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Varieties with minimal secant degree and linear systems of maximal dimension on surfaces

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Abstract

In this paper we prove, using a refinement of Terracini's Lemma, a sharp lower bound for the degree of (higher) secant varieties to a given projective variety, which extends the well known lower bound for the degree of a variety in terms of its dimension and codimension in projective space. Moreover we study varieties for which the bound is attained proving some general properties related to tangential projections, e.g. these varieties are rational. In particular we completely classify surfaces (and curves) for which the bound is attained. It turns out that these surfaces enjoy some maximality properties for their embedding dimension in terms of their degree or sectional genus. This is related to classical beautiful results of Castelnuovo and Enriques that we revise here in terms of adjunction theory.

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0. Introduction

In this paper, in which we work over the field of complex numbers, we touch, as the title suggests, two different themes, i.e. secant varieties and linear systems, and we try to indicate some new, rich, and to us unexpected, set of relations between them.

Let $X \subseteq \mathbb{P}^r$ be a reduced, irreducible, projective variety. Basic geometric objects related to X are its secant varieties $S^k(X)$, i.e. the varieties described by all projective subspaces $\mathbb{P}^k$ of $\mathbb{P}^r$ which are $(k+1)$-secant to X (see Section 1.3 for a formal definition: in Section 1 we collected all the notation and a bunch of useful preliminaries which we use in the paper). The presence of secant varieties in the study of projective varieties is ubiquitous, since a great deal of projective geometric properties of a variety is encoded in the behaviour of its secant varieties. However, the importance of secant varieties is not restricted to algebraic geometry only. Indeed, different important problems which arise in various fields of mathematics can be usefully translated in terms of secant varieties. Among these it is perhaps the case to mention polynomial interpolation problems, rank tensor computations and canonical forms, expressions of polynomials as sums of powers and Waring-type problems, algebraic statistics, etc. (see, for instance, [13,17,29,35]).

Going back to projective algebraic geometry, let us mention the first basic example of a property of a variety which is reflected in properties of a secant variety: it is well known, indeed, that a smooth variety $X \subseteq \mathbb{P}^r$ can be projected isomorphically to $\mathbb{P}^{r-m}$, with $m > 0$, if and only if its first secant variety $S(X) := S^1(X)$ has codimension at least m in $\mathbb{P}^r$. Furthermore, one can ask *how singular* a general projection of X to $\mathbb{P}^{r-m-1}$ from a general $\mathbb{P}^m$ is, if m is exactly the codimension of $S(X)$ in $\mathbb{P}^r$. One moment of reflection shows that a basic step in answering this question is to know in how many points $S(X)$ intersects a general $\mathbb{P}^m$ in $\mathbb{P}^r$, i.e. one has to know what is *degree* of $S(X)$. A related, more difficult problem, is to understand what is the structure of the cone of secant lines to X passing through a general point in $S(X)$, a classical question considered by various authors even in very recent times (see, for instance, [42]). Of course similar problems arise in relation with higher secant varieties $S^k(X)$ as well and lead to the important questions of understanding what is the *dimension* and the *degree* of $S^k(X)$ for any $k \geqslant 1$.

As well known, if X has dimension n, there is a basic upper bound for the dimension of $S^k(X)$ which is provided by a naive count of parameters (see (1.2) below). As often happens in many similar situations in algebraic geometry, one expects that *most* varieties achieve this upper bound, and that it should be possible to classify all the others, the so-called *k-defective varieties*, namely the ones for which the dimension of $S^k(X)$ is smaller than the expected. Unfortunately this viewpoint, which is in principle correct, is in practice quite hard to be successfully pursued. Indeed, while there are no defective curves and the classification of defective surfaces, though not at all trivial, is however classical (see [14,54,57] for a modern reference), the classification of defective threefolds is quite intricate and has only recently been completed (see [16]) after the classical work of Scorza [53] on 1-defective threefolds (see also [15]). As for higher-dimensional defective varieties, no complete classification result is available, though a number of beautiful theorems concerning some special classes of defective varieties is available (see [58]).

One of the objectives of the present paper is to address the other question we indicated above, i.e. the one concerning the determination of the degree of secant varieties. This question, though important, has never been systematically investigated in general, neither in the past, nor in more recent times, exceptions being, for instance, the paper [12] for the case of curves (see also [59]), and the computation of the degree of secant varieties to varieties of some particular classes, like one does in [50] (see also Section 5 below).

Of course, given any variety $X \subseteq \mathbb{P}^r$, one has a famous, classical lower bound for the degree of X (see (4.1) below), which says that the degree in question is bounded below by the codimension of X plus one. This bound is sharp, and the varieties achieving it, the so-called *varieties of minimal degree*, are completely classified, in particular they turn out to be rational (see [22]). The aforementioned bound of course applies to the secant varieties of X too, but, according to the classification of varieties of minimal degree, one immediately sees that it is never sharp in this case. Thus the question arises to give a *sharp* lower bound for the degree of $S^k(X)$. This is the problem that we solve in Section 4, where our main result, i.e. Theorem 4.2, is the bound (4.2) for the degree of $S^k(X)$. Moreover, we prove a similar bound (4.3) for the multiplicity of $S^k(X)$ at a general point of X. One of the main steps in the proof of Theorem 4.2 is the result in Section 3, namely Theorem 3.1, in which we give relevant informations about the tangent cone to $S^k(X)$ at the general point of $S^l(X)$, where $l < k$. This can be seen as a wide generalization of the famous Terracini's Lemma (see Theorem 1.1 below), which describes the general tangent space to $S^k(X)$.

The lower bound (4.2) for the degree of $S^k(X)$ is a generalization of the classical lower bound (4.1) for the degree of any variety, and, as well as the latter, it is sharp. Actually, in Theorem 4.2 we also show that varieties X such that $S^k(X)$ has the minimum possible degree, called varieties with *minimal k-secant degree* or $\mathcal{M}^k$*-varieties* (see Definition 4.4), enjoy important properties like: general m-internal projections X^m of X, i.e. projections of X from m general points on it, are also of *minimal k-secant degree*, general m-tangential projections X_m of X, i.e. projections of X from $m \leqslant k$ general tangent spaces, are of *minimal* $(k-m)$*-secant degree*, in particular, for $k = m$, projections X_k of X from k general tangent spaces are of *minimal degree*, hence they are rational. Since we know very well varieties of minimal degree, and a general *k-tangential projection* X_k of X is one of them, a natural question, at this point, arises: what is the structure of the projection $X - - \rightarrow X_k$? The interesting answer is that, if X is not k-defective then the map in question is generically finite and its degree is bounded above by $\mu_k(X)$ which, by definition, is the number of $(k+1)$-secant $\mathbb{P}^k$ to X passing through the general point of $S^k(X)$. In particular, if X is not k-defective, if $S^k(X)$ has minimal degree and $\mu_k(X) = 1$, then X, as well as X_k, is rational. The main ingredient for the proof of the bound on the degree of the k-tangential projection $X - - \rightarrow X_k$ is proved in Section 2 (see Theorem 2.7), where we exploit and generalize the technique, introduced in [18], of degeneration of projections, based on a beautiful idea of Franchetta (see [26,27]).

Notice that the condition $\mu_k(X) = 1$ is rather mild, i.e. one expects that *most* non k-defective varieties $X \subset \mathbb{P}^r$ enjoy this property if $S^k(X) \subsetneq \mathbb{P}^r$ (see Section 1.5, in particular Proposition 1.5 for a sufficient condition for this to happen). The varieties X, not k-defective, such that $S^k(X)$ has minimal degree and $\mu_k(X) = 1$ are called $\mathcal{MA}^{k+1}_{k-1}$-variety or $\mathcal{OA}^{k+1}_{k-1}$-variety according to whether $S^k(X)$ is strictly contained in $\mathbb{P}^r$ or not (see Definition 4.4), e.g. X is an $\mathcal{OA}^{k+1}_{k-1}$-variety if and only if $S^k(X) = \mathbb{P}^r$, $r = (k+1)n + k$ and there is only one $(k+1)$-secant $\mathbb{P}^k$ to X passing through the general point of $\mathbb{P}^r$, i.e. the general projection X' of X to $\mathbb{P}^{r-1}$ acquires a new $(k+1)$-secant $\mathbb{P}^{k-1}$ that X did not use to have. This was classically called an *apparent* $(k+1)$-*secant* $\mathbb{P}^{k-1}$ of X. It should be mentioned, at this point, the pioneering work of Bronowski on this subject: in his inspiring, but unfortunately very obscure, paper [6] he essentially states that the map $X - - \to X_k$ is birational if and only if X is either an $\mathcal{MA}^{k+1}_{k-1}$-variety or $\mathcal{OA}^{k+1}_{k-1}$-variety. As we said, one implication has been proved by us, the other is open in general, and we call it the kth Bronowski's conjecture (see Remark 4.6). The results of the present paper imply that Bronowski's conjecture holds for smooth surfaces (see Corollary 9.3), whereas the main theorem of [18] implies that the Bronowski's conjecture holds for smooth threefolds in $\mathbb{P}^7$ if $k = 1$. It would be extremely nice to shed some light on the validity of this conjecture in general, since, according to Bronowski, this would make the study and the classification of $\mathcal{MA}^{k+1}_{k-1}$ and $\mathcal{OA}^{k+1}_{k-1}$-varieties easier.

The existence of $\mathcal{M}^k$, $\mathcal{MO}^{k+1}_{k-1}$, and $\mathcal{MA}^{k+1}_{k-1}$-varieties, and therefore the sharpness of the bound proved in Theorem 4.2, is showed in Section 5, where several important classes of examples are exhibited. Among these one has: rational normal scrolls, some Veronese fibrations, some Veronese embeddings of the plane, defective surfaces, del Pezzo surfaces, etc.

With all the above apparatus at hand, the natural question is to look for classification theorems for $\mathcal{M}^k$, $\mathcal{MA}^{k+1}_{k-1}$, and $\mathcal{OA}^{k+1}_{k-1}$-varieties. This turns out to be a very intriguing but considerably difficult question to answer. Indeed the problem is non-trivial even in the case of curves, considered in Section 6: the classification theorem here, which follows by results of Catalano–Johnson, is that a curve is an $\mathcal{MA}^{k+1}_{k-1}$ or an $\mathcal{OA}^{k+1}_{k-1}$-variety if and only if it is a rational normal curve (see Theorem 6.1). Our proof is a slight variation of Catalano–Johnson's argument. The classification of $\mathcal{OA}^2_0$-varieties, also called *OADP*-varieties, which means *varieties with one apparent double point*, is a classical problem. The case of *OADP*-surfaces goes back to Severi [54], whereas examples and general considerations concerning the higher dimensional case can be found in papers by Edge [21] and Bronowski [6]. This latter author came to the consideration of this problem studying extended forms of the Waring problem for polynomials. Severi's incomplete argument has been recently fixed by the second author [51], and a different proof can be found in [18], where one provides the full classification of *OADP*-threefolds in $\mathbb{P}^7$. Finally, an attempt of classification of $\mathcal{OA}^{k+1}_{k-1}$-surfaces is again due to Bronowski [7], whose approach, based on his aforementioned unproved conjecture, was certainly not rigorous and led him, by the way, to an incomplete list.

The problem we started from, and which actually was the original motivation for this paper, was to verify and justify Bronowski's classification theorem of $\mathcal{OA}^{k+1}_{k-1}$-surfaces, without, unfortunately, having the possibility of fully relying on his still unproven conjecture. It was in considering this question that we understood we had to slightly change our viewpoint and first look at a different kind of problem. This leads us to the second theme of the present paper, i.e. linear system on surfaces, which occupies Section 7. We discovered in fact that the classification of $\mathcal{MA}^{k+1}_{k-1}$ and $\mathcal{OA}^{k+1}_{k-1}$-surfaces is closely related to a beautiful classical theorem of Castelnuovo [8] and Enriques [24] (see Theorem 7.3) which gives an upper bound for the dimension of a linear system $\mathcal{L}$ of curves of given geometric genus on a surface X, and classifies those pairs $(X, \mathcal{L})$ for which the bound is attained. Of course, Castelnuovo–Enriques' theorem has to do with the intrinsic birational geometry of surfaces. However, if one looks at the hyperplane sections linear systems, it becomes a theorem in projective geometry and our remark was that Castelnuovo–Enriques' list of extremal cases consisted of some k-defective surfaces and of $\mathcal{MA}^{k+1}_{k-1}$ and $\mathcal{OA}^{k+1}_{k-1}$-surfaces for some k. It became then apparent to us that there should have been a relationship between minimality properties of secant varieties encoded in the $\mathcal{M}^k$, $\mathcal{MA}^{k+1}_{k-1}$, and $\mathcal{OA}^{k+1}_{k-1}$-properties and the Castelnuovo–Enriques' maximality conditions on the dimension of the hyperplane sections linear system. The relation between the two items was underlined, in our view, by the fact that Castelnuovo and Enriques' beautiful original approach was based on iterated applications of tangential projections, a technique that, as we indicated above, enters all the time in the study of secant varieties. In fact, we do not reproduce here Castelnuovo–Enriques' original argument, which, based on the technical Proposition 1.6, is however hidden, as we will explain in a moment, in the proof of our classification theorems of $\mathcal{M}^k$, $\mathcal{MA}^{k+1}_{k-1}$, and $\mathcal{OA}^{k+1}_{k-1}$-surfaces given in Sections 8 and 9. We preferred instead to give an intrinsic, birational geometric, proof of Castelnuovo–Enriques' theorem, which enables us to prove a slightly more general statement than the original one and is also useful for extensions, like our Theorem 7.9, in which we classify those smooth surfaces in projective space such that their hyperplane linear system has dimension *close* to Casteluovo–Enriques' upper bound. The Castelnuovo-Enriques' upper bound (7.3) for smooth irreducible curves is essentially the main result of Hartshorne [33, Corollary 2.4, Theorems 3.5 and 4.1], where the classification of the extremal cases is not considered. Our simple and short proof, which we hope has some independent interest, relies on an application of Mori's Cone Theorem, namely Proposition 7.1, which has an independent interest and says that given a pair (X, D), where X is a smooth, irreducible, projective surface, and D is a nef divisor on it, one has that $K + D$ is also nef, unless one of the following facts occurs: either (X, D) is not *minimal*, i.e. there is an exceptional curve of the first kind E on X such that $D \cdot E = 0$, or (X, D) is a *h-scroll*, with $h \leqslant 1$, i.e. there is a rational curve F on X such that $F^2 = 0$ and $D \cdot F = h$, or (X, D) is a *d-Veronese*, with $d \leqslant 2$, i.e. $X = \mathbb{P}^2$ and D is a curve of degree $d \leqslant 2$. A slightly more general version of this last result, in the case D irreducible (smooth) curve, was obtained by Iitaka, see [36], and revised from the above point of view of the Cone Theorem by Dicks, see [20] Theorem 3.1. For weaker results of the same type, concerning the case D ample, see for example [38]. It should be stressed that, as indicated in Castelnuovo's paper [9], one can push these ideas further, thus giving suitable upper bounds for the dimension of certain linear systems on scrolls, or equivalently on the degree of curves on scrolls as in [33, Theorem 2.4 and Corollary

