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Regularity of Minimizers of Some Variational Integrals with Discontinuity

Maria Alessandra Ragusa and Atsushi Tachikawa

Abstract. We prove regularity properties in the vector valued case for minimizers of variational integrals of the form

$$\mathcal{A}(u) = \int_\Omega A(x, u, Du)\, dx$$

where the integrand $A(x, u, Du)$ is not necessarily continuous respect to the variable x, grows polinomially like $|\xi|^p$, $p \geq 2$.

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1. Introduction

In this note we consider the regularity problem of minimizers of the variational integral

$$\mathcal{A}(u) = \int_\Omega A(x, u, Du)\, dx \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^m$, $u : \Omega \to \mathbb{R}^n$ is a mapping in a suitable Sobolev space, $Du = (D_\alpha u^i)$ ($\alpha = 1, \ldots, m,\ i = 1, \ldots, n$). The nonnegative integrand function $A : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$ is in the class *VMO* with respect to the variable x, continuous in u and of class C^2 with respect to Du. It is also assumed that for some $p \geq 2$ there exist two constants λ_1 and Λ_1 such that

$$\lambda_1(1 + |\xi|^p) \leq A(x, u, \xi) \leq \Lambda_1(1 + |\xi|^p), \quad \forall (x, u, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}.$$

M. A. Ragusa: Dipartimento di Matematica, Università di Catania, Viale A. Doria, 6, 95125 Catania, Italia; maragusa@dmi.unict.it
A. Tachikawa: Department of Mathematics, Faculty of Science and Technology, Tokyo University of Science, Noda, Chiba, 278-8510, Japan ;
tachikawa_atsushi@ma.noda.tus.ac.jp

A minimizer for the functional $\mathcal{A}$ is a function $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ such that, for every $\varphi \in W_0^{1,p}(\Omega, \mathbb{R}^n)$,

$$\mathcal{A}(u; \mathrm{supp}\varphi) \leq \mathcal{A}(u + \varphi; \mathrm{supp}\varphi).$$

For the case that $A(x, u, \xi)$ is continuous in x, many sharp regularity results for minimizers of $\mathcal{A}$ have been already known (see, e.g., [7, 8, 10, 12]). On the other hand, when $A(\cdot, u, \xi)$ is assumed only to be L^∞, we can not expect the regularity of minimizers in general, as a famous example due to De Giorgi contained in [5] asserts. So, it seems to be natural to consider the regularity problems for $A(x, u, \xi)$ with *"mild"* discontinuity with respect to x. In 1996 Huang in [13] investigates regularity results for the elliptic system

$$-D_\alpha(a_{ij}^{\alpha\beta}(x) D_\beta u^j) \;=\; g_i(x) - \mathrm{div} f^i(x), \quad i, j = 1, \ldots, n;\ \alpha, \beta = 1, \ldots, m$$

assuming that $a_{ij}^{\alpha\beta}$ belong to the Sarason class VMO of vanishing mean oscillation functions. Then he generalizes Acquistapace's [1] and Campanato's results [7, p. 88, Theorem 3.2]. Campanato showed regularity properties under the assumption that the coefficients $a_{ij}^{\alpha\beta}$ are in $C^\alpha(\Omega)$. Acquistapace refined the results by Campanato, considering the coefficients in the class so-called "small multipliers of BMO".

In a recent study made by Daněček and Viszus [4], it is considered the following functional:

$$\int_\Omega \left\{ A_{ij}^{\alpha\beta}(x) D_\alpha u^i D_\beta u^j + g(x, u, Du) \right\} dx,$$

where $A_{ij}^{\alpha\beta}$ are in general discontinuous, more precisely belong to the vanishing mean oscillation class (VMO class) and satisfy a strong ellipticity condition while the lower order term g is a Charathéodory function satisfying the following growth condition:

$$|g(x, u, z)| \;\leq\; f(x) + H|z|^\kappa,$$

where $f \geq 0$, a.e. in Ω, $f \in L^p(\Omega)$, $2 < p \leq \infty$, $H \geq 0$, $0 \leq \kappa < 2$.

We also recall the paper by Di Gironimo, Esposito and Sgambati [6] where is treated the Morrey regularity for minimizers of the functional

$$\int_\Omega A_{ij}^{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^j \, dx,$$

where $(A_{ij}^{\alpha\beta}(x, u))$ are elliptic and of the VMO class in the variable x.

In [17] the authors extend the results of [4] and [6] to the case that the functional is given by

$$\int_\Omega \left\{ A_{ij}^{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^j + g(x, u, Du) \right\} dx.$$

In the note [18], it is studied the Morrey regularity for minimizer of the more general functionals

$$\mathcal{A}(u) = \int_\Omega A(x, u, Du)\, dx,$$

where $A(x, u, \xi)$ is a nonnegative function defined on $\Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$ which is of class VMO as a function of x, continuous in u and of class C^2 with respect to ξ. We point out that it is assumed that for some positive constants $\mu_0 \le \mu_1$,

$$\mu_0|\xi|^2 \le A(x, u, \xi) \le \mu_1|\xi|^2 \quad \forall (x, u, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}.$$

We point out that in the above mentioned papers concerning functionals given by integrals with VMO class integrands, we have considered quadratic growth functionals. The super quadratic cases with continuous coefficients are treated in [2] and [11].

In the present note we investigate the partial regularity of the minima of $\mathcal{A}$, defined by (1.1) under p-growth hypothesis of the integrand function A, $p \ge 2$. This study can be considered as an improving of [17] and [18] because of the growth condition is more general.

