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On the existence and uniqueness of minima and maxima on spheres of the integral functional of the calculus of variations

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Dedicated, with esteem, to Professor R.T. Rockafellar on his seventieth birthday

Abstract

Given a bounded domain $\Omega \subset \mathbf{R}^n$, we prove that if $f:\mathbf{R}^{n+1} \to \mathbf{R}$ is a C^1 function whose gradient is Lipschitzian in $\mathbf{R}^{n+1}$ and non-zero at 0, then, for each $r > 0$ small enough, the restriction of the integral functional $u \to \int_\Omega f(u(x), \nabla u(x))\,dx$ to the sphere $\{u \in H^1(\Omega)\colon \int_\Omega (|\nabla u(x)|^2 + |u(x)|^2)\,dx = r\}$ has a unique global minimum and a unique global maximum.

Keywords: Sobolev space; Integral functional; Minimum; Maximum; Sphere; Existence; Uniqueness

1. Introduction

Here and in the sequel, $\Omega \subset \mathbf{R}^n$ is a bounded domain, with smooth boundary, and $f:\Omega \times \mathbf{R}^{n+1} \to \mathbf{R}$ is a function such that, for each $y \in \mathbf{R}^{n+1}$, the function $f(\cdot, y)$ is measurable in Ω, while, for each $x \in \Omega$, $f(x, \cdot)$ is a C^1 function in $\mathbf{R}^{n+1}$ whose gradient is non-constant and Lipschitzian (with respect to the Euclidean metric), with Lipschitz constant L (independent of x). We also assume that

$$\sup_{x\in\Omega}\big(\big|f(x,0)\big| + \big|\nabla f(x,0)\big|\big) < +\infty. \tag{1}$$

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We will consider the Sobolev space $H^1(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega \left(\left|\nabla u(x)\right|^2 + \left|u(x)\right|^2 \right) dx \right)^{\frac{1}{2}}$$

which is induced by the scalar product

$$\langle u, v \rangle = \int_\Omega \left(\nabla u(x) \nabla v(x) + u(x) v(x) \right) dx.$$

The linear growth of $\nabla f(x, \cdot)$ (coming from its Lipschitzianity) and (1) imply that the functional

$$u \to J(u) := \int_\Omega f\left(x, u(x), \nabla u(x)\right) dx$$

is (well defined and) C^1 on $H^1(\Omega)$, with derivative given by

$$\left\langle J'(u), v \right\rangle = \int_\Omega \left(f_\xi\left(x, u(x), \nabla u(x)\right) v(x) + \nabla_\eta f\left(x, u(x), \nabla u(x)\right) \nabla v(x) \right) dx$$

for all $u, v \in H^1(\Omega)$ [3, p. 248].

Let $r > 0$. We are interested in minima and maxima of the restriction of the functional J to the sphere $S_r := \{u \in H^1(\Omega)\colon\ \|u\| = r\}$.

In the present setting, there is no evidence of their existence and uniqueness. In fact, with regard to the existence aspect, not only S_r is not weakly compact but also, if $f(x, \xi, \cdot)$ is neither convex nor concave in $\mathbf{R}^n$, the functional J is neither lower nor upper weakly semicontinuous. But, even when J is sequentially weakly continuous, it may happen that J has no minima and/or maxima on S_r.

In this connection, consider the following simple and enlightening situation. Assume that $J(u) = \int_\Omega f(u(x))\, dx$, where $f : \mathbf{R} \to \mathbf{R}$ has a unique global maximum in $\mathbf{R}$, say ξ_0. Then, it is clear that the constant function $x \to \xi_0$ is the unique maximum of the functional J. In this case, J turns out to be sequentially weakly continuous, thanks to the Rellich–Kondrachov theorem [3, p. 239]. Then, by [2, Lemma 2.1], the function $\rho \to \sup_{S_\rho} J$ is non-decreasing in $]0, +\infty[$. Consequently, if $r > |\xi_0|(\mathrm{meas}(\Omega))^{1/2}$, $J|_{S_r}$ has no maxima.