2.5]. This has been done already, in an independent way also in [49], but we hope to return on these matters in the future since we believe that some of the results in [9], see also [33] Sections 2 and 3, and in [49] can be slightly improved and perhaps related to projective geometry in the spirit of the present paper.

As we said, in Sections 8 and 9 we come back to the classification of $\mathcal{MA}_{k-1}^{k+1}$ and $\mathcal{OA}_{k-1}^{k+1}$-surfaces. Using the machinery of tangential projections and degeneration of projections we discover that the surfaces in question are either extremal with respect to Castelnuovo–Enriques' bound or they are close to be extremal, so that their classification can be at this point accomplished using the results of Section 7. Finally in Section 10 we prove, using the same ideas, a result, namely Theorem 10.1, which is a wide generalization of the famous theorem of Severi's saying that the Veronese surface in $\mathbb{P}^5$ is the only defective surface which is not a cone.

In conclusion we would like to mention that, though the above classification results for $\mathcal{M}^k$, $\mathcal{MA}_{k-1}^{k+1}$, and $\mathcal{OA}_{k-1}^{k+1}$-varieties are quite satisfactory and conclusive in low dimensions, i.e. for curves and surfaces, quite a lot of room is left open for the higher-dimensional case, where, except for the aforementioned result of [18], nothing, to the best of our knowledge, is known. We hope the ideas presented in this paper will be useful in this more general context too. Another interesting direction of research is to try to extend to higher-dimensional varieties Castelnuovo–Enriques' results in Section 7. This question is also widely open. The adjunction theoretical approach that we use in the surface case can in principle be extended, but it is not clear whether it leads to anything really useful. On the other hand Castelnuovo–Enriques tangential projection approach, in order to work, has to be modified, since one needs to make projections from osculating, rather than tangent, spaces. An interesting suggestion in this direction comes from the beautiful comments of Castelnuovo's to [8] in the volume of collected papers [10, pp. 186–188]. However, osculating projections present serious technical problems which make Castelnuovo's suggestion rather hard to be pursued. On the other hand, the specific problem which Castelnuovo was considering in his comments in [10, pp. 186–188], i.e. the classification of linear systems of rational surfaces in $\mathbb{P}^3$, has been recently successfully addressed by various authors, in particular by Mella [43], by using Mori's program. The interplay between intrinsic birational geometry, i.e. Mori's program, and extrinsic projective geometry, i.e. osculating projections and relations with secant varieties, is a very promising, uncharted territory to be explored.

1. Notation and preliminary results

1.1. Let $X \subseteq \mathbb{P}^r$ be a projective scheme over $\mathbb{C}$. We will denote by $\deg(X)$ the *degree* of X, by $\dim(X)$ the *dimension* of X, by $\operatorname{codim}(X) = r - \dim(X)$ its *codimension* and by $(X)_{\mathrm{red}}$ the reduced subscheme supported by X. We will mainly consider the case in which X is a reduced, irreducible variety.

If $Y \subset \mathbb{P}^r$ is a subset, we denote by $\langle Y \rangle$ the span of Y. We will say that Y is *non-degenerate* if $\langle Y \rangle = \mathbb{P}^r$.

1.2. Let $X \subseteq \mathbb{P}^r$ be a reduced, irreducible variety of dimension n. If $x \in X$ we will denote by $C_{X,x}$ the *tangent cone* to x at X, which is an n-dimensional cone with vertex at x. Note that $C_{X,x}$ has a natural structure of a subscheme of $\mathbb{P}^r$. We will denote by $\mathrm{mult}_x(X)$ the *multiplicity* of X at x. One has $\mathrm{mult}_x(X) = \deg(C_{X,x})$ and X is a *cone* if and only if X has some point x such that $\mathrm{mult}_x(X) = \deg(X)$. In this case x is a *vertex* of X and we will denote by $\mathrm{Vert}(X)$ the set of vertices of X, which is a linear subspace contained in X. It is well known that

$$\mathrm{Vert}(X) = \bigcap_{x \in X} T_{X,x}. \tag{1.1}$$

If x is a smooth point of X, then $C_{X,x}$ is an n-dimensional linear subspace of $\mathbb{P}^r$, i.e. the *tangent space* to X at x, which we will denote by $T_{X,x}$.

1.3. Let k be a non-negative integer and let $S^k(X)$ be the *k-secant variety* of X, i.e. the Zariski closure in $\mathbb{P}^r$ of the set:

$$\{x \in \mathbb{P}^r : x \text{ lies in the span of } k+1 \text{ independent points of } X\}.$$

Of course $S^0(X) = X$, $S^r(X) = \mathbb{P}^r$ and $S^k(X)$ is empty if $k \geqslant r+1$. We will write $S(X)$ instead of $S^1(X)$ and we will assume $k \leqslant r$ from now on.

Let $\mathrm{Sym}^h(X)$ be the hth symmetric product of X. One can consider the *abstract kth secant variety* S^k_X of X, i.e. $S^k_X \subseteq \mathrm{Sym}^k(X) \times \mathbb{P}^r$ is the Zariski closure of the set of all pairs $([p_0, \ldots, p_k], x)$ such that $p_0, \ldots, p_k \in X$ are linearly independent points and $x \in \langle p_0, \ldots, p_k \rangle$. One has the surjective map $p^k_X : S^k_X \to S^k(X) \subseteq \mathbb{P}^r$, i.e. the projection to the second factor. Hence

$$s^{(k)}(X) := \dim(S^k(X)) \leqslant \min\{r, \dim(S^k_X)\} = \min\{r, n(k+1)+k\}. \tag{1.2}$$

We will denote by $h^{(k)}(X)$ the codimension of $S^k(X)$ in $\mathbb{P}^r$, i.e. $h^{(k)}(X) := r - s^{(k)}(X)$.

The right-hand side of (1.2) is called the *expected dimension* of $S^k(X)$ and will be denoted by $\sigma^{(k)}(X)$. One says that X has a *k-defect*, or is *k-defective*, or is *defective of index k* when strict inequality holds in (1.2). One says that

$$\delta_k(X) := \sigma^{(k)}(X) - s^{(k)}(X)$$

is the *k-defect* of X.

Notice that the general fibre of p^k_X is pure of dimension $(k+1)n + k - s^{(k)}(X)$, which equals $\delta_k(X)$ when $r \geqslant n(k+1)+k$. We will denote by $\mu_k(X)$ the number of irreducible components of this fibre. In particular, if $s^{(k)}(X) = (k+1)n+k$, then p^k_X is generically finite and $\mu_k(X)$ is the degree of p^k_X, i.e. it is the number of $(k+1)$-secant $\mathbb{P}^k$'s to X passing through the general point of $S^k(X)$.

If $s^{(k)}(X) = (k+1)n + k$, we will denote by $\nu_k(X)$ the number of $(k+1)$-secant $\mathbb{P}^k$'s to X meeting the general $\mathbb{P}^{h^{(k)}(X)}$ in $\mathbb{P}^r$. Of course one has

$$\nu_k(X) = \mu_k(X) \cdot \deg(S^k(X)) \tag{1.3}$$

and therefore

$$\nu_k(X) = \mu_k(X) \quad \text{if} \quad r = s^{(k)}(X) = (k+1)n + k. \tag{1.4}$$

1.4. Let $X \subset \mathbb{P}^r$ be an irreducible, projective variety. Let k be a positive integer and let $p_1, \ldots, p_k$ be general points of X. We denote by $T_{X,p_1,\ldots,p_k}$ the span of T_{X,p_i}, $i = 1, \ldots, k$.

If $X \subset \mathbb{P}^r$ is a projective variety, Terracini's lemma describes the tangent space to $S^k(X)$ at a general point of it (see [56] or, for modern versions, [1,14,19,58])

Theorem 1.1 (*Terracini's lemma*). *Let $X \subset \mathbb{P}^r$ be an irreducible, projective variety. If $p_0, \ldots, p_k \in X$ are general points and $x \in \langle p_0, \ldots, p_k \rangle$ is a general point, then*

$$T_{S^k(X),x} = T_{X,p_0,\ldots,p_k}.$$

If X is k-defective, then the general hyperplane H containing $T_{X,p_0,\ldots,p_k}$ is tangent to X along a variety $\Sigma_{p_0,\ldots,p_k}$ of pure, positive dimension $n_k(X)$ containing $p_0, \ldots, p_k$. Moreover one has

$$k \leqslant \dim(\langle \Sigma_{p_0,\ldots,p_k} \rangle) \leqslant k n_k(X) + k + n_k(X) - \delta_k(X).$$

Consider the projection of X with centre $T_{X,p_1,\ldots,p_k}$. We call this a *general k-tangential projection* of X, and we will denote it by $\tau_{X,p_1,\ldots,p_k}$ or simply by $\tau_{X,k}$. We will denote by X_k its image. By Terracini's lemma, the map $\tau_{X,k}$ is generically finite to its image if and only if $s^{(k)}(X) = (k+1)n + k$. In this case we will denote by $d_{X,k}$ its degree.

In the same situation, the projection of X with centre the space $\langle p_1, \ldots, p_k \rangle$ is called a *general k-internal projection* of X, and we will denote it by $t_{X,p_1,\ldots,p_k}$ or simply by $t_{X,k}$. We denote by X^k its image. We set $X_0 = X^0 = X$. Notice that the maps $t_{X,k}$ are birational to their images as soon as $k < r - n = \operatorname{codim}(X)$.

Sometimes we will use the symbols X_k [resp., X^k] for k-tangential projections [resp., k-internal projections] relative to *specific*, rather than *general*, points. In this case we will explicitly specify this, thus we hope no confusion will arise for this reason.

1.5. We recall from [14] the definition of a *k-weakly defective* variety, i.e. a variety $X \subset \mathbb{P}^r$ such that if $p_0, \ldots, p_k \in X$ are general points, then the general hyperplane H containing $T_{X,p_0,\ldots,p_k}$ is tangent to X along a variety $\Sigma_{p_0,\ldots,p_k}$ of pure, positive dimension $n_k(X)$ containing $p_0, \dots, p_k$. By Terracini's lemma, a k-defective variety is also k-weakly defective, but the converse does not hold in general (see [14]).

Remark 1.2. A curve is never k-weakly defective for any k. A variety is 0-weakly defective if and only if its dual variety is not a hypersurface. In the surface case this happens if and only if the surface is *developable*, i.e. if and only if the surface is either a cone or the tangent developable to a curve.

The two next results are consequences of Theorem 1.4 of [14] that we partially recall here.

Theorem 1.3. *Let $X \subset \mathbb{P}^r$ be an irreducible, projective, non-degenerate variety of dimension n. Assume X is not k-weakly defective for a given k such that $r \geqslant (n+1)(k+1)$. Then, given $p_0, \dots, p_k$ general points on X, the general hyperplane H containing $T_{X,p_0,\dots,p_k}$ is tangent to X only at $p_0, \dots, p_k$. Moreover such a hyperplane H cuts on X a divisor with ordinary double points at $p_0, \dots, p_k$.*

The first consequence we are interested in is the following:

Lemma 1.4. *Let $X \subset \mathbb{P}^r$ be an irreducible, projective, non-degenerate variety of dimension n, which is not k-weakly defective for a fixed $k \geqslant 1$ such that $r \geqslant (k+1)(n+1)$. Then a general k-tangential projection of X is birational to its image, i.e. $d_{X,k} = 1$. In particular, if $r \geqslant 2n+2$, the general tangential projection of X is birational to its image.*

Proof. Since X is not k-weakly defective, it is not l-defective for all $l \leqslant k$. Thus we have $s^{(l)}(X) = (l+1)n + l$ for all $l \leqslant k$, so that by Terracini's lemma $\tau_{X,p_1,\dots,p_l}$ is generically finite onto X_l for every $l \leqslant k$ and $p_1, \dots, p_l$ general points on X. In particular this is true for $l = k$.