2. Definitions and preliminary tools

In the sequel we set

$$Q(x, R) = \left\{ y \in \mathbb{R}^m \; : \; |y^\alpha - x^\alpha| < R, \; \alpha = 1, \dots, m \right\}$$

a generic cube in $\mathbb{R}^m$ having center x and side $2R$.

Let us now give some useful definitions, starting to the Morrey space $L^{p,\lambda}$.

Definition 2.1. (see [16]). Let $1 \le p < \infty, 0 \le \lambda < m$. A measurable function $G \in L^p(\Omega, \mathbb{R}^n)$ belongs to the Morrey class $L^{p,\lambda}(\Omega, \mathbb{R}^n)$ if

$$\|G\|_{L^{p,\lambda}(\Omega)} = \sup_{\substack{0<\rho<\operatorname{diam}\Omega \\ x\in\Omega}} \frac{1}{\rho^\lambda} \int_{\Omega\cap Q(x,\rho)} |G(y)|^p dy < +\infty,$$

where $Q(x, \rho)$ ranges in the class of the cubes of $\mathbb{R}^m$.

Definition 2.2. Let $H \in L^1(\Omega, \mathbb{R}^n)$. The integral average $H_{x,R}$ is defined by

$$H_{x,R} = \fint_{\Omega\cap Q(x,R)} H(y)\, dy = \frac{1}{|\Omega \cap Q(x, R)|} \int_{\Omega\cap Q(x,R)} H(y)\, dy,$$

where $|\Omega \cap Q(x, R)|$ is the Lebesgue measure of $\Omega \cap Q(x, R)$. In the case that we are not interested in specifying which the center is considered, we simply write H_R.

Let us introduce the Bounded Mean Oscillation class.

Definition 2.3 ([15]). Let $H \in L^1_{\text{loc}}(\mathbb{R}^m)$. We say that H belongs to $BMO(\mathbb{R}^m)$ if

$$\|H\|_* \equiv \sup_{Q(x,R)} \frac{1}{|Q(x,R)|} \int_{Q(x,R)} |H(y) - H_{x,R}| dy < \infty.$$

Let us now introduce the space of vanishing mean oscillation functions.

Definition 2.4 ([19]). If $H \in BMO(\mathbb{R}^m)$ and

$$\eta(H;R) = \sup_{\rho \le R} \frac{1}{|Q(x,\rho)|} \int_{Q(x,\rho)} |H(y) - H_\rho| dy$$

We define that $H \in VMO(\Omega)$ if $\lim_{R \to 0} \eta(H;R) = 0$.

Throughout the present paper we consider $p \ge 2$ and $u : \Omega \to \mathbb{R}^n$ a minimizer of the functional

$$\mathcal{A}(u) = \int_\Omega A(x,u,Du)\, dx$$

where the hypothesis on the integrand function $A(x,u,\xi)$ are the following.

(A-1) For every $(u,\xi) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $A(\cdot,u,\xi) \in VMO(\Omega)$ and the mean oscillation of $\frac{A(\cdot,u,\xi)}{|\xi|^p}$ vanishes uniformly with respect to u, ξ in the following sense: there exist a positive number ρ_0 and a function $\sigma(z,\rho)$: $\mathbb{R}^m \times [0,\rho_0) \to [0,\infty)$ with

$$\lim_{R \to 0} \sup_{\rho < R} \fint_{Q(0,\rho) \cap \Omega} \sigma(z,\rho) dz = 0,$$

such that $A(\cdot,u,\xi)$ satisfies, for every $x \in \overline{\Omega}$ and $y \in Q(x,\rho_0) \cap \Omega$,

$$\left| A(y,u,\xi) - A_{x,\rho}(u,\xi) \right| \le \sigma(x-y,\rho)(1+|\xi|^2)^{\frac{p}{2}} \quad \forall (u,\xi) \in \mathbb{R}^n \times \mathbb{R}^{mn},$$

where $A_{x,\rho}(u,\xi) = \fint_{Q(x,\rho) \cap \Omega} A(y,u,\xi) dy$.

(A-2) For every $x \in \Omega$, $\xi \in \mathbb{R}^{mn}$ and $u, v \in \mathbb{R}^n$,

$$\left| A(x,u,\xi) - A(x,v,\xi) \right| \le (1+|\xi|^2)^{\frac{p}{2}} \omega(|u-v|^2),$$

where ω is some monotone increasing concave function with $\omega(0) = 0$.

(A-3) For almost all $x \in \Omega$ and all $u \in \mathbb{R}^n$, $A(x,u,\cdot) \in C^2(\mathbb{R}^{mn})$.

(A-4) There exist positive constants λ_1, Λ_1 such that

$$\lambda_1(1+|\xi|^p) \le A(x,u,\xi) \le \Lambda_1(1+|\xi|^p)$$

$$\lambda_1(1+|\eta|^p) \le \frac{\partial^2 A(x,u,\xi)}{\partial \xi^i_\alpha \partial \xi^j_\beta} \eta^i_\alpha \eta^j_\beta \le \Lambda_1(1+|\eta|^p)$$

for all $(x,u,\xi,\eta) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \times \mathbb{R}^{mn}$.

Let us state the main theorem of the paper concerning the partial regularity of the minimizers of the functionals $\mathcal{A}$.