Nevertheless, we will show that if $\int_\Omega \nabla f(x, 0)\, dx \neq 0$ then $J|_{S_r}$ possesses exactly one minimum and exactly one maximum for each $r > 0$ small enough.

2. The result

To shorten the statement of our result, let us introduce some further notations. In the sequel, $g : \Omega \times \mathbf{R}^{n+1} \to \mathbf{R}$ is another non-negative function, with $g(x, 0) = 0$, such that, for each $y \in \mathbf{R}^{n+1}$, the function $g(\cdot, y)$ is measurable in Ω, while, for each $x \in \Omega$, $g(x, \cdot)$ is a C^1 function in $\mathbf{R}^{n+1}$ whose gradient is Lipschitzian, with Lipschitz constant $\nu < 2$ (independent of x). We also assume

$$\sup_{x \in \Omega} \left(\left|g(x, 0)\right| + \left|\nabla g(x, 0)\right| \right) < +\infty.$$

We set

$$I(u)=\int_{\Omega} g\big(x,u(x),\nabla u(x)\big)\,dx$$

for all $u \in H^1(\Omega)$.

Moreover, V is a closed linear subspace of $H^1(\Omega)$ with the following property: there exists $v_0 \in V$ such that

$$\int_{\Omega} \big(f_\xi(x,0)v_0(x)+\nabla_\eta f(x,0)\nabla v_0(x)\big)\,dx \neq 0. \tag{2}$$

Further, we denote by S the set (possibly empty) of all global minima of the restriction to V of the functional

$$u \to \|u\|^2+I(u)+\frac{2-\nu}{L}J(u).$$

Finally, given a set $C \subseteq V$, we say that the problem of minimizing (respectively maximizing) J over C is weakly (respectively strongly) well posed if the following two properties hold:

- the restriction of J to C has a unique global minimum (respectively maximum), say u^*;
- if $\{u_n\}$ is any sequence in C such that $\lim_{n\to\infty} J(u_n)=\inf_C J$ (respectively $\lim_{n\to\infty} J(u_n)=\sup_C J$) then $\{u_n\}$ converges weakly (respectively strongly) to u^*.

Then, with the convention $\inf\emptyset=+\infty$, our result reads as follows:

Theorem 1. *Under the above assumptions, one has*

$$\delta := \inf_{u\in S}\big(\|u\|^2+I(u)\big)>0$$

and, for each $r \in\,]0,\delta[$, the problem of minimizing the functional J over the set

$$C_r := \big\{u \in V\colon\ \|u\|^2+I(u)=r\big\}$$

is weakly well posed.

Proof. Let $\mu \geqslant 0$ and let $u,v,w \in H^1(\Omega)$, with $\|w\|=1$. Using Cauchy–Schwartz and Hölder inequalities, we have

$$\begin{aligned}
&\big|\big\langle I'(u)+\mu J'(u)-I'(v)-\mu J'(v),\,w\big\rangle\big|\\
&\quad\leqslant \int_{\Omega}\big|\big(g_\xi(x,u,\nabla u)-g_\xi(x,v,\nabla v)\big)w+\big(\nabla_\eta g(x,u,\nabla u)-\nabla_\eta g(x,v,\nabla v)\big)\nabla w\big|\,dx\\
&\qquad+\mu\int_{\Omega}\big|\big(f_\xi(x,u,\nabla u)-f_\xi(x,v,\nabla v)\big)w\\
&\qquad+\big(\nabla_\eta f(x,u,\nabla u)-\nabla_\eta f(x,v,\nabla v)\big)\nabla w\big|\,dx\\
&\quad\leqslant \int_{\Omega}\big(\big|g_\xi(x,u,\nabla u)-g_\xi(x,v,\nabla v)\big|^2+\big|\nabla_\eta g(x,u,\nabla u)-\nabla_\eta g(x,v,\nabla v)\big|^2\big)^{\frac12}\\
&\qquad\times\big(|w|^2+|\nabla w|^2\big)^{\frac12}\,dx
\end{aligned}$$