Suppose now that $d_{X,k} > 1$. Then, given a general point $p_0 \in X$ there is a point $q \in X \setminus (T_{X,p_1,\dots,p_k} \cap X)$, $q \neq p_0$, such that $\tau_{X,p_1,\dots,p_k}(p_0) = \tau_{X,p_1,\dots,p_k}(q) := x \in X_k$. This would imply that $T_{X,p_0,p_1,\dots,p_k}$ and $T_{X,q,p_1,\dots,p_k}$ coincide, since both these spaces project via $\tau_{X,p_1,\dots,p_k}$ onto $T_{X_k,x}$. In particular, the general hyperplane tangent to X at $p_0, p_1, \dots, p_k$ is also tangent at q. This contradicts Theorem 1.3. $\square$

We also note that Terracini's lemma and Theorem 1.3 imply that

Proposition 1.5. *Let $X \subset \mathbb{P}^r$ be an irreducible, projective variety which is not k-weakly defective. If $r \geqslant (n+1)(k+1)$, then $\mu_k(X) = 1$.*

In the sequel we will also need the following technical:

Proposition 1.6. *Let $X \subset \mathbb{P}^r$ be a smooth, irreducible, projective, non-degenerate surface, which is not $(k-1)$-weakly defective for a fixed $k \geqslant 1$ such that $r \geqslant 3k+2$. Let $p_1, \dots, p_k \in X$ be general points and assume that the linear system $\mathcal{L}$ of hyperplane sections of X tangent at $p_1, \dots, p_k$ has a not empty fixed part $F = \sum_{i=1}^{h} n_i \Gamma_i$, with*

Γ_i distinct, irreducible curves and $n_i > 0$, for all $i = 1, \dots, h$. Let $\mathcal{M}$ be the movable part of $\mathcal{L}$ and let M be its general curve. Then F is reduced, i.e. $n_i = 1$ for all $i = 1, \dots, h$ and

(i) *either $h = 1$, F is a smooth, rational curve containing $p_1, \dots, p_k$, whereas $\mathcal{M}$ has simple base points at $p_1, \dots, p_k$ and $M \cdot F = k$, hence $M \in \mathcal{M}$ general meets F transversally at $p_1, \dots, p_k$ and nowhere else;*

(ii) *or $h = k$, Γ_i is a smooth, rational curve containing p_i for $i = 1, \dots, k$, $\Gamma_i \cap \Gamma_j = \emptyset$ if $1 \leqslant i < j \leqslant k$, $\mathcal{M}$ has simple base points at $p_1, \dots, p_k$ and $M \cdot \Gamma_i = 1$, hence $M \in \mathcal{M}$ general meets Γ_i transversally at p_i and nowhere else, for all $i = 1, \dots, k$.*

Moreover, if $r \geqslant 3k + 3$ and if the general k-tangential projection X_k of X, has rational hyperplane sections, then the general curve $M \in \mathcal{M}$ is rational.

Proof. Let C be a general curve in $\mathcal{L}$, so that $C = F + M$. By Theorem 1.3, we know that C has nodes at $p_1, \dots, p_k$ and is otherwise smooth. This implies that

- F is reduced;
- all the curves Γ_i, $i = 1, \dots, h$, are smooth off $p_1, \dots, p_k$, where they can have at most nodes;
- Γ_i and Γ_j, for $1 \leqslant i < j \leqslant h$, may intersect only at some of the points $p_1, \dots, p_k$, where only two of them may meet transversally;
- M is smooth off $p_1, \dots, p_k$ where it can have at most nodes, and may intersect the curves Γ_i only at $p_1, \dots, p_k$, where it may meet only one of them transversally;
- if the point p_i, $i = 1, \dots, k$, is a node for a curve Γ_j, $i = 1, \dots, h$, then it does not belong neither to M, nor to Γ_j, $j \neq i$;
- if the point p_i, $i = 1, \dots, k$, is a node for M, then it does not belong to F;
- if the point p_i, $i = 1, \dots, k$, is a smooth point for a curve Γ_j, $i = 1, \dots, h$, then it belongs either to M, or to a curve Γ_j, $j \neq i$, but not to both.

We prove the assertion in various steps.

Claim 1.7. *Every irreducible component Γ_i of F contains some of the points $p_1, \dots, p_k$.*

Otherwise we would have $\Gamma_i \cap \overline{C - \Gamma_i} = \emptyset$, and C would be disconnected, a contradiction since it is very ample on X.

Claim 1.8. *F contains all the points $p_1, \dots, p_k$.*

In fact, if $p_1 \notin F$, then, by changing the role of the points $p_1, \dots, p_k$, none of the points $p_1, \dots, p_k$ is in F, contradicting Claim 1.7.

Claim 1.9. *F is smooth.*

We know F can be singular only at some of the points $p_1, \dots, p_k$. Suppose this is the case. Then by symmetry, it is singular at any one of the points in question. But then we would have $M \cap F = \emptyset$, which leads to a contradiction as above.

Claim 1.10. *Let Γ_1 be the irreducible component of F through p_1. Then either also $p_2, \dots, p_k \in \Gamma_1$, or none of the points $p_2, \dots, p_k$ lies on Γ_1. In the former case $\Gamma_1 = F$. In the latter each of the points $p_i, i = 1, \dots, k$, belongs to one and only one component Γ_i of F.*

Suppose Γ_1 contains $p_1, \dots, p_i$, with $1 < i < k$. By changing the role of the points $p_1, \dots, p_k$, any i among the points $p_1, \dots, p_k$ lie on some irreducible component of F. Then F would be singular, contradicting Claim 1.9. This proves the first part of the Claim.

Assume $p_1, \dots, p_k \in \Gamma_1$. Then Claims 1.7 and 1.9 imply that $F = \Gamma$. Suppose instead only p_1 lies on Γ. Then by changing the role of the points $p_1, \dots, p_k$, each of the other points $p_i, i = 2, \dots, k$, also lies on one and only one component of F.

Claim 1.11. *Every irreducible component Γ_i of F is rational.*

By projecting X from $T_{X, p_1, \dots, p_{k-1}}$, we get an irreducible surface $X_{k-1} \subset \mathbb{P}^{r-3k+3}$, with $r - 3k + 3 \geqslant 5$, which is birational to X by Lemma 1.4 and which is not 0-weakly defective. Let q be the image on X_{k-1} of a general point p_k of X. Notice that the general tangent hyperplane section to X_{k-1} at q, which is the image of C, is reducible containing M', the image of M, and Γ', the image of Γ_k, both passing through q. Notice that M' is the movable part of the linear system of hyperplane sections of X_{k-1} tangent at q, whereas Γ' is the fixed part. Then X_{k-1} is either the Veronese surface in $\mathbb{P}^5$ or a non-developable scroll over a curve (see for instance [46]). Hence Γ' is rational. Since $\tau_{X, p_1, \dots, p_{k-1}}$ is birational by Lemma 1.4, then Γ_k is birational to Γ', and is therefore rational. If $\Gamma_k = F$ there is nothing else to prove. Otherwise, by changing the role of the points p_i, we see that Γ_i is rational for any $i = 1, \dots, k$.

The above claims imply (i) and (ii). As for the last assertion, it follows from Lemma 1.4. □

1.6. If $X, Y \subset \mathbb{P}^r$ are closed subvarieties we denote by $J(X, Y)$ the *join* of X and Y, i.e. the Zariski closure of the union of all lines $\langle x, y \rangle$, with $x \in X, y \in Y, x \neq y$. If X is a linear subspace, then $J(X, Y)$ is the cone over Y with vertex X. With this notation, for every $k \geqslant 1$ one has

$$S^k(X) = J(S^l(X), S^h(X)) \tag{1.5}$$

if $l + h = k - 1$, $l \geqslant 0$, $h \geqslant 0$.

We record the following:

Lemma 1.12. *Let $X, Y \subset \mathbb{P}^r$ be closed, irreducible, subvarieties and let Π be a linear subspace of dimension n which does not contain either X or Y. Let $\pi : \mathbb{P}^r \dashrightarrow \mathbb{P}^{r-n-1}$ be the projection from Π and let X', Y' be the images of X, Y via π. Then*:

$$\pi(J(X, Y)) = J(X', Y').$$

In particular, if Π does not contain X, then for any non-negative integer k one has

$$\pi(S^k(X)) = S^k(X').$$

Proof. It is clear that $\pi(J(X,Y)) \subseteq J(X',Y')$. Let $x' \in X'$, $y' \in Y'$ be general points. Then there are $x \in X$, $y \in Y$ such that $\pi(x) = x'$, $\pi(y) = y'$. Thus $\pi(\langle x, y\rangle) = \langle x', y'\rangle$, proving that $J(X',Y') \subseteq \pi(J(X,Y))$, i.e. the first assertion. The rest of the statement follows by (1.5) with $l = 0$, by making induction on k. □

The following lemma is an application of Terracini's lemma:

Lemma 1.13. *Let $X \subset \mathbb{P}^r$ be an irreducible, projective variety. For all $i = 1, \ldots, k$ one has*

$$h^{(k-i)}(X_i) = h^{(k)}(X),$$

whereas for all $i \geqslant 1$ one has

$$h^{(k)}(X^i) = \max\{0, h^{(k)}(X) - i\}.$$

Proof. Let $p_0, \ldots, p_k \in X$ be general points. Terracini's lemma says that $T_{X,p_0,\ldots,p_k}$ is a general tangent space to $S^k(X)$ and that its projection from $T_{X,p_{k-i+1},\ldots,p_k}$ is the general tangent space to $S^{k-i}(X_i)$. This implies the first assertion.

To prove the second assertion, note that it suffices to prove it for $i < h^{(k)}(X)$. Indeed, if $i \geqslant h^{(k)}(X)$ then, by Lemma 1.12 one has $h^{(k)}(X^i) = 0$ since already $h^{(k)}(X^{h^{(k)}}) = 0$. Thus, suppose $i < h^{(k)}(X)$. Let $p_0, \ldots, p_k \in X$ be general points and take i general points $q_1, \ldots, q_i$ in $X \setminus (X \cap T_{X,p_0,\ldots,p_k})$. Then the projection of $T_{X,p_0,\ldots,p_k}$ from $\langle q_0, \ldots, q_i\rangle$ is the tangent space to $S^k(X^i)$. Furthermore $i < h^{(k)}(X)$ yields $\langle q_0, \ldots, q_i\rangle \cap T_{X,p_0,\ldots,p_k} = \emptyset$. This implies the second assertion. □

1.7. Let $0 \leqslant a_1 \leqslant a_1 \leqslant \cdots \leqslant a_n$ be integers and set $\mathbb{P}(a_1, \ldots, a_n) := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n))$. We will denote by H a divisor in $|\mathcal{O}_{\mathbb{P}(a_1,\ldots,a_n)}(1)|$ and by F a fibre of the structure morphism $\pi : \mathbb{P}(a_1, \ldots, a_n) \to \mathbb{P}^1$. Notice that the corresponding divisor classes, which we still denote by H and F, freely generate $\mathrm{Pic}(\mathbb{P}(a_1, \ldots, a_n))$.

Set $r = a_1 + \cdots + a_n + n - 1$ and consider the morphism

$$\phi := \phi_{|H|} : \mathbb{P}(a_1, \ldots, a_n) \to \mathbb{P}^r$$

whose image we denote by $S(a_1, \ldots, a_n)$. As soon as $a_n > 0$, the morphism ϕ is birational to its image. Then the dimension of $S(a_1, \ldots, a_n)$ is n and its degree is $a_1 + \cdots + a_n = r - n + 1$, thus $S(a_1, \ldots, a_n)$ is a *rational normal scroll*, which is smooth if and only if $a_1 > 0$. Otherwise, if $0 = a_1 = \cdots = a_i < a_{i+1}$, then $S(a_1, \ldots, a_n)$ is the cone over $S(a_{i+1}, \ldots, a_n)$ with vertex a $\mathbb{P}^{i-1}$. One uses the simplified notation $S(a_1^{h_1}, \ldots, a_m^{h_m})$ if a_i is repeated h_i times, $i = 1, \ldots, m$.

We will sometimes use the notation H and F to denote the Weil divisors in $S(a_1, \ldots, a_n)$ corresponding to the ones on $\mathbb{P}(a_1, \ldots, a_n)$. Of course this is harmless if $a_1 > 0$, since then $\mathbb{P}(a_1, \ldots, a_n) \simeq S(a_1, \ldots, a_n)$.

Recall that rational normal scrolls, the Veronese surface in $\mathbb{P}^5$ and the cones on it, and the quadrics, can be characterized as those non-degenerate, irreducible varieties $X \subset \mathbb{P}^r$ in a projective space having minimal degree $\deg(X) = \operatorname{codim}(X) + 1$ (see [22]).