Theorem 2.5. *Assume that* $\Omega \subset \mathbb{R}^m$ *is a bounded domain with sufficiently smooth boundary* $\partial\Omega$ *and that* $p \geq 2$. *Let* $u \in H^{1,p}(\Omega, \mathbb{R}^n)$ *a minimizer of the functional* $\mathcal{A}(u,\Omega) = \int_\Omega A(x,u,Du)\,dx$ *in the class* $X_g(\Omega) = \{u \in H^{1,p}(\Omega)\ ;\ u - g \in H_0^{1,p}(\Omega)\}$ *for a given boundary data* $g \in H^{1,s}(\Omega)$ *with* $s > p$. *Suppose that assumptions* (A-1), (A-2), (A-3)*and* (A-4) *are satisfied. Then, for some positive* ε, *for every* $0 < \tau < \min\{2+\varepsilon, m(1-\frac{p}{s})\}$ *we have*

$$D\,u \in L^{p,\tau}(\Omega_0, \mathbb{R}^{mn}),$$

where Ω_0 *is a relatively open subset of* $\overline{\Omega}$ *which satisfies*

$$\overline{\Omega} \setminus \Omega_0 = \left\{ x \in \Omega\colon\ \liminf_{R\to 0} \frac{1}{R^{m-p}} \int_{\Omega(x,R)} |Du(y)|^p dy > 0 \right\}.$$

Moreover, we have $\mathcal{H}^{m-p-\delta}(\overline{\Omega} \setminus \Omega_0) = 0$ *for some* $\delta > 0$, *where* $\mathcal{H}^r$ *denotes the* r*-dimensional Hausdorff measure.*

As a corollary of the above theorem we have the following partial Hölder regularity result.

Corollary 2.6. *Let* g, u *and* Ω_0 *be as in Theorem* 2.5. *Assume that* $p+2 \geq m$ *and that* $s > \max\{m,p\}$. *Then, for some* $\alpha \in (0,1)$, *we have*

$$u \in C^{0,\alpha}(\Omega_0, \mathbb{R}^n).$$

Moreover, as a corollary of the proof of Theorem 2.5, we have the following full-regularity result for the case that A does not depend on u.

Corollary 2.7. *Assume that* A *and* g *satisfy all assumptions of Theorem* 2.5 *and that* A *does not depend on* u. *Let* u *be a minimizer of* $\mathcal{A}$ *in the class* X_g *then*

$$D\,u \in L^{p,\tau}(\Omega, \mathbb{R}^{mn}). \tag{2.1}$$

Moreover, if $p+2 \geq m$ *and* $s > \max\{m,p\}$, *we have full-Hölder regularity of* u, *namely* $u \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^n)$.

3. Preliminary lemmas and proof of the main results

Throughout the paper we use the following notation:

$$\begin{aligned}
Q^+(x,R) &= \{y \in \mathbb{R}^m\ ;\ |y^\alpha - x^\alpha| < R,\ \alpha = 1,\dots,m,\quad y^m > 0\} \\
&\qquad \text{for } x \in \mathbb{R}^m \cap \{x\ ;\ x^m = 0\},\quad R > 0, \\
\Omega(x,R) &= Q(x,R) \cap \Omega \\
\Gamma(x,R) &= Q(x,R) \cap \partial\Omega.
\end{aligned}$$

When the center x is understood, we sometimes omit the center and write simply $Q(R)$, $Q^+(R)$ etc. For the sake of simplicity, we always assume that $0 < R < 1$ in the following.

We can always reduce locally to the case of flat boundary, by means of a diffeomorphism which does not change properties of the functional assumed in the conditions (A-1)–(A-4). More precisely, we can choose a positive constant R_1 depending only on $\partial\Omega$ which has the following properties:

1. A finite number of cubes $\{Q(x, R_1)\}$ centered at $x \in \partial\Omega$ cover the boundary, namely $\partial\Omega \subset \bigcup_{k=1}^N Q(x_k, R_1), \quad x_k \in \partial\Omega,\ k = 1, \dots, N$.
2. For every $Q(x_k, 2R_1)$, by means of a suitable diffeomorphism, we can assume that $x_k = 0$ and that

$$\Gamma(x_k, 2R_1) = Q(0, 2R_1) \cap \partial\Omega \subset \{x \in \mathbb{R}^m \ ; x^m = 0\}$$
$$Q(x_k, 2R_1) \cap \Omega = Q^+(0, 2R_1) = \{x \in \mathbb{R}^m \ ; \ |x| < 2R_1,\ x^m > 0\}.$$

Let us define a so-called *frozen functional.* For some fixed point $x_0 \in \Omega$ and $R > 0$ let us define $A^0(\xi)$ and $\mathcal{A}^0(u)$ by

$$A^0(\xi) = A_R(u_R, \xi) := \fint_{\Omega(x_0,R)} A(y, u_R, \xi)\, dy$$
$$\mathcal{A}^0(u, \Omega(x_0, R)) := \int_{\Omega(x_0,R)} A^0(Du)\, dx,$$

where $u_R = u_{x_0,R} = \fint_{\Omega(x_0,R)} u(y)dy$.

For weak solutions of the Euler-Lagrange equation of $\mathcal{A}^0$, we have the following regularity results.

For interior points, we have the following (see [2, Theorem 3.1]).