$$+\mu\int_\Omega\left(\left|f_\xi(x,u,\nabla u)-f_\xi(x,v,\nabla v)\right|^2\left|\nabla_\eta f(x,u,\nabla u)-\nabla_\eta f(x,v,\nabla v)\right|^2\right)^{\frac{1}{2}}$$
$$\times\left(|w|^2+|\nabla w|^2\right)^{\frac{1}{2}}dx$$
$$\leqslant\left(\int_\Omega\left(\left|g_\xi(x,u,\nabla u)-g_\xi(x,v,\nabla v)\right|^2+\left|\nabla_\eta g(x,u,\nabla u)-\nabla_\eta g(x,v,\nabla v)\right|^2\right)dx\right)^{\frac{1}{2}}$$
$$+\mu\left(\int_\Omega\left(\left|f_\xi(x,u,\nabla u)-f_\xi(x,v,\nabla v)\right|^2\right.\right.$$
$$\left.\left.+\left|\nabla_\eta f(x,u,\nabla u)-\nabla_\eta f(x,v,\nabla v)\right|^2\right)dx\right)^{\frac{1}{2}}$$
$$\leqslant(\nu+\mu L)\|u-v\|.$$

Hence, the derivative of the functional $I+\mu J$ is Lipschitzian, with constant $\nu+\mu L$. As a consequence, if $0\leqslant\mu<\frac{2-\nu}{L}$, the functional $u\to\|u\|^2+I(u)+\mu J(u)$ is strictly convex and coercive. To see this, it is enough to show that its derivative is strongly monotone [4, pp. 247–248]. Indeed, if $\Phi(\cdot):=\|\cdot\|^2$, we have for all $u,v\in H^1(\Omega)$

$$\left\langle\Phi'(u)+I'(u)+\mu J'(u)-\Phi'(v)-I'(v)-\mu J'(v),u-v\right\rangle$$
$$\geqslant 2\|u-v\|^2-\left\|I'(u)-I'(v)+\mu\left(J'(u)-J'(v)\right)\right\|\|u-v\|$$
$$\geqslant(2-\nu-\mu L)\|u-v\|^2.$$

Clearly, this shows also the convexity of the functional $\Phi+I+\frac{2-\nu}{L}J$. Assume $S\neq\emptyset$. Then, S is closed and convex, and so there exists a unique $\hat{u}\in S$ such that

$$\|\hat{u}\|^2+I(\hat{u})=\delta.$$

Observe that $\|u\|^2+I(u)>0$ for all $u\in V\setminus\{0\}$. So, $\delta\geqslant 0$. Arguing by contradiction, assume $\delta=0$. Then, it would follow $\hat{u}=0$. Hence, since $0\in S$, we would have

$$\left\langle\Phi'(0)+I'(0)+\frac{2-\nu}{L}J'(0),v\right\rangle=0$$

for all $v\in V$ and so, since $\Phi'(0)+I'(0)=0$ (being 0 the global minimum of $\Phi+I$), it would follow

$$\int_\Omega\left(f_\xi(x,0)v(x)+\nabla_\eta f(x,0)\nabla v(x)\right)dx=0$$

for all $v\in V$, against condition (2). Hence, we have proven that $\delta>0$. Now, fix $r\in\,]0,\delta[$ and consider the function $\Psi:V\times[\frac{L}{2-\nu},+\infty[\,\to\mathbf{R}$ defined by

$$\Psi(u,\lambda)=J(u)+\lambda\left(\|u\|^2+I(u)-r\right)$$

for all $(u,\lambda)\in V\times[\frac{L}{2-\nu},+\infty[$. As we have seen above, $\Psi(\cdot,\lambda)$ is continuous and convex for all $\lambda\geqslant\frac{L}{2-\nu}$ and coercive for all $\lambda>\frac{L}{2-\nu}$, while $\Psi(u,\cdot)$ is continuous and concave for all $u\in V$, with $\lim_{\lambda\to+\infty}\Psi(0,\lambda)=-\infty$. So, we can apply to Ψ a classical saddle-point theorem [4, Theorem 49.A] which ensures the existence of $(u^*,\lambda^*)\in V\times[\frac{L}{2-\nu},+\infty[$ such that

$$J(u^*)+\lambda^*\big(\|u^*\|^2+I(u^*)-r\big)=\inf_{u\in V}\big(J(u)+\lambda^*\big(\|u\|^2+I(u)-r\big)\big)$$
$$=J(u^*)+\sup_{\lambda\geqslant\frac{L}{2-\nu}}\lambda\big(\|u^*\|^2+I(u^*)-r\big).$$