Let $X = S(a_1, \ldots, a_n) \subset \mathbb{P}^r$ be as above. We leave to the reader to see that:

$$X^1 = S(b_1, \ldots, b_n), \quad \text{where} \quad \{b_1, \ldots, b_n\} = \{a_1, \ldots, a_n - 1\}. \tag{1.6}$$

One can also consider the projection X' of X from a general $\mathbb{P}^{n-1}$ of the ruling of X. This is not birational to its image if $a_1 = 0$ and one sees that if $a_1 = \cdots = a_i = 0 < a_{i+1}$, then:

$$X' = S(c_1, \ldots, c_{n-i}), \quad \text{where} \quad \{c_1, \ldots, c_{n-i}\} = \{a_{i+1} - 1, \ldots, a_n - 1\}. \tag{1.7}$$

A general tangential projection of $X = S(a_1, \ldots, a_n)$ is the composition of the projection of X from a general $\mathbb{P}^{n-1}$ of the ruling of X and of a general internal projection of X'. Therefore, by putting (1.6) and (1.7) together, one deduces that if $a_1 = \cdots = a_i = 0 < a_{i+1}$, then:

$$X_1 = S(d_1, \ldots, d_{n-i}), \quad \text{where} \quad \{d_1, \ldots, d_{n-i}\} = \{a_{i+1} - 1, \ldots, a_n - 2\}. \tag{1.8}$$

As a consequence we have

Proposition 1.14. *Let $X = S(a_1, \ldots, a_n) \subset \mathbb{P}^r$ be a rational normal scroll as above. Then*:

$$\dim(S^k(X)) = \min\left\{ r, r + k + 1 - \sum_{1 \leqslant j \leqslant n;\, k \leqslant a_j} (a_j - k) \right\}.$$

In particular, if $r \geqslant (k+1)n + k$, then $s^{(k)}(X) = (k+1)n + k$ if and only if $a_1 \geqslant k$.

Proof. It follows by induction using (1.8) and Terracini's lemma. We leave the details to the reader. □

A different proof of the same result can be obtained by writing the equations of $S^k(X)$ (see [11,50] for this point of view).

1.8. Given positive integers $0 < m_1 \leqslant \cdots \leqslant m_h$ we will denote by $\mathrm{Seg}(\mathbb{P}^{m_1}, \ldots, \mathbb{P}^{m_h})$, or simply by $\mathrm{Seg}(m_1, \ldots, m_h)$ the *Segre variety* of type $(m_1, \ldots, m_h)$, i.e. the image of $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_h}$ in $\mathbb{P}^r$, $r = (m_1+1)\cdots(m_h+1)-1$, under the *Segre embedding*. Notice that, if $\mathbb{P}^{m_i} = \mathbb{P}(V_i)$, where V_i is a complex vector space of dimension m_i+1, $i = 1, \ldots, h$, then $\mathbb{P}^r = \mathbb{P}(V_1 \otimes \cdots \otimes V_h)$ and $\mathrm{Seg}(m_1, \ldots, m_h)$ is the set of equivalence classes of indecomposable tensors in $\mathbb{P}^r$. We use the shorter notation $\mathrm{Seg}(m_1^{k_1}, \ldots, m_s^{k_s})$ if m_i is repeated k_i times, $i = 1, \ldots, s$.

Recall that $\mathrm{Pic}(\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_h}) \simeq \mathrm{Pic}(\mathrm{Seg}(m_1, \ldots, m_h)) \simeq \mathbb{Z}^h$, is freely generated by the line bundles $\xi_i = pr_i^*(\mathcal{O}_{\mathbb{P}^{m_i}}(1))$, $i = 1, \ldots, h$, where $pr_i : \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_h} \to \mathbb{P}^{m_i}$ is the projection to the ith factor. A divisor D on $\mathrm{Seg}(m_1, \ldots, m_h)$ is said to be of *type* $(\ell_1, \ldots, \ell_h)$ if $\mathcal{O}_{\mathrm{Seg}(m_1,\ldots,m_h)}(D) \simeq \xi_1^{\ell_1} \otimes \cdots \otimes \xi_h^{\ell_h}$. The line bundle $\xi_1^{\ell_1} \otimes \cdots \otimes \xi_h^{\ell_h}$ on $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_h}$ is also denoted by $\mathcal{O}_{\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_h}}(\ell_1, \ldots, \ell_h)$. The hyperplane divisor of $\mathrm{Seg}(m_1, \ldots, m_h)$ is of type $(1, \ldots, 1)$.

It is useful to recall what are the defects of the Segre varieties $\mathrm{Seg}(m_1, m_2)$ with $m_1 \leqslant m_2$. As above, let V_i be complex vector spaces of dimension $m_i + 1$, $i = 1, 2$. We can interpret the points of $\mathbb{P}(V_1 \otimes V_2)$ as the equivalence classes of all $(m_1+1) \times (m_2+1)$ complex matrices and $\mathrm{Seg}(m_1, m_2) = \mathrm{Seg}(\mathbb{P}(V_1), \mathbb{P}(V_2))$ as the subscheme of $\mathbb{P}(V_1 \otimes V_2)$ formed by the equivalence classes of all matrices of rank 1. Similarly $S^k(\mathrm{Seg}(m_1, m_2))$ can be interpreted as the subscheme of $\mathbb{P}(V_1 \otimes V_2)$ formed by the equivalence classes of all matrices of rank less than or equal to $k+1$. Therefore $S^k(\mathrm{Seg}(m_1, m_2)) = \mathbb{P}(V_1 \otimes V_2)$ if and only if $k \geqslant m_1$. In the case $k < m_1$ one has instead:

$$\mathrm{codim}(S^k(\mathrm{Seg}(m_1, m_2))) = (m_1 - k)(m_2 - k)$$

(see [2, p. 67]). As a consequence one has

$$\delta_k(\mathrm{Seg}(m_1, m_2)) = k(k+1)$$

if $k < m_1 \leqslant m_2$.

The degree of $S^k(\mathrm{Seg}(m_1, m_2))$, with $k < m_1 \leqslant m_2$, are computed by a well known formula by Giambelli [30], apparently already known to Segre (see [50, p. 42], [28, 14.4.9], for a modern reference). The case $k = m_1 - 1$, which is the only one we will use later, is not difficult to compute (see [32, p. 243]) and reads

$$\deg(S^{m_1-1}(\mathrm{Seg}(m_1, m_2))) = \binom{m_2+1}{m_1}.$$

1.9. We will recall now some definition and result due to Kempf [39], which we are going to use later.

Let V_1, V_2, V_3 finite-dimensional complex vector spaces. A pairing

$$\phi : V_1 \otimes V_2 \to V_3$$

is said to be 1-*generic* if $0 \neq v \in V_1$ and $0 \neq u \in V_2$ implies $\phi(v \otimes u) \neq 0$. From a projective geometric point of view, ϕ determines a projection $\varphi : \mathbb{P}(V_1 \otimes V_2) - - \to \mathbb{P}(V_3)$ and the 1-genericity condition translates into the fact that the centre of the projection φ does not intersect $\mathrm{Seg}(\mathbb{P}(V_1), \mathbb{P}(V_2))$.

If ϕ is surjective, then we may regard ϕ as specifying a linear space of linear transformations:

$$V_3^* \subseteq \mathrm{Hom}(V_1, V_2^*) \simeq V_1^* \otimes V_2^*.$$

One says that V_3^* is 1-*generic* if ϕ is.

Let $m_i + 1 = \dim(V_i)$ and suppose $m_1 \leqslant m_2$. For each k such that $0 \leqslant k \leqslant m_1$, let $(V_3^*)_k$ be the subscheme of V_3^* of all matrices in V_3^* with rank less than or equal to $k+1$, i.e. the scheme-theoretic intersection of V_3^* with the scheme $\mathrm{Hom}(V_1, V_2^*)_k$ of all matrices with rank less than or equal to $k+1$ in $\mathrm{Hom}(V_1, V_2^*)$. Of course $(V_3^*)_k$ is a cone, hence it gives rise to a closed subscheme $\mathbb{P}((V_3^*)_k)$ of $\mathbb{P}(V_3^*)$ which is the scheme theoretic intersection of $\mathbb{P}(V_3^*)$ with $S^k(\mathrm{Seg}(\mathbb{P}(V_1^*), \mathbb{P}(V_2^*))$. Notice that the expected codimension of $\mathbb{P}((V_3^*)_k)$ in $\mathbb{P}(V_3^*)$ is:

$$m_1 m_2 - k(m_1 + m_2) + k^2 = \dim(\mathbb{P}(V_1^* \otimes V_2^*)) - s^{(k)}(\mathrm{Seg}(\mathbb{P}(V_1^*), \mathbb{P}(V_2^*))).$$

This is also the expected codimension of $(V_3^*)_k$ in V_3^*. We can now state Kempf's theorem:

Theorem 1.15. *If $V_3^* \subseteq V_1^* \otimes V_2^*$ is 1-generic, then $(V_3^*)_{m_1-1}$ is reduced, irreducible and of the expected codimension $m_2 - m_1 + 1$ in V_3^*. The same is true for $\mathbb{P}((V_3^*)_{m_1-1})$, whose degree is $\binom{m_2+1}{m_1}$.*

1.10. Given positive integers n, d, we will denote by $V_{n,d}$ the image of $\mathbb{P}^n$ under the *d-Veronese embedding* of $\mathbb{P}^n$ in $\mathbb{P}^{\binom{n+d}{d}-1}$.

1.11. If X is a variety of dimension n and Y a subvariety of X, we will denote by $\mathrm{Bl}_Y(X)$ the blow-up of X along Y. If Y is a finite set $\{x_1, \ldots, , x_n\}$ we denote the blow-up by $\mathrm{Bl}_{x_1,\ldots,x_n}(X)$.

With the symbol $\equiv$ we will denote the linear equivalence of divisors on X. The symbol $\sim$ will instead denote numerical equivalence. If $\mathcal{L}$ is a linear system of divisors on X, of dimension r, we will denote by $\phi_{\mathcal{L}} : X - - \to \mathbb{P}^r$ the rational map defined by $\mathcal{L}$.

If D is a divisor on the variety X, we denote by $|D|$ the complete linear series of D. If $X \subset \mathbb{P}^r$ is an irreducible, projective variety, and D is a hyperplane section of X, one says that X is *linearly normal* if the linear series cut out on X by the hyperplanes of $\mathbb{P}^r$ is complete, i.e. if the natural map

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(X, \mathcal{O}_X(D))$$

is surjective.

If D [resp., $\mathcal{D}$] is a divisor [resp., a line bundle] on X, we will say that D [resp., $\mathcal{D}$] is *effective* if $h^0(X, \mathcal{O}_X(D)) > 0$ [resp., $h^0(X, \mathcal{L}) > 0$]. We will say that D [resp., $\mathcal{D}$] is *nef* if for every curve C on X, one has $D \cdot C \geqslant 0$ [resp., $\mathcal{D} \cdot C \geqslant 0$]. A nef divisor D [resp., a nef line bundle $\mathcal{D}$] is *big* if $D^n > 0$ [resp., $\mathcal{D}^n > 0$].

1.12. Let X be a smooth, irreducible, projective surface. As customary, we will use the following notation $q := q(X) := h^1(X, \mathcal{O}_X)$ for the *irregularity*, $\kappa := \kappa(X)$ for the *Kodaira dimension* of X. We will denote by $K := K_X$ a *canonical divisor* on X and, as usual, $p_g := p_g(X) := h^0(X, \mathcal{O}_X(K))$ is the *geometric genus*.

If C is a curve on X, it will be called a $(-n)$*-curve*, if $C \simeq \mathbb{P}^1$ and $C^2 = -n$. Recall that a famous theorem of Castelnuovo's identifies the (-1)-curves as the exceptional divisors of blow-ups.

Let D be a Cartier divisor on an irreducible, projective surface X. We denote by $p_a(D)$ the arithmetic genus of D. We will say that D is a *curve* on X if it is effective. If D is reduced curve on X, we will consider $p_g(D)$ the *geometric genus* of D, i.e. the arithmetic genus of the normalization of D.

A curve D on X will be called m-connected if for every decomposition $D = A + B$, with A, B non-zero curves on X, one has $A \cdot B \geqslant m$. If D is 1-connected one has $h^0(D, \mathcal{O}_D) = 1$ and $h^1(D, \mathcal{O}_D) = p_a(D) \geqslant 0$ (see [4]). If D is a big and nef curve on X, then D is 1-connected (see [44, Lemma (2.6)]).

If X is smooth, we will say that the pair (X, D) is:

- *effective* [resp., *nef*, *big*, *ample*, *very ample*] if D is such;
- *minimal* if there is no (-1)-curve C on X such that $D \cdot C = 0$;
- a *h-scroll*, with $h \geqslant 0$ an integer, if there is a smooth rational curve F on X such that $F^2 = 0$ and $D \cdot F = h$;
- a *del Pezzo pair* if $K \sim -D$ and (X, D) is big and nef.

A 1-scroll will be simply called a *scroll*.

Notice that if (X, D) is a del Pezzo pair, then X is rational and $K \equiv -D$. Indeed $-K$ is nef and big, thus $\kappa(X) = -\infty$ and $q = h^1(X, \mathcal{O}_X) = h^1(X, \mathcal{O}_X(K - K)) = 0$ by Ramanujam's vanishing theorem (see [48]).

If $\mathcal{L}$ is a linear system on X and $D \in \mathcal{L}$ is its general divisor, we will say that $(X, \mathcal{L})$ is *nef*, *big*, *ample*, *minimal*, *a h-scroll*, etc. if (X, D) is such. One says that $(X, \mathcal{L})$ is *very ample* if $\phi_{\mathcal{L}}$ is an isomorphism of X to its image.