Lemma 3.1. *Let $u \in H^{1,p}(\Omega, \mathbb{R}^n)$ $p \geq 2$, be a solution of the system*

$$D_\alpha a_i^\alpha(Du) = 0, \quad i = 1, \dots, n, \quad \textit{in } \Omega,$$

in the sense that $\int_\Omega a_i^\alpha(Du) D_\alpha \varphi^i\, dx = 0$, for all $\varphi \in C_0^\infty(\Omega, \mathbb{R}^n)$, under the conditions

(1) *$a_i^\alpha(0) = 0$;*

(2) *there exist two constants $\nu > 0$ and $M > 0$ such that, for all $x \in \Omega$ and for all $\xi, \zeta \in \mathbb{R}^{mn}$,*

$$\|A(\xi)\| \leq M \cdot (1 + \|\xi\|^2)^{\frac{p-2}{2}}$$
$$A_{ij}^{\alpha\beta}(\xi)\zeta_\alpha^i \zeta_\beta^j \geq \nu \cdot (1 + \|\xi\|^2)^{\frac{p-2}{2}} \|\zeta\|^2,$$

where $A = (A_{ij}^{\alpha\beta})$ and $A_{ij}^{\alpha\beta}(\xi) = \frac{\partial a_i^\alpha(\xi)}{\partial \xi_\beta^j}$.

Then, for all $Q(\sigma) = Q(x_0, \sigma) \subset\subset \Omega$ *and for all* $t \in (0, 1)$,

$$\int_{Q(t\sigma)} |Du|^p \, dx \le ct^{\lambda_0} \int_{Q(\sigma)} |Du|^p \, dx, \quad \lambda_0 = \min\{2 + \varepsilon_0, m\},$$

for some positive constants ε_0 *and* c *which do not depend on* t, σ *and* x^0.

In the neighborhood of the boundary, by the proof of [2, Theorem 7.1], we have the following.

Lemma 3.2. *Let* $a_i^\alpha(\xi)$ *and* λ_0 *be as in Lemma* 3.1 *and* $v \in H^{1,p}(Q^+(0, R))$ *be a solution of the problem*

$$\begin{cases} \displaystyle\int_{Q^+(0,R)} a_i^\alpha(Dv + Dg) D_\alpha \varphi^i \, dx = 0 \quad \forall \varphi \in C_0^\infty(Q^+(0, R)) \\ \qquad\qquad\qquad\qquad\qquad\qquad v = 0 \quad \text{on } \Gamma(0, R), \end{cases} \tag{3.1}$$

where g *is a given function with* $Dg \in L^s(Q^+(0, R))$ *for some* $s > p$. *Then, for every* $x_0 \in \Gamma(0, R)$ *and* τ_0 *with* $0 < \tau_0 < \min\{\lambda_0, m(1 - \frac{p}{s})\}$, *there exist a constant* $c > 0$ *such that*

$$\begin{aligned} &\int_{Q^+(x_0,t\sigma)} |W(Dv)|^2 \, dx \\ &\le ct^{\tau_0} \int_{Q^+(x_0,\sigma)} |W(Dv)|^2 \, dx + c\sigma^{\tau_0} \left(\int_{Q^+(x_0,\sigma)} |W(Dg)|^{\frac{2s}{p}} \, dx \right)^{\frac{p}{s}}, \end{aligned} \tag{3.2}$$

for any $\sigma \in (0, R - |x_0|]$ *and* $t \in (0, 1)$, *where* $W(\xi) = (1 + |\xi|^2)^{\frac{p-2}{4}} \xi$.

Outline of the proof. Since (3.1) is exactly (7.6) of [2], we can proceed as in [2, pp. 148–150] and get the following estimates:

$$\begin{aligned} &\int_{Q^+(x_0,t\sigma)} |W(Dv)|^2 \, dx \\ &\le c_1 t^\lambda \int_{Q^+(x_0,\sigma)} |W(Dv)|^2 \, dx + c_1 \int_{Q^+(x_0,\sigma)} \big(1 + |Dv| + |Dg|\big)^{p-2} |Dg|^2 \, dx \end{aligned}$$

$$\begin{aligned} &\int_{Q^+(x_0,\sigma)} \big(1 + |Dv| + |Dg|\big)^{p-2} |Dg|^2 \, dx \\ &\le c_2 \int_{Q^+(x_0,\sigma)} |W(Dg)|^2 dx + c_2 \int_{Q^+(x_0,\sigma)} |Dv|^{p-2} |Dg|^2 dx \end{aligned}$$

$$\begin{aligned} &\int_{Q^+(x_0,\sigma)} |Dv|^{p-2} |Dg|^2 dx \\ &\le \Big(1 - \frac{2}{p}\Big) \delta \int_{Q^+(x_0,\sigma)} |W(Dv)|^2 \, dx + \frac{2}{p} \, \delta^{1-\frac{p}{2}} \int_{Q^+(x_0,\sigma)} |W(Dg)|^2 \, dx \end{aligned}$$

for any $\delta > 0$. These estimates are nothing else than (17)–(19) of [2]. Combining them, we get

$$\begin{aligned}\int_{Q^+(x_0,t\sigma)} |W(Dv)|^2\,dx &\le c_1\left\{t^\lambda + c_2\Big(1-\frac{2}{p}\Big)\delta\right\}\int_{Q^+(x_0,\sigma)}|W(Dv)|^2\,dx \\ &\quad + c_1c_2\left(1+\frac{2}{p}\,\delta^{1-\frac{p}{2}}\right)\int_{Q^+(x_0,\sigma)}|W(Dg)|^2\,dx \\ &\le c_1\left\{t^\lambda + c_1c_2\Big(1-\frac{2}{p}\Big)\delta\right\}\int_{Q^+(x_0,\sigma)}|W(Dv)|^2\,dx \\ &\quad + c_3(p,\delta)\sigma^{m(1-\frac{p}{s})}\left(\int_{Q^+(x_0,\sigma)}|W(Dg)|^{\frac{2s}{p}}\,dx\right)^{\frac{p}{s}}.\end{aligned}$$

Now, using "A useful lemma" of [8, p. 44], we get (3.2). □

Moreover, we have the following L^q-estimate for u.