Of course, we have $\|u^*\|^2+I(u^*)\leqslant r$, since the sup is finite. But, if it were $\|u^*\|^2+I(u^*)<r$, we would have $\lambda^*=\frac{L}{2-\nu}$. This, in turn, would imply that $u^*\in S$, against the fact that $r<\delta$. Hence, we have $\|u^*\|^2+I(u^*)=r$. Consequently

$$J(u^*)+\lambda^* r=\inf_{u\in V}\big(J(u)+\lambda^*\big(\|u\|^2+I(u)\big)\big).$$

From this, we infer that $\lambda^*>\frac{L}{2-\nu}$ (since $r<\delta$), that u^* is a global minimum of $J|_{C_r}$ and that each global minimum of $J|_{C_r}$ is a global minimum in V of the functional $u\to\|u\|^2+I(u)+\frac{1}{\lambda^*}J(u)$. Since $\lambda^*>\frac{L}{2-\nu}$, this functional is strictly convex and coercive, and so u^* is its unique global minimum in V towards which every minimizing sequences weakly converges [1, p. 3]. The proof is complete. □

Remark 1. It is almost superfluous to remark that the conclusion of Theorem 1 may fail if condition (2) is not satisfied. In this connection, consider, for instance, the case $f(\sigma)=-|\sigma|^2$, with $g=0$. Condition (2), however, serves only to ensure that $\delta>0$. So, it becomes superfluous, in particular, when $S=\emptyset$. In other words, we have the following corollary:

Theorem 2. *Under the assumptions of Theorem* 1*, but condition* (2)*, if the set S is empty, then, for each $r>0$, the problem of minimizing the functional J over the set*

$$\big\{u\in V:\ \|u\|^2+I(u)=r\big\}$$

is weakly well posed.

Now, denote by S_1 the set (possibly empty) of all global minima of the restriction to V of the functional

$$u\to\|u\|^2+I(u)-\frac{2-\nu}{L}J(u).$$

Clearly, applying Theorem 1 also to $-f$, we get

Theorem 3. *Under the assumptions of Theorem* 1*, one has*

$$\delta_1:=\min\Big\{\inf_{u\in S}\big(\|u\|^2+I(u)\big),\ \inf_{u\in S_1}\big(\|u\|^2+I(u)\big)\Big\}>0$$

and, for each $r\in\,]0,\delta_1[$, the problems of minimizing and maximizing the functional J over the set

$$\big\{u\in V:\ \|u\|^2+I(u)=r\big\}$$

are both weakly well posed.

So, taking Remark 1 into account, we also get

Theorem 4. *Under the assumptions of Theorem* 1*, but condition* (2)*, if the sets S and S_1 are empty, then, for each $r > 0$, the problems of minimizing and maximizing the functional J over the set*

$$\{u \in V\colon\ \|u\|^2 + I(u) = r\}$$

are both weakly well posed.

Remark 2. It is easy to realize that, when the functional I is sequentially weakly lower semicontinuous (so, in particular, when $I = 0$), the weak well-posedness of the optimization problems considered in the previous results implies the strong well-posedness of them.

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References

[1] A.L. Dontchev, T. Zolezzi, Well-Posed Optimization Problems, Lecture Notes in Math., vol. 1543, Springer-Verlag, 1993.
[2] M. Schechter, K. Tintarev, Spherical maxima in Hilbert space and semilinear elliptic eigenvalue problems, Differential Integral Equations 3 (1990) 889–899.
[3] M. Struwe, Variational Methods, Springer-Verlag, 1996.
[4] E. Zeidler, Nonlinear Functional Analysis and Its Applications, vol. III, Springer-Verlag, 1985.