Suppose the linear system $\mathcal{L}$ has no fixed curve and the general curve in $\mathcal{L}$ is irreducible. Then, by blowing up the base points of $\mathcal{L}$, we see that there is a unique pair $(X', \mathcal{L}')$, where X' is a surface with a birational morphism $f : X' \to X$ and a $\mathcal{L}'$ is linear system on X' such that:

- $\mathcal{L}'$ is the strict transform of $\mathcal{L}$ on X';
- $\mathcal{L}'$ is base point free, and therefore its general curve D' is smooth and irreducible;
- $\mathcal{L}'$ is *f-relatively minimal*, i.e. if E is a (-1)-curve on X' such that $D' \cdot E = 0$ then E is not contracted by f.

We will call the pair $(X', \mathcal{L}')$ the *resolution* of the pair (X, D).

If $X \subseteq \mathbb{P}^r$ is an irreducible, projective surface, one considers $f : X' \to X \subseteq \mathbb{P}^r$ a minimal desingularization of X and $\mathcal{L}$ the linear system on X' such that $f = \phi_{\mathcal{L}}$. The pair $(X', \mathcal{L})$ is big, nef and minimal. One says that X is a *scroll* if the pair $(X', \mathcal{L})$ is a scroll.

If $X \simeq \mathbb{P}^2$ and R is a line, the pair (X, D) with $D \equiv dR$ will be called a *d-Veronese pair*. If $X = \mathbb{F}_a := \mathbb{P}(0, a)$ is the Hirzebruch surface with $a \geqslant 0$, we let E be a $(-a)$-curve on $\mathbb{F}_a$ and F a fibre of the ruling on $\mathbb{P}^1$, so that $F^2 = 0$ and $E \cdot F = 1$. Then a pair (X, D) with $X = \mathbb{F}_a$ and $D \equiv \alpha E + \beta F$ will be called a (a, α, β)*-pair* or an (α, β)*-pair* on $\mathbb{F}_a$.

Consider a pair (X, D) as above. Let $x_1, \ldots, x_n$ be distinct points on X. Consider the blow-up $p : \mathrm{Bl}_{x_1,\ldots,x_n}(X) \to X$ at the given points. On $\mathrm{Bl}_{x_1,\ldots,x_n}(X)$ we have the exceptional divisors $E_1, \ldots, E_n$ corresponding to $x_1, \ldots, x_n$. Consider the divisor $D_{x_1,\ldots,x_n} := p^*(D) - E_1 - \cdots - E_n$. The pair $(\mathrm{Bl}_{x_1,\ldots,x_n}(X), D_{x_1,\ldots,x_n})$ will be called the *internal projection* of (X, D) from $x_1, \ldots, x_n$.

In the same setting, the pair $(\mathrm{Bl}_{x_1,\ldots,x_n}(X), p^*(D))$ will be called a *blow-up* of (X, D).

Similarly, consider the divisor $D_{2x_1,\ldots,2x_n} := p^*(D) - 2E_1 - \cdots - 2E_n$. The pair $(\mathrm{Bl}_{x_1,\ldots,x_n}(X), D_{2x_1,\ldots,2x_n})$ will be called the *tangential projection* of (X, D) from $x_1, \ldots, x_n$.

2. Degeneration of projections

In this section we generalize some of the ideas presented in Sections 3 and 4 of [18], to which we will constantly refer. This will enable us to prove an extension of Theorem 4.1 of [18], which will be useful later.

Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate projective variety of dimension n. We fix $k \geqslant 1$, we assume that X is not k-defective and that $s^{(k)}(X) = (k+1)n + k$.

Let us fix an integer s such that $r - s^{(k)}(X) \leqslant s \leqslant r - s^{(k-1)}(X) - 2$, so that $s^{(k-1)}(X) + 1 \leqslant r - s - 1 \leqslant s^{(k)}(X) - 1$. Let $L \subset \mathbb{P}^r$ be a general projective subspace of dimension s and let us consider the projection morphism $\pi_L : S^{k-1}(X) \to \mathbb{P}^{r-s-1}$ of X from L. Notice that, under our assumptions on s, one has

$$\pi_L(S^k(X)) = \mathbb{P}^{r-s-1}, \quad \pi_L(S^{k-1}(X)) \subset \mathbb{P}^{r-s-1}.$$

Let $p_1, \ldots, p_k \in X$ be general points and let $x \in \langle p_1, \ldots, p_k \rangle$ be a general point, so that $x \in S^{k-1}(X)$ is a general point and $T_{S^{k-1}(X),x} = T_{X,p_1,\ldots,p_k}$. We will now study how the projection $\pi_L : S^{k-1}(X) \to \mathbb{P}^{r-s-1}$ *degenerates* when its centre L tends to a general s-dimensional subspace L_0 containing x, i.e. such that $L_0 \cap S^{k-1}(X) = L_0 \cap T_{X,p_1,\ldots,p_k} = \{x\}$. To be more precise we want to describe the *limit* of a certain *double point scheme* related to π_L in such a degeneration.

Let us describe in detail the set up in which we will work. We let T be a general $\mathbb{P}^{s^{(k-1)}(X)+s+1}$ which is tangent to $S^{k-1}(X)$ at x, i.e. T is a general $\mathbb{P}^{s^{(k-1)}(X)+s+1}$ containing $T_{X,p_1,\ldots,p_k}$. Then we choose a general line ℓ inside T containing x, and we also choose Σ a general $\mathbb{P}^{s-1}$ inside T. For every $t \in \ell$, we let L_t be the span of t and Σ. For $t \in \ell$ a general point, L_t is a general $\mathbb{P}^s$ in $\mathbb{P}^r$. For a general $t \in \ell$, we denote by $\pi_t : S^{k-1}(X) \to \mathbb{P}^{r-s-1}$ the projection morphism of $S^{k-1}(X)$ from L_t. We want to study the limit of π_t when t tends to x. We will suppose from now on that $k \geqslant 2$, since the case $k = 1$ has been considered in [18].

In order to perform our analysis, consider a neighborhood U of x in ℓ such that π_t is a morphism for all $t \in U \setminus \{x\}$. We will fix a local coordinate on ℓ so that x has the coordinate 0, thus we may identify U with a disk around $x = 0$ in $\mathbb{C}$. Consider the products:

$$\mathcal{X}_1 = X \times U, \quad \mathcal{X}_2 = S^{k-1}(X) \times U, \quad \mathbb{P}_U^{r-s-1} = \mathbb{P}^{r-s-1} \times U.$$

The projections π_t, for $t \in U$, fit together to give a morphism $\pi_1 : \mathcal{X}_1 \to \mathbb{P}_U^{r-s-1}$ and a rational map $\pi_2 : \mathcal{X}_2 - - \to \mathbb{P}_U^{r-s-1}$, which is defined everywhere except at the pair $(x, x) = (x, 0)$. In order to extend it, we have to blow up $\mathcal{X}_2$ at $(x, 0)$. Let $p : \tilde{\mathcal{X}}_2 \to \mathcal{X}$ be this blow-up and let $Z \simeq \mathbb{P}^{s^{(k-1)}(X)}$ be the exceptional divisor. Looking at the obvious morphism $\phi : \tilde{\mathcal{X}}_2 \to U$, we see that this is a flat family of varieties over U. The fibre over a point $t \in U \setminus \{0\}$ is isomorphic to $S^{k-1}(X)$, whereas the fibre over $t = 0$ is of the form $\tilde{S} \cup Z$, where $\tilde{S} \to S^{k-1}(X)$ is the blow up of $S^{k-1}(X)$ at x, and $\tilde{S} \cap Z = E$ is the exceptional divisor of this blow up, the intersection being transverse.

On $\tilde{\mathcal{X}}_2$ the projections π_t, for $t \in U$, fit together now to give a *morphism* $\tilde{\pi} : \tilde{\mathcal{X}}_2 \to \mathbb{P}_U^{r-s-1}$.

By abusing notation, we will denote by π_0 the restriction of $\tilde{\pi}$ to the central fibre $\tilde{S} \cup Z$. The restriction of π_0 to $\tilde{S}$ is determined by the projection of $S^{k-1}(X)$ from the subspace L_0: notice in fact that, since $L_0 \cap S^{k-1}(X) = L_0 \cap T_{X, p_1, \ldots, p_k} = \{x\}$, this projection is not defined on $S^{k-1}(X)$ but it is well defined on $\tilde{S}$.

As for the action of π_0 on the exceptional divisor Z, this is explained by the following lemma, whose proof is analogous to the proof of [18, Lemma 3.1], and therefore we omit it:

Lemma 2.1. *In the above setting, π_0 maps isomorphically Z to the $s^{(k-1)}(X)$-dimensional linear space Θ which is the projection of T from L_0.*

Now we consider $\mathcal{X}_1 \times_U \tilde{\mathcal{X}}_2$, which has a natural projection map $\psi : \mathcal{X}_1 \times_U \tilde{\mathcal{X}}_2 \to U$. One has a commutative diagram:

$$\begin{array}{ccc} \mathcal{X}_1 \times_U \tilde{\mathcal{X}}_2 & \stackrel{\overline{\pi}}{\to} & \mathbb{P}_U^{r-s-1} \\ \psi \downarrow & & \downarrow \\ U & \stackrel{id_U}{\to} & U, \end{array}$$

where $\overline{\pi} = \pi \times \tilde{\pi}$. For the general $t \in U$, the fibre of ψ over t is $X \times S^{k-1}(X)$, and the restriction $\overline{\pi}_t : X \times S^{k-1}(X) \to \mathbb{P}^{r-s-1}$ of $\overline{\pi}$ to it is nothing but $\pi_{t|X} \times \pi_{t|S^{k-1}(X)}$. We denote by $\Delta_t^{(s,k)}$ the double point scheme of $\overline{\pi}_t$. Notice that $\dim(\Delta_t^{(s,k)}) \geqslant s^{(k)}(X)+s-r$ and, by the generality assumptions, we may assume that equality holds for all $t \neq 0$. Finally consider the flat limit $\tilde{\Delta}_0^{(s,k)}$ of $\Delta_t^{(s,k)}$ inside $\Delta_0^{(s,k)}$. We will call it the *limit double point scheme* of the map $\overline{\pi}_t$, $t \neq 0$. We want to give some information about it. Notice the following lemma, whose proof is similar to the one of [18, Lemma 3.2], and therefore we omit it:

Lemma 2.2. *In the above setting, every irreducible component of $\Delta_0^{(s,k)}$ of dimension $s^{(k)}(X)+s-r$ sits in the limit double point scheme $\tilde{\Delta}_0^{(s,k)}$.*

Let us now denote by

- X_T the scheme cut out by T on X. X_T is cut out on X by $r-s^{(k-1)}(X)-s-1$ general hyperplanes tangent to X at $p_1, \ldots, p_k$. We call X_T a *general* $(r-s^{(k-1)}(X)-s-1)$*-tangent section* to X at $p_1, \ldots, p_k$. Remark that each component of X_T has dimension at least $n-(r-s^{(k-1)}(X)-s-1) = s^{(k)}(X)+s-r$;
- Y_T the image of X_T via the restriction of π_0 to X. By Lemma 2.1, Y_T sits in $\Theta = \pi_0(Z)$, which is naturally isomorphic to Z. Hence we may consider Y_T as a subscheme of Z;
- $Z_T \subset X \times Z$ the set of pairs (x, y) with $x \in X_T$ and $y = \pi_0(x) \in Y_T$. Notice that $Z_T \simeq X_T$;
- $\Delta_0'^{(s,k)}$ the double point scheme of the restriction of π_0 to $\tilde{S} \times X$.

With this notation, the following lemma is clear (see [18, Lemma 3.3]):

Lemma 2.3. *In the above setting, $\Delta_0^{(s,k)}$ contains as irreducible components $\Delta_0'^{(s,k)}$ on $X \times \tilde{S}$ and Z_T on $X \times Z$.*

As an immediate consequence of Lemmas 2.2 and 2.3, we have the following proposition (see [18, Proposition 3.4]):

Proposition 2.4. *In the above setting, every irreducible component of X_T, off $T_{X,p_1,\ldots,p_k}$, of dimension $s^{(k)}(X)+s-r$ gives rise to an irreducible component of Z_T which is contained in the limit double point scheme $\tilde{\Delta}_0^{(s,k)}$.*

Remark 2.5. We notice that the implicit hypothesis "off $T_{X,x}$" has to be added also in the statement of [18, Proposition 3.4]. Actually in the applications in [18] this hypothesis is always fulfilled.

So far we have essentially extended word by word the contents of Section 3 of [18]. This is not sufficient for our later applications. Indeed we need a deeper understanding of the relation between the double points scheme $\Delta_t^{(s,k)}$ and $(k+1)$-secant $\mathbb{P}^k$'s to X meeting the centre of projection L_t and related degenerations when t goes to 0. We will do this in the following remark.

Remark 2.6. (i) It is interesting to give a different geometric interpretation for the general double point scheme $\Delta_t^{(s,k)}$, for $t \neq 0$. Notice that, by the generality assumption, $L_t \cap S^k(X)$ is a variety of dimension $s^{(k)}(X) + s - r$, which we can assume to be irreducible as soon as $s^{(k)}(X) + s - r > 0$. Take the general point w of it if $s^{(k)}(X) + s - r > 0$, or any point of it if $s^{(k)}(X) + s - r = 0$. Then this is a general point of $S^k(X)$. This means that $w \in \langle q_0, \dots, q_k \rangle$, with $q_0, \dots, q_k$ general points on X. Now, for each $i = 0, \dots, k$, there is a point $r_i \in \langle q_0, \dots, \hat{q}_i, \dots, q_k \rangle$ which is collinear with w and q_i. Each pair (q_i, r_i), $i = 0, \dots, k$, is a general point of a component of $\Delta_t^{(s,k)}$. Conversely the general point of any component of $\Delta_t^{(s,k)}$ arises in this way.