Lemma 3.3. *Assume that $u \in H^{1,p}(Q^+(0,R))$ satisfies*

$$\mathcal{A}(u, Q^+(0,R)) \le \mathcal{A}(u+\varphi, Q^+(0,R)), \quad \varphi \in H_0^{1,p}(Q^+(0,R)),$$

and that $u = g$ on $\Gamma(0,R)$ for some $g \in H^{1,q_1}(Q^+(0,R))$ with $q_1 > p$. Then there exists an exponent $q \in (p, q_1]$ such that $u \in H^{1,q}(Q^+(0,r))$ for any $r < R$. Moreover, if $x_0 \in Q^+(0,r) \cup \Gamma(0,r)$ and $\rho < R - r$, we have the estimate

$$\begin{aligned}&\left(\fint_{Q(x_0,\rho/2)\cap Q^+(0,R)}(1+|Du|^2)^{\frac{q}{2}}\,dx\right)^{\frac{1}{q}} \\ &\le \left(\fint_{Q(x_0,\rho)\cap Q^+(0,R)}(1+|Du|^2)^{\frac{p}{2}}dx\right)^{\frac{1}{p}} + c\left(\fint_{Q(x_0,\rho)\cap Q^+(0,R)}(1+|Dg|^2)^{\frac{q}{2}}dx\right)^{\frac{1}{q}}.\end{aligned} \tag{3.3}$$

In addition, if $Q(x_0,\rho) \subset\subset Q^+(0,R)$, then we have

$$\left(\fint_{Q(x_0,\frac{\rho}{2})}(1+|Du|^2)^{\frac{q}{2}}\,dx\right)^{\frac{1}{q}} \le c\left(\fint_{Q(x_0,\rho)}(1+|Du|^2)^{\frac{p}{2}}\,dx\right)^{\frac{1}{p}}. \tag{3.4}$$

Outline of the Proof. For the case that $Q(x_0,\rho) \subset\subset Q^+(0,R)$, we can proceed as in the proof of [9, Theorem 4.1] to get (3.4). For general case, mentioning the difference on the growth conditions, we can proceed as in the proof of [14, Lemma 1]. □

Mention that the above lemma is valid for minimizers of $\mathcal{A}^0$ also.

For bounded domain D with smooth boundary, covering ∂D with a finite number of cubes and using the above local estimates we get the following global L^q-estimates for a minimizer.

Corollary 3.4. *Let $D \subset \mathbb{R}^m$ be an open set with smooth boundary ∂D, and let $v \in H^{1,p}(D)$ be a minimizer for the functional $\mathcal{A}$ (or $\mathcal{A}^0$) in the class*

$$X_g := \{w \in H^{1,p}(D); w - g \in H_0^{1,p}(D)\}$$

for a given map $g \in H^{1,q_1}(D)$, $q_1 > p$. Then $Dv \in L^q(D)$ for some $q \in (p, q_1)$ and

$$\int_D \left(1 + |Dv|^2\right)^{\frac{q}{2}} dx \leq c \int_D \left(1 + |Dg|^2\right)^{\frac{q}{2}} dx.$$

We show the partial regularity of u by comparing u with v. For this purpose, we need the following lemma which can be shown as [11, Theorem 4.2, (4.8)].

Lemma 3.5. *Let $v \in H^{1,p}(\Omega(x_0, r))$ is a minimizer for $\mathcal{A}^0(w, \Omega(x_0, r))$ in the class $\{w \in H^{1,p}(\Omega(x_0, r))\ ;\ w - u \in H_0^{1,p}(\Omega(x_0, r))\}$ for a given function $u \in H^{1,p}(\Omega(x_0, r))$. Then we have*

$$\int_{\Omega(x_0,r)} |Du - Dv|^p \, dx \leq c\big\{\mathcal{A}^0(u; \Omega(x_0, r)) - \mathcal{A}^0(v; \Omega(x_0, r))\big\}.$$

Now, we can prove our main theorem.

Proof of Theorem 2.5. Assume that $Q(R) = Q(x_0, R) \subset\subset \Omega$. Let $v \in H^{1,p}(Q(R))$ be a minimizer of $\mathcal{A}^0(\tilde{v}, Q(R))$ in the class $\{\tilde{v} \in H^{1,p}(Q(R))\ ;\ u - \tilde{v} \in H_0^{1,p}(Q(R))\}$, and let $w = u - v$. First we will estimate $\int_{Q(R)} |Dw|^p \, dx$. By Lemma 3.5 we can see that

$$\begin{aligned}
\int_{Q(R)} |Dw|^p \, dx &= c\big\{\mathcal{A}^0(u) - \mathcal{A}^0(v)\big\} \\
&\leq c \int_{Q(R)} \big|A_R(u_R, Du) - A(x, u_R, Du)\big| \, dx \\
&\quad + c \int_{Q(R)} \big|A(x, u_R, Du) - A(x, u, Du)\big| \, dx \\
&\quad + c \int_{Q(R)} \big|A(x, v, Dv) - A(x, u_R, Dv)\big| \, dx \\
&\quad + c \int_{Q(R)} \big|A(x, u_R, Dv) - A_R(u_R, Dv)\big| \, dx.
\end{aligned}$$

