepsilon|s|^p$ for all $s \in [-s_0, s_0]$. Then, if $u \in X$ and $\|u\| \le \frac{s_0}{c_\infty}$, one has

$$F(u(x)) \le \varepsilon|u(x)|^p \quad \text{for every } x \in \Omega.$$

So,

$$J(u) \le \varepsilon\|\alpha\|_{L^1}\|u\|_\infty^p \le \varepsilon\|\alpha\|_{L^1}c_\infty^p\|u\|^p,$$

which implies at once

$$\limsup_{u\to 0} \frac{J(u)}{\|u\|^p} \le 0. \tag{3.3}$$

We claim to apply Theorem 2.1 to the functional J defined above which, in view of (3.2) and (3.3), verifies

$$\max\left\{\limsup_{u\to 0} \frac{J(u)}{\|u\|^p}, \ \limsup_{\|u\|\to\infty} \frac{J(u)}{\|u\|^p}\right\} \le 0.$$

Denote by $\delta = \text{dist}(K, \partial\Omega)$ and define $u_c : \Omega \to \mathbb{R}$ as it follows:

$$u_c(x) = \begin{cases} c & \text{if dist}(x, K) = 0 \\ c\left(1 - \dfrac{\text{dist}(x, K)}{\delta}\right) & \text{if } 0 < \text{dist}(x, K) \le \delta \\ 0 & \text{if dist}(x, K) > \delta, \end{cases} \tag{3.4}$$

where c is from assumption (iii). Notice that the function $u_c \in X$ is Lipschitzian and $|\nabla u_c(x)| \le \frac{c}{\delta}$ for almost every $x \in \Omega$. One has that

$$\|u_c\|^p = \int_{\Omega\setminus K} |\nabla u_c|^p \le \frac{c^p}{\delta^p}m(\Omega \setminus K).$$

Also, since K is the support of the function α, one has

$$J(u_c) = \int_\Omega \alpha(x)F(u_c) = F(c)\|\alpha\|_{L^1}, \tag{3.5}$$

and assumption (iv) implies at once that

$$\|u_c\|^p < pJ(u_c). \tag{3.6}$$

From assumption (iii) we have also that u_c is not a global maximum of J. Indeed, if d is a positive number such that $F(d) > F(c)$, then defining u_d as in (3.4), we have that

$$J(u_d) = F(d)\|\alpha\|_{L^1} > F(c)\|\alpha\|_{L^1} = J(u_c). \tag{3.7}$$

It is easily seen that

$$\left\{u \in X : \ \|u\| \le (pS_F\|\alpha\|_{L^1})^{\frac{1}{p}}\right\} \subseteq \left\{u \in X : \ \|u\|_\infty \le k_\infty(pS_F\|\alpha\|_{L^1})^{\frac{1}{p}}\right\}.$$

So, if $u \in X$ and $\|u\| \le (pS_F\|\alpha\|_{L^1})^{\frac{1}{p}}$, then, due to assumption (iii),

$$J(u) \le J(u_c),$$

that is, together with the fact that $\|u_c\| \le (pS_F\|\alpha\|_{L^1})^{\frac{1}{p}}$ (cf. (iv)),

$$J(u_c) = \max_{\bar{B}\left(0,(pS_F\|\alpha\|_{L^1})^{\frac{1}{p}}\right)} J, \tag{3.8}$$

where $\bar{B}(0,\sigma)$ denotes the closed ball in X centered at zero of radius σ.

Assumptions (j)–(jjj) of Theorem 2.1 are immediately verified with the choices $\gamma = S_F\|\alpha\|_{L^1}$, $\rho = \nu = 0$ and $u_1 = u_2 = u_c$. Indeed, condition (j) follows from (3.1) while (jj) is a consequence of (3.6). Finally, (3.7) and (3.8) imply at once (jjj). All the assumptions of Theorem 2.1 are satisfied. Then, there exists $\hat{\lambda} > 1$ such that the functional

$$u \to \frac{1}{p}\|u\|^p - \hat{\lambda}J(u)$$

has at least three non-zero critical points. As critical points of this functional are non-negative weak solutions of problem $(P_{\hat{\lambda}})$, our conclusion is achieved. □

Remark 3.1. As we already pointed out in the Introduction, without α having compact support K in Ω we meet several obstacles to apply Theorem 2.1. Indeed, in order to ensure the existence of a non-zero, local (but not global) maximum point $u_2 = u_c$ for J, careful estimates are needed with specific truncation functions in X, see relations (3.5), (3.7) and (3.8); otherwise, the aforementioned expressions will become much involved. Notice that if in (P_λ) we have a zero Neumann boundary condition instead of the zero Dirichlet boundary condition, the arguments become much simpler and no restriction is needed on α.

We propose now an example of nonlinearity satisfying the assumptions of Theorem 1.1.