(ii) Now we specialize to the case $t = 0$. More precisely, consider $Z_T \subset X \times Z$ and a general point (p, q) on an irreducible component of it of dimension $s^{(k)}(X) + s - r$, which therefore sits in the limit double point scheme $\tilde{\Delta}_0^{(s,k)}$. Hence, there is a 1-dimensional family $\{(p_t, q_t)\}_{t \in U}$ of pairs of points such that $(p_t, q_t) \in \Delta_t^{(s,k)}$ and $p_0 = p, q_0 = q$.

By (i) of the present remark, we can look at each pair (p_t, q_t), $t \neq 0$, as belonging to a $(k+1)$-secant $\mathbb{P}^k$ to X, denoted by Π_t, forming a flat family $\{\Pi_t\}_{t \in U \setminus \{0\}}$ and such that $\Pi_t \cap L_t \neq \emptyset$. Consider then the flat limit Π_0, for $t = 0$, of the family $\{\Pi_t\}_{t \in U \setminus \{0\}}$. Since $q \in Z$, clearly Π_0 contains x. Moreover it also contains p. This implies that Π_0 is the span of p with one of the k-secant $\mathbb{P}^{k-1}$'s to X containing $x \in S^{k-1}(X)$.

As an application of the previous remark, we can prove the following crucial theorem, which extends [18, Theorem 4.1]:

Theorem 2.7. *Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, projective variety such that $s^{(k)}(X) = (k+1)n + k$. Then*

$$d_{X,k} \cdot \deg(X_k) \leqslant v_k(X).$$

In particular

(i) *if $r \geqslant (k+1)(n+1)$ and X is not k-weakly defective, then*:

$$\deg(X_k) \leqslant v_k(X);$$

(ii) *if $r = (k+1)n + k$ then*:

$$d_{X,k} \leqslant \mu_k(X).$$

Proof. We let $s = h^{(k)}(X) = r - s^{(k)}(X)$ and we apply Remark 2.6 to this situation. Then X_T has $d_{X,k} \cdot \deg(X_k)$ isolated points, which give rise to as many flat limits of

$(k+1)$-secant $\mathbb{P}^k$'s to X meeting a general $\mathbb{P}^s$. By the definition of $v_k(X)$ the first assertion follows. Then (i) follows from Lemma 1.4 and (ii) follows by (1.3). $\square$

3. Tangent cones to higher secant varieties

In this section we describe the tangent cone to the variety $S^k(X)$, at a general point of $S^l(X)$, where $0 \leqslant l < k$, and $X \subset \mathbb{P}^r$ is an irreducible, projective variety of dimension n. Our result is the following theorem, which can be seen as a generalization of Terracini's lemma:

Theorem 3.1. *Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, projective variety and let $l, m \in \mathbb{N}$ be such that $l+m = k-1$. If $z \in S^l(X)$ is a general point, then the cone $J(T_{S^l(X),z}, S^m(X))$ is an irreducible component of $(C_{S^k(X),z})_{\mathrm{red}}$. Furthermore one has*

$$\mathrm{mult}_z(S^k(X)) \geqslant \deg(J(T_{S^l(X),z}, S^m(X))) \geqslant \deg(S^m(X_{l+1})).$$

Proof. We assume that $S^l(X) \neq \mathbb{P}^r$, otherwise the assertion is trivially true.

The scheme $C_{S^k(X),z}$ is of pure dimension $s^{(k)}(X)$. Let now $w \in S^m(X)$ be a general point. By Terracini's lemma and by the generality of $z \in S^l(X)$, we get

$$\begin{aligned} \dim(J(T_{S^l(X),z}, S^m(X))) &= \dim(J(T_{S^l(X),z}, T_{S^m(X),w})) \\ &= \dim(J(S^l(X), S^m(X))) = \dim(S^k(X)) = s^{(k)}(X). \end{aligned}$$

Thus, since $J(T_{S^l(X),z}, S^m(X))$ is irreducible and reduced, it suffices to prove the inclusion $J(T_{S^l(X),z}, S^m(X)) \subseteq (C_{S^k(X),z})_{\mathrm{red}}$.

Let again $w \in S^m(X)$ be a general point. We claim that $w \notin T_{S^l(X),z}$. Indeed $S^l(X) \neq \mathbb{P}^r$ and by (1.1)

$$\mathrm{Vert}(S^l(X)) := \bigcap_{y \in S^l(X)} T_{S^l(X),y}$$

is a proper linear subspace of $\mathbb{P}^r$. If the general point of $S^m(X)$ would be contained in $\mathrm{Vert}(S^l(X))$, then $X \subseteq S^m(X) \subseteq \mathrm{Vert}(S^l(X))$ and X would be degenerate, contrary to our assumption.

Since $w \notin T_{S^l(X),z}$, then z is a smooth point of the cone $J(w, S^l(X))$. We deduce that:

$$\langle w, T_{S^l(X),z}\rangle = T_{J(w,S^l(X)),z} = C_{J(w,S^l(X)),z} \subseteq C_{J(S^m(X),S^l(X)),z} = C_{S^k(X),z}.$$

By the generality of $w \in S^m(X)$ we finally have $J(T_{S^l(X),z}, S^m(X)) \subseteq C_{S^k(X),z}$. This proves the first part of the theorem.

To prove the second part, we remark that

$$\mathrm{mult}_z(S^k(X)) = \deg(C_{S^k(X),z}) \geqslant \deg(J(T_{S^l(X),z}, S^m(X))).$$

Now, if $p_0, \ldots, p_l \in X$ are general points, then $J(T_{S^l(X),z}, S^m(X))$ is the cone with vertex $T_{S^l(X),z}$ over $\tau_{X,p_0,\ldots,p_l}(S^m(X))$, and, by Lemma 1.12 we have that $\tau_{X,p_0,\ldots,p_l}(S^m(X)) = S^m(X_{l+1})$. Thus $\deg(J(T_{S^l(X),z}, S^m(X))) \geqslant \deg(S^m(X_{l+1}))$, proving the assertion. $\square$

4. A lower bound on the degree of secant varieties

As we recalled in Section 1, the degree d of an irreducible non-degenerate variety $X \subset \mathbb{P}^r$ verifies the lower bound

$$d \geqslant \mathrm{codim}(X) + 1. \tag{4.1}$$

Varieties whose degree is equal to this lower bound are called varieties of *minimal degree*. As well known, they have nice geometric properties, e.g. they are rational (see [22]). In the present section we will prove a lower bound on the degree of the k-secant variety to a variety X. This bound generalizes (4.1) and we will see that varieties X attaining it have interesting features which resemble the properties of minimal degree varieties.

Before proving the main result of this section, we need a useful lemma. For an irreducible variety $Z \subseteq \mathbb{P}^N$ we defined $t_{Z,p}$ as the projection from the general point $p \in Z$ restricted to Z, i.e. $t_{Z,p} : Z - - \to t_{Z,p}(Z) = Z^1$, see Section 1.4. In this section, we shall sometimes abuse notation by considering an arbitrary $p \in Z$ and also in this case we shall indicate by Z^1 the projection from p.

Lemma 4.1. *Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, projective variety, let $k \geqslant 0$ be an integer such that $S^k(X) \neq \mathbb{P}^r$ and let $p \in X$ be an arbitrary point. Then one has*

(i) $t_{S^k(X),p}(S^k(X)) = S^k(X^1)$;
(ii) *the general point in X does not belong to* $\mathrm{Vert}(S^k(X))$;
(iii) *if $p \in X \setminus (X \cap \mathrm{Vert}(S^k(X))$, in particular if $p \in X$ is a general point, then $t_{S^k(X),p}$ is generically finite to its image $S^k(X^1)$ and $s^{(k)}(X) = s^{(k)}(X^1)$;*
(iv) *if X is not k-defective and $p \in X \setminus (X \cap \mathrm{Vert}(S^k(X))$, then X^1 is also not k-defective;*
(v) *if $p \in X \setminus (X \cap \mathrm{Vert}(S^k(X))$ and if $\theta_k(X)$ denotes the degree of $t_{S^k(X),p}$, then*

$$\begin{aligned}\deg(S^k(X)) &= \theta_k(X) \cdot \deg(S^k(X^1)) + \mathrm{mult}_p(S^k(X)) \\ &\geqslant \deg(S^k(X^1)) + \mathrm{mult}_p(S^k(X))\end{aligned}$$

and

$$\mu_k(X^1) = \theta_k(X) \cdot \mu_k(X).$$

In particular

(vi) *if* $p \in X \setminus (X \cap \mathrm{Vert}(S^k(X))$ *and if*

$$\deg(S^k(X)) = \deg(S^k(X^1)) + \mathrm{mult}_p(S^k(X))$$

then $\theta_k(X) = 1$, *i.e.* $t_{S^k(X),p} : S^k(X) - - \to S^k(X^1)$ *is birational and then* $\mu_k(X^1) = \mu_k(X)$;

(vii) *if, in addition,* $\mu_k(X^1) = 1$ *then also* $\mu_k(X) = 1$ *and* $\theta_k(X) = 1$.

Proof. (i) follows by Lemma 1.12.

Since $S^k(X)$ is a proper subvariety in $\mathbb{P}^r$, then $\mathrm{Vert}(S^k(X))$ is a proper linear subspace of $\mathbb{P}^r$. This implies (ii). (iii) is immediate.

Since $S^k(X) \neq \mathbb{P}^r$, if X is not k-defective, we have $s^{(k)}(X) = (k+1)n + k < r$. By (iii) we have also $s^{(k)}(X^1) = (k+1)n + k \leqslant r - 1$, i.e. X^1 is also not k-defective. This proves (iv).

The first assertion of (v) is immediate. Furthermore, we have a commutative diagram of rational maps:

$$\begin{array}{ccc} S^k_X & \overset{t}{- - \to} & S^k_{X^1} \\ p^k_X \downarrow & & \downarrow p^k_{X^1} \\ S^k(X) & \overset{t_{S^k(X),p}}{- - \to} & S^k(X^1), \end{array}$$

where t is determined, in an obvious way, by $t_{S^k(X),p}$. By the hypothesis, $t_{S^k(X)}$ has degree $\theta_k(X)$, whereas t is easily seen to be birational. Hence the conclusion follows. (vi) and (vii) are now obvious. $\square$

Now we come to the main result of this section:

Theorem 4.2. *Let* $X \subset \mathbb{P}^r$ *be an irreducible, non-degenerate, projective variety and let* $h := \mathrm{codim}(S^k(X)) > 0$. *Then*

$$\deg(S^k(X)) \geqslant \binom{h+k+1}{k+1} \tag{4.2}$$

and, if $l = 0, \ldots, k$ *and* $x \in S^l(X)$ *is any point, then*

$$\mathrm{mult}_x(S^k(X)) \geqslant \binom{h+k-l}{k-l}. \tag{4.3}$$

Suppose equality holds in (4.2) *and* $h \geqslant 1$. *Then*

(i) *if* $x \in X$ *is a general point, one has*

$$C_{S^k(X),x} = J(T_x(X), S^{k-1}(X)), \quad \mathrm{mult}_x(S^k(X)) = \binom{k+h}{k};$$

(ii) *for every* m *such that* $1 \leqslant m \leqslant h$, *one has*

$$\deg(S^k(X^m)) = \binom{h-m+k+1}{k+1};$$

(iii) *for every* m *such that* $1 \leqslant m \leqslant h$, *the projection from a general point* $x \in X^{m-1}$

$$t_{S^k(X^{m-1}),x} : S^k(X^{m-1}) - - \to S^k(X^m)$$

is birational;

(iv) *for every* m *such that* $1 \leqslant m \leqslant k$ *one has*

$$\deg(S^{k-m}(X_m)) = \binom{h+k-m+1}{k-m+1};$$

in particular X_k *is a variety of minimal degree*;

(v) *if* X *is not* k*-defective, then, for every* m *such that* $1 \leqslant m \leqslant h$, *also* X^m *is not* k*-defective and* $\mu_k(X) = \mu_k(X^m)$;

(vi) *if* X *is not* k*-defective then*

$$d_{X,k} \leqslant \mu_k(X).$$

Proof. We make induction on both k and h. For $k = 0$ we have the bound 4.1 for the minimal degree of an algebraic variety, while for $h = 0$ the assertion is obvious for every k. Let us project X and $S^k(X)$ from a general point $x \in X$. By Lemmas 4.1 and 1.13, Theorem 3.1, and by induction we get

$$\begin{aligned}
\deg(S^k(X)) &\geqslant \deg(S^k(X^1)) + \mathrm{mult}_x(S^k(X)) \\
&\geqslant \deg(S^k(X^1)) + \deg(S^{k-1}(X_1)) \\
&\geqslant \binom{k+h}{k+1} + \binom{k+h}{k} = \binom{k+h+1}{k+1}
\end{aligned}$$

whence (4.2) follows. Let now $x \in S^l(X)$ be a general point, then by Theorem 3.1, Lemma 1.13 and by (4.2) one has

$$\mathrm{mult}_x(S^k(X)) \geqslant \deg(S^{k-l-1}(X_{l+1})) \geqslant \binom{k+h-l}{k-l}$$

proving (4.3) in this case. Of course (4.3) also holds if $x \in S^l(X)$ is any point.