Here we have used the minimality of u. So, using the assumptions on A, we get

$$\begin{aligned}
\int_{Q(R)} |Dw|^p \, dx \leq &\int_{Q(R)} \big\{\sigma(x, R) + \omega(|u - u_R|^2)\big\}\big(1 + |Du(x)|^2\big)^{\frac{p}{2}} dx \\
&+ \int_{Q(R)} \big\{\sigma(x, R) + \omega(|v - u_R|^2)\big\}\big(1 + |Dv(x)|^2\big)^{\frac{p}{2}} dx.
\end{aligned} \tag{3.5}$$

Using Hölder's inequality, Lemma 3.3, (3.4) and the boundedness of ω and σ, we have

$$\begin{aligned}
&\int_{Q(R)} \{\sigma(x,R)+\omega(|u-u_R|^2)\}\big(1+|Du(x)|^2\big)^{\frac{p}{2}}\,dx \\
&\le C\left\{\left(\fint_{Q(R)}\sigma(x,R)\,dx\right)^{\frac{q-p}{q}}+\left(\fint_{Q(R)}\omega(|u-u_R|^2)\,dx\right)^{\frac{q-p}{q}}\right\} \qquad (3.6)\\
&\quad\times\int_{Q(2R)}\big(1+|Du(x)|^2\big)^{\frac{p}{2}}\,dx.
\end{aligned}$$

Using Corollary 3.4, and (3.4) we get similarly

$$\begin{aligned}
&\int_{Q(R)} \{\sigma(x,R)+\omega(|v-u_R|^2)\}\big(1+|Dv(x)|^2\big)^{\frac{p}{2}}\,dx \\
&\le C\left\{\left(\fint_{Q(R)}\sigma(x,R)\,dx\right)^{\frac{q-p}{q}}+\left(\fint_{Q(R)}\omega(|v-u_R|^2)\,dx\right)^{\frac{q-p}{q}}\right\} \qquad (3.7)\\
&\quad\times\int_{Q(2R)}\big(1+|Du(x)|^2\big)^{\frac{p}{2}}\,dx.
\end{aligned}$$

By virtue of concavity of ω, using Jensen's inequality and Poincaré inequality, we have

$$\left.\begin{aligned}
&\fint_{Q(R)}\omega(|u-u_R|^2)\,dx\\
&\fint_{Q(R)}\omega(|v-u_R|^2)\,dx
\end{aligned}\right\}\le C\omega\left(R^{p-m}\int_{Q(R)}|Du|^p\,dx\right). \qquad (3.8)$$

Combining (3.5) – (3.8), we obtain

$$\begin{aligned}
\int_{Q(R)}|Dw|^p\,dx \le C&\left\{\left(\fint_{Q(R)}\sigma(x,R)dx\right)^{\frac{q-p}{q}}+\omega\left(R^{p-m}\int_{Q(R)}|Du|^p dx\right)^{\frac{q-p}{q}}\right\}\\
&\times\int_{Q(2R)}(1+|Du(x)|^2)^{\frac{p}{2}}\,dx.
\end{aligned}$$

Now, from Lemma 3.1 and the above inequality, we get

$$\begin{aligned}
\int_{Q(r)}|Du|^p dx &\le \int_{Q(r)}\big(|Dv|^p+|Dw|^p\big)dx\\
&\le C\left\{\left(\frac{r}{R}\right)^{\lambda}+\left(\fint_{Q(R)}\sigma(x,R)\,dx\right)^{\frac{q-p}{q}}\right. \qquad (3.9)\\
&\quad\left.+\,\omega\left(R^{p-m}\int_{Q(R)}|Du|^2 dx\right)^{\frac{q-p}{q}}\right\}\int_{Q(2R)}\big(1+Du(x)|^p\big)^{\frac{p}{2}}\,dx.
\end{aligned}$$

Let us consider the behavior of u near the boundary. Let $Q(x_l, 2R_1)$ be a member of the covering $\{Q(x_k, 2R_1)\}$ which is introduced at the beginning of this section. Then, u satisfies

$$\begin{cases} \mathcal{A}(u, Q^+(x_l, 2R_1)) \le \mathcal{A}(u+\varphi, Q^+(x_l, 2R_1)) \quad \forall \varphi \in H_0^{1,p}(Q^+(x_l, 2R_1)) \\ \qquad\qquad u = g \quad \text{on} \quad \Gamma(x_l, 2R_1). \end{cases}$$

Fix a point $x_0 \in \Gamma(x_l, R_1)$ and a positive number $R < R_1$ arbitrarily (here, mention that $Q^+(x_0, R) \subset Q^+(x_l, 2R_1)$). Let $v \in H^{1,p}(Q^+(x_0, R))$ be a minimizer of $\mathcal{A}^0(v, Q^+(x_0, R))$ in the class $\{v \in H^{1,p}(Q^+(x_0, R))\ ;\ u - v \in H_0^{1,p}(Q^+(x_0, R))\}$, and put $w = u - v$. Then, using Lemma 3.5, we can proceed as in the interior case and get

$$\begin{aligned} \int_{Q^+(R)} |Dw|^p\,dx \le& \int_{Q^+(R)} \{\sigma(x, R) + \omega(|u - u_R|^2)\}\big(1 + |Du(x)|^2\big)^{\frac{p}{2}}\,dx \\ &+ \int_{Q^+(R)} \{\sigma(x, R) + \omega(|v - u_R|^2)\}\big(1 + |Dv(x)|^2\big)^{\frac{p}{2}}\,dx. \end{aligned}$$