Example 3.1. Let $q > p > n$, $\alpha \in L^1(\Omega)$ be a non-negative function with compact support K satisfying

$$\|\alpha\|_{L^1} > \frac{m(\Omega\setminus K)}{p\mathrm{dist}(K,\partial\Omega)^p}\frac{\pi^{p-1}}{(\sqrt{2})^{q+1}}.$$

Assume also that m is an integer, verifying

$$m > \frac{k_\infty}{(2\pi)^{1-\frac{1}{p}}}(p\|\alpha\|_{L^1})^{\frac{1}{p}}.$$

Define $f : [0,+\infty[\to \mathbb{R}$ by

$$f(s) = \begin{cases} \mathrm{sign}(\pi - s)|\sin s|^{q-1} & \text{if } s \in [0, 2\pi[\\ 2|\sin s|^{q-1}\chi_{[2m\pi,(2m+1)\pi]}(s) & \text{if } s \in [2\pi, +\infty[\end{cases}$$

where $\chi_{[2m\pi,(2m+1)\pi]}$ is the characteristic function of the interval $[2m\pi, (2m+1)\pi]$.

First, for $\lambda \in]0, c_\infty^{-p}\|\alpha\|_{L^1}^{-1}[$, problem (P_λ) has only the zero solution. On the other hand, according to Theorem 1.1, for every $\lambda > 1$ problem (P_λ) has at least two non-zero weak solutions, and there exists $\hat{\lambda} > 1$ such that problem $(P_{\hat{\lambda}})$ has at least three non-zero, non-negative solutions.

In the previous result and example we treated the case of bounded primitive F. It is also possible to handle the case when F is unbounded under more technical assumptions. We conclude the present paper by stating such a result without giving its explicit proof.

Theorem 3.1. *Let $p > n$, $\alpha \in L^\infty(\Omega)$ be a non-negative, non-zero function with compact support K. Assume that*

(i) $\sup_{[0,+\infty[} F = +\infty$;
(ii) $\limsup_{s\to 0^+} \frac{F(s)}{s^p} \le 0$; $\limsup_{s\to+\infty} \frac{F(s)}{s^p} \le 0$.

Moreover, assume that there exist $\gamma > 0$, $\nu \in [0, \frac{1}{p}[$ and $c > 0$ such that

(iii) $F(c) = \max_{[0,k_\infty\left(\frac{p\gamma}{1-p\nu}\right)^{\frac{1}{p}}]} F$;
(iv) $\frac{F(c)}{c^p} > \frac{m(\Omega\setminus K)}{p\mathrm{dist}(K,\partial\Omega)^p\|\alpha\|_{L^1}}$;
(v) $\|\alpha\|_{L^\infty}F(s) - \nu(k_\infty m(\Omega))^{-1}s^p \le \frac{\gamma}{m(\Omega)}$ *for all* $s \ge 0$.

Then, there exists $\hat{\lambda} > 1$ such that problem $(P_{\hat{\lambda}})$ has at least three non-zero, non-negative solutions.

Acknowledgments

The present study has been initiated during the visit of A. Kristály at the Università di Catania, Catania, Italy, supported partially by the Istituto Nazionale di Alta Matematica (INdAM). He is also supported by grant CNCSIS PCCE-55/2008 "Sisteme diferenţiale î n analiza neliniară şi aplicaţii" and by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. The authors would like to thank Professor B. Ricceri for the stimulating discussions during the preparation of the manuscript.

References

[1] D. Averna, G. Bonanno, A three critical points theorem and its applications to the ordinary Dirichlet problem, Topol. Methods Nonlinear Anal. 22 (1) (2003) 93–103.
[2] G. Bonanno, Some remarks on a three critical points theorem, Nonlinear Anal. 54 (2003) 651–665.
[3] G. Bonanno, G. Molica Bisci, Three weak solutions for elliptic Dirichlet problems, J. Math. Anal. Appl. 382 (1) (2011) 1–8.
[4] F. Cammaroto, A. Chinnì, B. Di Bella, Multiple solutions for a Dirichlet problem involving the p-Laplacian, Dynam. Systems Appl. 16 (4) (2007) 673–679.
[5] F.O. De Paiva, Multiple solutions for a class of quasilinear problems, Discrete Contin. Dyn. Syst. 15 (2006) 669–680.
[6] N.S. Papageorgiou, E.M. Rocha, A multiplicity theorem for a variable exponent Dirichlet problem, Glasg. Math. J. 50 (2) (2008) 335–349.
[7] B. Ricceri, On a three critical points theorem, Arch. Math. (Basel) 75 (2000) 220–226.
[8] X. Wu, J. Chen, On existence and multiplicity of solutions for elliptic equations involving the p-Laplacian, NoDEA Nonlinear Differential Equations Appl. 15 (6) (2008) 745–755.
[9] B. Ricceri, A further three critical points theorem, Nonlinear Anal. 71 (9) (2009) 4151–4157.
[10] J. Wei, Exact multiplicity for some nonlinear elliptic equations in balls, Proc. Amer. Math. Soc. 125 (11) (1997) 3235–3242.
[11] P. Korman, Y. Li, T. Ouyang, Exact multiplicity results for boundary value problems with nonlinearities generalizing cubic, Proc. Roy. Soc. Edinburgh Sect. A 126 (1996) 599–616.
[12] G.M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 (1988) 1203–1219.
[13] B. Ricceri, A class of nonlinear eigenvalue problems with four solutions, J. Nonlinear Convex Anal. 11 (2010) 503–511.
[14] G. Talenti, Some inequalities of Sobolev type on two-dimensional spheres, in: W. Walter (Ed.), General Inequalities, Vol. 5, in: Internat. Ser. Numer. Math., vol. 80, 1987, pp. 401–408.