If equality holds in (4.2), one immediately obtains assertions (i)–(iv) for $m = 1$. By an easy induction one sees that (i)–(iv) hold in general.

Assertion (v) follows by Lemma 4.1. As for (vi), consider the following commutative diagram:

$$\begin{array}{ccc} X & \overset{\tau_{X,k}}{- - \to} & X_k \\ t_{X,h} \downarrow & & \downarrow t_{X_k,h} \\ X^h & \overset{\tau_{X^h,k}}{- - \to} & \mathbb{P}^n. \end{array}$$

Notice that the vertical maps $t_{X,h}, t_{X_k,h}$ are birational being projections from h general points on a variety of codimension bigger than h. Thus one has

$$d_{X,k} = d_{X^h,k}.$$

On the other hand, by Theorem 2.7 and Lemma 4.1 one has

$$d_{X^h,k} \leqslant \mu_k(X^h) = \mu_k(X)$$

which proves the assertion. □

Remark 4.3. It is possible to improve the previous result. For example, using Lemma 4.1, one sees that (i) holds not only if $x \in X$ is general, but also if x is any smooth point of X not lying on $\mathrm{Vert}(S^k(X))$. Similar improvements can be found for (ii)–(v). We leave this to the reader, since we are not going to use it later.

Definition 4.4. Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, projective variety of dimension n. Let k be a positive integer.

Let $k \geqslant 2$ be an integer. One says that X is *k-regular* if it is smooth and if there is no subspace $\Pi \subset \mathbb{P}^r$ of dimension $k-1$ such that the scheme cut out by Π on X contains a finite subscheme of length $\ell \geqslant k+1$. By definition 1-*regularity* coincides with smoothness.

We say that X *has minimal k-secant degree*, briefly X is an $\mathcal{M}^k$-variety, if $r = s^{(k)}(X) + h$, $h := \mathrm{codim}(S^k(X)) > 0$, and $\deg(S^k(X)) = \binom{h+k+1}{k+1}$ (compare with Theorem 4.2).

We say that X is a variety *with the minimal number of apparent* $(k+1)$*-secant* $\mathbb{P}^{k-1}$*'s*, briefly X is an $\mathcal{MA}_{k-1}^{k+1}$-variety, if $s^{(k)}(X) = (k+1)n + k$, $r = s^{(k)}(X) + h$, $h := \operatorname{codim}(S^k(X)) > 0$, and if $v_k(X) = \binom{h+k+1}{k+1}$ (compare with Theorems 4.2 and 1.3). In other words X is an $\mathcal{MA}_{k-1}^{k+1}$-variety if and only if it is not k-defective, is an $\mathcal{M}^k$-variety and $\mu_k(X) = 1$. For example, an $\mathcal{M}^k$-variety which is not k-weakly defective is an $\mathcal{MA}_{k-1}^{k+1}$-variety (see Proposition 1.5).

We say that X is a variety *with one apparent* $(k+1)$*-secant* $\mathbb{P}^{k-1}$, briefly X is an $\mathcal{OA}_{k-1}^{k+1}$-variety, if $r = s^{(k)}(X) = (k+1)n + k$ and $\mu_k(X) = 1$.

The terminology introduced in the previous definition is motivated by the fact that, for example, $\mathcal{OA}_{k-1}^{k+1}$-varieties are an extension of *varieties with one apparent double point* or OADP-varieties, classically studied by Severi [54] (for a modern reference see [18]).

With this definitions in mind, we have:

Corollary 4.5. *Let k be a positive integer. Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, projective variety of dimension n and let $h := \operatorname{codim}(S^k(X)) \geqslant 0$. One has*

(i) *if X is a $\mathcal{M}^k$-variety then for every m such that $1 \leqslant m \leqslant h$, the variety X^m is again a $\mathcal{M}^k$-variety;*

(ii) *if X is a $\mathcal{MA}_{k-1}^{k+1}$-variety then for every m such that $1 \leqslant m \leqslant h-1$, the variety X^m is again a $\mathcal{MA}_{k-1}^{k+1}$-variety and X^h is a $\mathcal{OA}_{k-1}^{k+1}$-variety;*

(iii) *if X is either an $\mathcal{MA}_{k-1}^{k+1}$-variety or an $\mathcal{OA}_{k-1}^{k+1}$-variety then $\tau_{X,k} : X - - \to X_k \subseteq \mathbb{P}^{n+h}$ is birational and X_k is a variety of dimension n of minimal degree $h+1$. In particular, X is a rational variety and the general member of the movable part of the linear system of k-tangent hyperplane sections is a rational variety.*

Proof. (i) follows by Theorem 4.2, (ii). (ii) follows by Theorem 4.2, (ii) and (v). In (iii), the birationality of $\tau_{X,k}$ follows by Theorem 2.7, (ii). The rest of the assertion follows by Theorem 4.2, (iv). □

Remark 4.6. In the papers [6,7], Bronowski considers the case $k = 1$, $h = 0$ and the case $k \geqslant 2, n = 2, h = 0$. He claims there, without giving a proof, that the converse of Corollary 4.5 holds for $h = 0$. We will call this the *kth Bronowski's conjecture*, a generalized version of which, for any $h \geqslant 0$, can be stated as follows: *Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, projective variety of dimension n. Set $h := \operatorname{codim}(S^k(X))$. If $\tau_{X,k} : X - - \to X_k \subseteq \mathbb{P}^{n+h}$ is birational and X_k is a variety of dimension n and of minimal degree $h+1$, then X is either an $\mathcal{MA}_{k-1}^{k+1}$-variety or an $\mathcal{OA}_{k-1}^{k+1}$-variety, according to whether h is positive or zero*. We call this the *kth generalized Bronowski's conjecture*.

Even the curve case $n = 1$ of this conjecture is still open in general. The results in [18,51,54], imply that the above conjecture is true for X smooth if $k = 1, h = 0$ and $1 \leqslant n \leqslant 3$. The general smooth surface case $n = 2$, $k \geqslant 1$, $h \geqslant 0$ follows by the results in Sections 8 and 9 (see Corollary 9.3). This interesting conjecture is quite open in general.

Bronowski's conjecture would, for example, imply that the converse of (ii) of Corollary 4.5 holds. The following result gives partial evidence for this:

Proposition 4.7. *Let k be a positive integer. Let $X \subset \mathbb{P}^{r+1}$, with $r = (k+1)n + k$, be an irreducible, non-degenerate, not k-defective, projective variety of dimension n. If the general internal projection X^1 of X is a $\mathcal{OA}^{k+1}_{k-1}$-variety, then X is a $\mathcal{MA}^{k+1}_{k-1}$-variety.*

Proof. By (vii) of Lemma 4.1, we have that $\mu_k(X) = 1$ and $\theta_k(X) = 1$. Let $d = \deg(S^k(X))$ and let $p \in X$ be a general point. Then $t_{S^k(X),p} : S^k(X) - - \to \mathbb{P}^r$ is a birational map and therefore $\mathrm{mult}_x(S^k(X)) = d - 1$. Let $p_0, \ldots, p_{k+1}$ be general points of X. Since $S^{k+1}(X) = \mathbb{P}^{r+1}$, then $S^k(X)$ does not contain $\Pi := \langle p_0, \ldots, p_{k+1} \rangle$. Therefore $S^k(X)$ intersects Π in a hypersurface of degree d with multiplicity $d-1$ at $p_0, \ldots, p_{k+1}$. This implies that $d \leqslant k+2$. On the other hand $d \geqslant k+2$ by Theorem 4.2. This proves the assertion. □

It is interesting to remark that the $\mathcal{M}^k$, $\mathcal{OA}^{k+1}_{k-1}$ and $\mathcal{MA}^{k+1}_{k-1}$-properties are essentially preserved under flat limits:

Proposition 4.8. *Let $X, X' \subset \mathbb{P}^r$ be reduced, irreducible, non-degenerate, projective varieties of dimension n, such that $s^{(k)}(X) = s^{(k)}(X')$. Suppose that X' is a flat limit of X and that X is a $\mathcal{M}^k$-variety* [*resp., a $\mathcal{OA}^{k+1}_{k-1}$-variety, a $\mathcal{MA}^{k+1}_{k-1}$-variety*]. *Then X' is also a $\mathcal{M}^k$-variety* [*resp., a $\mathcal{OA}^{k+1}_{k-1}$-variety, a $\mathcal{MA}^{k+1}_{k-1}$-variety*] *and if* $\mathrm{codim}(S^k(X)) = \mathrm{codim}(S^k(X')) > 0$, *then $S^k(X')$ is the flat limit of $S^k(X)$.*

Proof. Suppose X is a $\mathcal{M}^k$-variety, so that $\mathrm{codim}(S^k(X)) = \mathrm{codim}(S^k(X')) > 0$. Let Σ be the flat limit of $S^k(X)$ when X tends to X'. Of course $S^k(X')$ is an irreducible component of Σ, thus by Theorem 4.2 we have

$$\binom{k+h+1}{k+1} \leqslant \deg(S^k(X')) \leqslant \deg(\Sigma) = \deg(S^k(X)) = \binom{k+h+1}{k+1}$$

and therefore the equality has to hold, proving the assertion.

Suppose then X is a $\mathcal{MA}^{k+1}_{k-1}$-variety. The above argument proves that $S^k(X')$ is the flat limit of $S^k(X)$. Hence $\mu_k(X') \leqslant \mu_k(X) = 1$, proving that also $\mu_k(X') = 1$, namely the assertion.

The case in which X is a $\mathcal{OA}^{k+1}_{k-1}$-variety is similar and can be left to the reader. □

Finally we point out the following:

Proposition 4.9. *Let $X \subset \mathbb{P}^r$ be a variety with $\mu_k(X) = 1$, which is k-regular and not k-defective. Then X is linearly normal.*

Proof. Suppose X is not linearly normal. Then there is a variety $X' \subset \mathbb{P}^{r+1}$ and a point $p \notin X'$ such that the projection π from p determines an isomorphism $\pi : X' \to X$. Now we remark that $p \notin S^k(X')$ because of the k-regularity assumption on X. Furthermore, the assumption $\mu_k(X) = 1$ implies that $\pi : S^k(X') \to S^k(X)$ is also birational.

Set, as usual, $h = \operatorname{codim}(S^k(X))$. Then, by Theorem 4.2 we deduce

$$\binom{k+h+1}{h+1} = \deg(S^k(X)) = \deg(S^k(X')) \geqslant \binom{k+h+2}{h+1}$$

a contradiction. □

5. Examples

In this section we give several examples of $\mathcal{MA}^{k+1}_{k-1}$ and $\mathcal{OA}^{k+1}_{k-1}$-varieties.

Example 5.1. *Rational normal scrolls*. Let $X = S(a_1, \ldots, a_n)$ be an n-dimensional rational normal scroll in $\mathbb{P}^r$. We keep the notation introduced in Section 1.7.

We will assume $\sum_{1\leqslant j\leqslant n;\; k\leqslant a_j}(a_j - k) - k - 1 \geqslant 0$, otherwise, according to Proposition 1.14, one has $S^k(X) = \mathbb{P}^r$, a case which is trivial for us.

Claim 5.2. *If* $\sum_{1\leqslant j\leqslant n;\; k\leqslant a_j}(a_j - k) - k - 1 \geqslant 0$, *then* $X = S(a_1, \ldots, a_n)$ *is an* $\mathcal{M}^k$-*variety*.

Proof of Claim 5.2. In order to see this, one may generalize Room's specialization argument (see [50, p. 257]). Indeed, one has a description of $S^k(X) \subset \mathbb{P}^r$ as a determinantal variety as follows (see [11]): the homogeneous ideal of $S^k(X)$ is generated by the minors of order $k+2$ of a suitable matrix of type $(k+2)\times\sum_{1\leqslant j\leqslant n;\; k\leqslant a_j}(a_j - k)$ of linear forms, i.e. a suitable *Hankel* matrix of linear forms. Since by Proposition 1.14 one has $h := \operatorname{codim}(S^k(X)) = \sum_{1\leqslant j\leqslant n;\; k\leqslant a_j}(a_j - k) - k - 1$, then $S^k(X)$ has, as a determinantal variety, the expected dimension. Therefore it is a specialization of the variety defined by the $k+2$ minors of a general matrix of type $(k+2)\times\sum_{1\leqslant j\leqslant n;\; k\leqslant a_j}(a_j - k)$ of linear forms, which, as well known (see [2, Chapter II, Section 5]), has degree equal to $\binom{\sum_{1\leqslant j\leqslant n;\; k\leqslant a_j}(a_j-k)}{k+1}$. As a consequence we have

$$\deg(S^k(X)) = \binom{\sum_{1\leqslant j\leqslant n;\; k\leqslant a_j}(a_j - k)}{k+1} = \binom{h+k+1}{k+1}$$

which proves Claim 5.2. □

Next we assume that X is not k-defective, i.e., according to Proposition 1.14, that $a_1 \geqslant k$. First we will consider the case in which $r = (k+1)n + k$, i.e. $a_1 + \cdots +$

$a_n = kn + k + 1$, $h := \operatorname{codim}(S^k(X)) = 0$, namely $S^k(X) = \mathbb{P}^r$. Then we make the following:

Claim 5.3. *If $a_1 \geqslant k$ and $a_1 + \cdots + a_n = kn + k + 1$, then $X = S(a_1, \ldots, a_n)$ is a $\mathcal{O}A^{k+1}_{k-1}$-variety.*

Proof of Claim 5.3. What we have to prove is that $\mu_k(X) = 1$, i.e. that there is a unique $(k+1)$-secant $\mathbb{P}^k$ to X passing through a general point of $\mathbb{P}^r$.