Moreover, using (3.3) instead of (3.4) and proceeding as in the interior case, we have

$$\begin{aligned} &\int_{Q^+(R)} |Dw|^p\,dx \\ &\le C\left\{\left(⨍_{Q^+(R)} \sigma(x, R)\,dx\right)^{\frac{q-p}{q}} + \omega\left(R^{p-m}\int_{Q^+(R)} |Du|^p\,dx\right)^{\frac{q-p}{q}}\right\} \qquad (3.10) \\ &\times \int_{Q^+(2R)} \big(1 + |Du(x)|^2\big)^{\frac{p}{2}}\,dx + CR^{m\frac{q-p}{q}}\left(\int_{Q^+(2R)} \big(1 + |Dg|^2\big)^{\frac{q}{2}}\,dx\right)^{\frac{p}{q}}. \end{aligned}$$

Now, combining (3.2) and (3.10), we obtain

$$\begin{aligned} \int_{Q^+(r)} |Du|^p\,dx \le C\Bigg\{&\left(\frac{r}{R}\right)^{\tau_0} + \left(⨍_{Q^+(R)} \sigma(x, R)\,dx\right)^{\frac{q-p}{q}} \\ &+ \omega\left(R^{p-m}\int_{Q^+(R)} |Du|^p\,dx\right)^{\frac{q-p}{q}}\Bigg\} \\ &\times \int_{Q^+(2R)} \big(1 + |Du(x)|^2\big)^{\frac{p}{2}}\,dx \qquad (3.11) \\ &+ cR^{\tau_0}\left(\int_{Q^+(R)} \big(1 + |Dg|^2\big)^{\frac{s}{2}}\,dx\right)^{\frac{p}{s}} \\ &+ CR^{m\frac{q-p}{q}}\left(\int_{Q^+(2R)} \big(1 + |Dg|^2\big)^{\frac{q}{2}}\,dx\right)^{\frac{p}{q}}. \end{aligned}$$

Since we are assuming that $Dg \in L^s$ for some $s > p$, and we can choose $q > p$ sufficiently near to p, without loss of generality we can assume that $s > q > p$. So, we can estimate the last term of (3.11) as follows:

$$R^{m\frac{q-p}{q}} \left(\int_{Q^+(2R)} \left(1 + |Dg|^2\right)^{\frac{q}{2}} dx \right)^{\frac{p}{q}} \leq CR^{m(1-\frac{p}{s})} \left(\int_{Q^+(2R)} \left(|1 + |Dg|^2\right)^{\frac{s}{2}} dx \right)^{\frac{p}{s}}.$$

Here, we can assume that $R < 1$, so the above estimates hold even if $m(1 - \frac{p}{s})$ can be replaced by the smaller constant τ_0. Mentioning the above fact and combining the above estimate with (3.11), we get the following estimate:

$$\begin{aligned} \int_{Q^+(r)} |Du|^p \, dx \leq C \Bigg\{ \left(\frac{r}{R}\right)^{\tau_0} &+ \left(-\!\!\!\!\!\!\int_{Q^+(R)} \sigma(x, R) \, dx \right)^{\frac{q-p}{q}} \\ &+ \omega \left(R^{p-m} \int_{Q^+(R)} |Du|^p \, dx \right)^{\frac{q-p}{q}} \Bigg\} \\ &\times \int_{Q^+(2R)} \left(1 + |Du(x)|^2\right)^{\frac{p}{2}} dx + C(g) R^{\tau_0}. \end{aligned} \tag{3.12}$$

By the assumption (A-1), we have $-\!\!\!\!\!\!\int_{Q(R)} \sigma(x, R) \, dx \to 0$ as $R \to 0$. So, using "a useful Lemma" on p. 44 of [8] for (3.9) and (3.12), and putting

$$\Phi(x, r) = \int_{\Omega(x,r)} \left(1 + |Du|^2\right)^{\frac{p}{2}} dx,$$

we can see that for any τ with $0 < \tau < \tau_0 (< \lambda_0)$ there exist positive constants δ, M and R_0 ($R_0 < \frac{R_1}{2}$) with the following properties:

`Interior Case.` If $r_1, r_1^{p-m} \Phi(x, r_1) < \delta$ for some $r_1 \in (0, R_0)$ with $Q(x, r_1) \subset\subset \Omega$, then for $0 < \rho < r < r_1$ we have

$$\Phi(x, \rho) \leq M \left(\frac{\rho}{r}\right)^{\tau} \Phi(x, r).$$

`Boundary Case.` For $x \in \partial\Omega$, if r_1, $r_1^{p-m} \Phi(x, r_1) < \delta$ for some $r_1 \in (0, R_0)$, then we have

$$\Phi(x, \rho) \leq M \left(\frac{\rho}{r}\right)^{\tau} \Phi(x, r) + M \rho^{\tau}.$$

Now, we can proceed as in Giusti's book [12, pp. 318–319] to show partial Morrey-type regularity of u. Namely, there exist positive constants δ and M with the following properties: for any $x \in \Omega$, if r_0, $r_0^{p-m} \Phi(x, r_0) \leq \delta$ for some $r_0 > 0$, then $\rho^{-\tau} \Phi(x, \rho) \leq \tilde{M}$. So, we get the assertion. □

Proof of Corollary 2.6. When $p + 2 \geq m$ and $s > \max\{m, p\}$, we can take τ sufficiently near to $\min\{2 + \varepsilon, m(1 - \frac{p}{s})\}$ so that $\tau > m - p$. So, Corollary 2.6 is a direct consequence of Theorem 2.5 and Morrey's theorem on the growth of the Dirichlet integral (see, for example, [8, p.43]). □

Proof of Corollary 2.7. When $A(x,u,\xi)$ does not depend on u, we can proceed as in the proof of Theorem 2.5 without the term with ω and get, instead of (3.9) and (3.11),

$$\int_{Q(x_0,r)} |Du|^p\,dx \le C\left\{\left(\frac{r}{R}\right)^\lambda + \left(\fint_{Q(R)} \sigma(x,R)\,dx\right)^{\frac{q-p}{q}}\right\} \times \int_{Q(2R)} \left(1+|Du(x)|^2\right)^{\frac{p}{2}} dx.$$

for $Q(2R) = Q(x_0,2R) \subset\subset \Omega$ and

$$\int_{Q^+(x_0,r)} |Du|^p\,dx \le C\left\{\left(\frac{r}{R}\right)^\lambda + \left(\fint_{Q^+(R)} \sigma(x,R)\,dx\right)^{\frac{q-p}{q}}\right\} \times \int_{Q^+(2R)} \left(1+|Du(x)|^2\right)^{\frac{p}{2}} dx + C(g)R^\tau,$$

for $x_0 \in \partial\Omega$. So, we can proceed as in the last part of Theorem 2.5 without assuming that

$$r_1^{p-m}\Phi(x,r_1) = r_1^{p-m}\int_{\Omega(x,r_1)} \left(1+|Du|^2\right)^{\frac{p}{2}} dx < \delta.$$

and see that $\rho^{-\tau}\Phi(x,\rho) \le \tilde{M}$ for all $x \in \Omega$. Thus we get the assertions. □

Remark 3.6. Without any restriction on the dimension of the domain, it is not possible to obtain a Hölder regularity result in all the domain Ω as showed by V. Šverak and X. Yan in a counterexample contained in [20].

References

[1] Acquistapace, P., On BMO-regularity for linear elliptic systems. *Ann. Mat. Pura Appl.* 161 (1992)(4), 231 – 269.

[2] Campanato, S., Elliptic systems with nonlinearity q greater or equal to two. Regularity of the solution of the Dirichlet problem. *Ann. Mat. Pura Appl.* 147 (1987)(4), 117 – 150.

[3] Campanato, S., A maximum principle for nonlinear elliptic systems: boundary fundamental estimates. *Adv. in Math.* 66 (1987), 291 – 317.

[4] Daněček, J. and Viszus, E., $L^{2,\lambda}$-regularity for minima of variational integrals. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.* (8) 6-B (2003), 39 – 48.

[5] De Giorgi, E., Un esempio di estremali discontinue per un problema variazionale di tipo ellittico (in Italian). *Boll. Unione Mat. Ital.* (4) 1 (1968), 135 – 137.

[6] Di Gironimo, P., Esposito, L. and Sgambati, L., A remark on $L^{2,\lambda}$-regularity for minimizers of quasilinear functionals. *Manuscripta Math.* 113 (2004), 143 – 151.

[7] Giaquinta, M., *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems.* Ann. Math. Studies 105. Princeton (NJ): Princeton Univ. Press 1983.

[8] Giaquinta, M., *Introduction to Regularity Theory for Nonlinear Elliptic Systems.* Lectures in Math. ETH Zürich. Basel: Birkhäuser 1993.

[9] Giaquinta, M. and Giusti, E., On the regularity of the minima of variational integrals. *Acta Math.* 148 (1982), 31 – 46.

[10] Giaquinta, M. and Martinazzi, L., *An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs.* Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) 2. Pisa: Edizioni della Normale 2005.

[11] Giaquinta, M. and Modica, G., Remarks on the regularity of the minimizers of certain degenerate functionals. *Manuscripta Math.* 57 (1986), 55 – 99.

[12] Giusti, E., *Direct Method in the Calculus of Variations.* River Edge (NJ): World Scientific 2003.

[13] Huang, Q., Estimates on the generalized Morrey Spaces $L_{\phi}^{p,\lambda}$ and BMO for linear elliptic systems. *Indiana Univ. Math. Journal* 45 (1996), 397 – 439.

[14] Jost, J. and Meier, M. Boundary regularity for minima of certain quadratic functionals. *Math. Ann.* 262 (1983), 549 – 561.

[15] John, F. and Nirenberg, L., On functions of bounded mean oscillation. *Comm. Pure Appl. Math.* 14 (1961), 415 – 476.

[16] Kufner, A., John, O. and Fučik, S., *Functions Spaces.* Prague: Academia 1977.

[17] Ragusa, M. A. and Tachikawa, A., Partial regularity of the minimizers of quadratic functionals with VMO coefficients. *J. London Math. Soc.* 72 (2005), 609 – 620.

[18] Ragusa, M. A. and Tachikawa, A., On continuity of minimizers for certain quadratic functionals. *J. Math. Soc. Japan* 57 (2005)(3), 691 – 700.

[19] Sarason, D., On functions of vanishing mean oscillation. *Trans. Amer. Math. Soc.* 207 (1975), 391 – 405.

[20] Šverak, V. and Yan, X., A singular minimizer of a smooth strongly convex functional in the three dimension. *Calc. Var. Partial Diff. Eqs.* 10 (2000), 213 – 221.

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