Since $a_1 \geqslant k$, then $|H - kF|$ is generated by global sections and $h^0(X, \mathcal{O}_X(H - kF)) = \sum_{i=1}^n (a_i + 1 - k) = k(n+1) + 1 - n(k-1) = k + n + 1$. Let

$$\phi_1 = \phi_{|kF|} : X \to \mathbb{P}^k = \mathbb{P}(V_1)$$

and

$$\phi_2 = \phi_{|H-kF|} : X \to \mathbb{P}^{k+n} = \mathbb{P}(V_2).$$

where $V_1 = H^0(X, \mathcal{O}_X(kF))^*$, $V_2 = H^0(X, \mathcal{O}_X(H - kF))^*$. Clearly $\phi_2(X) = S(a_1 - k, \ldots, a_n - k)$, hence $\deg(\phi_2(X)) = k + 1$. Let $\phi = \phi_1 \times \phi_2$. We get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \mathbb{P}^k \times \mathbb{P}^{k+n} \\ \downarrow & & \downarrow \\ \mathbb{P}^r & \hookrightarrow & \mathbb{P}^{(k+1)(k+n+1)-1} := \mathbb{P}_{n,n+k}. \end{array}$$

where the right vertical map is the Segre embedding.

Recall that $\mathbb{P}_{n,n+k} = \mathbb{P}(V_1 \otimes V_2) = \mathbb{P}(\operatorname{Hom}(V_1^*, V_2))$. Thus one has a rational map $\psi : \mathbb{P}_{n,n+k} - - \to \mathbb{G}(k, n+k)$ which associates to the class of a rank $k+1$ homomorphism $\xi : V_1^* \to V_2$ the subspace $\mathbb{P}(\operatorname{Im}(\xi))$ of $\mathbb{P}^{n+k} = \mathbb{P}(V_2)$.

One has a natural $\mathrm{GL}(V_1) = \mathrm{GL}(k+1, \mathbb{C})$-action on $V_1 \otimes V_2$, which descends to a linear $\mathrm{PGL}(k+1, \mathbb{C})$-action on $\mathbb{P}_{n,n+k}$. From the above description of the map ψ, it is clear that the general fibre of ψ is a linear space of dimension $k^2 + 2k$, which is also the closure of a general orbit of this $\mathrm{PGL}(k+1, \mathbb{C})$-action. More precisely, if $x \in \mathbb{P}_{k,n+k}$ is a general point, then x is the class of a homomorphism $\xi : V_1^* \to V_2$, i.e. of a linear embedding $\iota_\xi : \mathbb{P}^k = \mathbb{P}(V_1^*) \to \mathbb{P}^{n+k} = \mathbb{P}(V_2)$. If we denote by $\mathbb{P}^k_x$ the image of ι_ξ, then the closure $\Psi_x \simeq \mathbb{P}^{k^2+2k}$ of the fibre of ψ through x can be interpreted as the linear span of $\operatorname{Seg}(k,k) = \mathbb{P}^k \times \mathbb{P}^k_x \subset \operatorname{Seg}(k, n+k)$. One moment of reflection shows that this $\operatorname{Seg}(k,k) = \mathbb{P}^k \times \mathbb{P}^k_x$ is an *entry locus* in the sense of [58], i.e. it is the closure of the locus of points of $\operatorname{Seg}(k, n+k)$ described by its intersection with the $(k+1)$-secant $\mathbb{P}^k$'s to $\operatorname{Seg}(k, n+k)$ passing through x.

Remark now that ψ is well defined along $\mathbb{P}^r \subset \mathbb{P}_{k,n+k}$. Indeed, up to projective transformations, we may assume that $\phi(X)$ contains $k+1$ given general points of

$\mathbb{P}^k \times \mathbb{P}^{k+n}$. Hence, we can assume that $\mathbb{P}^r$ contains an arbitrarily given point of $S^k(\mathrm{Seg}(k, n+k)) = \mathbb{P}_{n,n+k}$, e.g. a point where ψ is defined. A different proof can be obtained as an application of Kempf's Theorem 1.15 (see Example 5.5 below, we leave the details to the reader). Let us denote by $\widetilde{\psi} : \mathbb{P}^r - - \to \mathbb{G}(k, n+k)$ the restriction of ψ to $\mathbb{P}^r$.

We claim that $\widetilde{\psi}$ is dominant. In fact, take Π a general k-dimensional subspace of $\mathbb{P}^{n+k} = \mathbb{P}(V_2)$. Then Π cuts $\phi_2(X)$ at $k+1$ points $p_0, \dots, p_k$, which, by the way, can be interpreted as $k+1$ general points of X. Consider the points $q_i := \phi_2(p_i) \in \mathbb{P}^k = \mathbb{P}(V_1)$, $i = 0, \dots, k$. Then one has the embedding $\mathbb{P}^k = \mathbb{P}(V_1^*) \to \Pi \subset \mathbb{P}^{n+k} = \mathbb{P}(V_2)$, which, for every $i = 0, \dots, k$, maps the hyperplane $\langle q_0, \dots, q_{i-1}, q_{i+1}, \dots, q_k \rangle$ to the point p_i. As we saw above, the span $\mathbb{P}^k \times \Pi$ is the fibre of ψ over the point of $\mathbb{G}(k, n+k)$ corresponding to Π. We thus see that it intersects $X \subset \mathbb{P}_{k,n+k}$ at the points $p_0, \dots, p_k$.

By the theorem of the dimension of the fibres, the general fibre of $\widetilde{\psi}$ has dimension k. Actually its closure is the intersection of the linear space $\mathbb{P}^r$ with the general fibre of ψ, which is also a linear space of dimension $k^2 + 2k$. Hence we see that this intersection is transversal, i.e. the closure of the general fibre of $\widetilde{\psi}$ is a $\mathbb{P}^k$. By the previous analysis we see that it is in fact a $(k+1)$-secant $\mathbb{P}^k$ to X and that the general such $\mathbb{P}^k$ arises in this way.

In conclusion, since the general $(k+1)$-secant $\mathbb{P}^k$ to X is the fibre of the rational map $\widetilde{\psi} : \mathbb{P}^r - - \to \mathbb{G}(k, n+k)$, we see that there is a unique $(k+1)$-secant $\mathbb{P}^k$ to X passing through the general point of $\mathbb{P}^r$, i.e. $\mu_k(X) = 1$. □

Finally, we consider the case $a_1 \geqslant k$ and $r > (k+1)n+k$, i.e. $a_1+\cdots+a_n > kn+k+1$, $h := h^{(k)}(X) > 0$, thus $S^k(X) \neq \mathbb{P}^r$. In this case we make the

Claim 5.4. *If $a_1 \geqslant k$ and $a_1 + \cdots + a_n > kn + k + 1$, then $X = S(a_1, \dots, a_n)$ is a $\mathcal{M}A^{k+1}_{k-1}$-variety.*

Proof of Claim 5.4. Since X is not defective, by Claim 5.2 all what we have to prove is that $\mu_k(X) = 1$. This easily follows by Lemma 4.1 (vii), and Claim 5.3, by making a sequence of general internal projections. □

Example 5.5. 2-*Veronese fibrations of dimension n and their internal projections from h points*, $1 \leqslant h \leqslant n+1$. Consider $\mathbb{P}(a_1, \dots, a_n)$, with $0 \leqslant a_1 \leqslant \cdots \leqslant a_n$ and $\sum_{i=1}^n a_i \geqslant 2$. Set $k+1 = \sum_{i=1}^n a_i + n$ and consider the map:

$$\phi_1 := \phi_{|H|} : \mathbb{P}(a_1, \dots, a_n) \to S(a_1, \dots, a_n) \subset \mathbb{P}^k.$$

Notice that, since $n \leqslant k-1$, one has $S(a_1, \dots, a_n) \neq \mathbb{P}^k$. Furthermore $|H+F|$ is very ample on $\mathbb{P}(a_1, \dots, a_n)$ and we can consider the embedding:

$$\phi_2 := \phi_{|H+F|} : \mathbb{P}(a_1, \dots, a_n) \to S(a_1 + 1, \dots, a_n + 1) \subset \mathbb{P}^{k+n}$$

Finally let

$$\phi_3 := \phi_{|2H+F|} : \mathbb{P}(a_1, \ldots, a_n) \to \mathbb{P}^r,$$

where

$$\begin{aligned} r &= h^0(\mathbb{P}(a_1, \ldots, a_n), \mathrm{Sym}^2(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)) \otimes \mathcal{O}_{\mathbb{P}^1}(1)) - 1 \\ &= (n+1)\sum_{i=1}^{n} a_i + n(n+1) = (n+1)(k+1) - 1 = (k+1)n + k. \end{aligned}$$

We set $\phi_3(\mathbb{P}(a_1, \ldots, a_n)) = X_{(a_1,\ldots,a_n)}$.

Claim 5.6. *$X := X_{(a_1,\ldots,a_n)}$ is a $\mathcal{OA}^{k+1}_{k-1}$-variety.*

Proof of Claim 5.6. The verification is conceptually similar to the case of rational normal scrolls we worked out in the previous example. Indeed we have a diagram:

$$\begin{array}{ccc} \mathbb{P}(a_1, \ldots, a_n) & \overset{\phi=\phi_1\times\phi_2}{\rightarrow} & \mathbb{P}^k \times \mathbb{P}^{k+n} \\ \downarrow & & \downarrow \\ \mathbb{P}^{nk+n+k} & \hookrightarrow & \mathbb{P}^{(k+1)(k+n+1)-1} := \mathbb{P}_{k,k+n}. \end{array}$$

Consider the restriction $\widetilde{\psi}$ to $\mathbb{P}^{nk+n+k}$ of the rational map $\psi : \mathbb{P}_{k,k+n} - - \to \mathbb{G}(k, k+n)$.

Let us apply Kempf's Theorem 1.15 to the vector spaces $V_1 = H^0(X, \mathcal{O}_X(H))$, $V_2 = H^0(X, \mathcal{O}_X(H+F))$ and $V_3 = H^0(X, \mathcal{O}_X(2H+F))$, where the pairing $V_1 \otimes V_2 \to V_3$ is the obvious multiplication map. By interpreting the elements of V_1, V_2, V_3 as sections of vector bundles on $\mathbb{P}^1$, one immediately sees that the pairing is 1-generic and surjective: we leave the details to the reader. Then the linear span of $\phi(X)$ under the Segre embedding is $\mathbb{P}(V_3^*)$. Moreover, the intersection scheme of $S^{k-1}(\mathrm{Seg}(k, k+n)) = S^{k-1}(\mathrm{Seg}(\mathbb{P}^k, \mathbb{P}^{k+n}))$ and $\mathbb{P}^{k(n+1)+k} = \mathbb{P}(V_3^*)$ is irreducible, reduced, of codimension $n+1$ and of degree $\binom{k+n+1}{k}$ in $\mathbb{P}^{kn+n+k}$.

In particular the restriction of ψ is well defined on $\mathbb{P}^{kn+n+k}$. Then one sees that $S^k(X) = \mathbb{P}^{nk+n+k}$ and $\mu_k(X) = 1$ because the general fibre of $\widetilde{\psi}$ is a general $(k+1)$-secant $\mathbb{P}^k$ of X. $\square$

Actually we can prove more:

Claim 5.7. *One has*

(i) *$X := X_{(a_1,\ldots,a_n)}$ is an $\mathcal{MA}^k_{k-2}$-variety;*
(ii) *the internal projection X^h of X from h points, $1 \leqslant h \leqslant n$, is an $\mathcal{MA}^k_{k-2}$-variety;*
(iii) *the internal projection X^{n+1} of X from $n+1$ points is an $\mathcal{OA}^k_{k-2}$-variety.*

Proof of Claim 5.7. By Corollary 4.5, we need to prove only (i). For this it suffices to observe that, as a consequence of the proof of Claim 5.6, one has that $S^{k-1}(X)$ is a subscheme of the intersection scheme of $S^{k-1}(\mathbb{P}^k, \mathbb{P}^{k+n})$ and of $\mathbb{P}^{kn+n+k}$. Since these two schemes are reduced, irreducible and of the same dimension, they coincide. This yields the desired result

$$\deg(S^{k-1}(X)) = \binom{k+n+1}{k} = \binom{k-1+\operatorname{codim}(S^{k-1}(X))+1}{k-1+1}. \qquad \square$$

We notice that, for $n=2$, we have conic bundles. Actually $\mathbb{P}(a_1, a_2) \simeq \mathbb{F}_a$, where $a = a_2 - a_1$, and $H = E + a_2 F$. Then $2H + F \equiv 2E + (2a_2+1)F = 2E + (a+k)F$, where E is a $(-a)$-curve and F is a ruling, so that $a + k \equiv 1 \pmod 2$.

Example 5.8. 5-*Veronese embedding of* $\mathbb{P}^2$ *and its tangential projections*. In this example we show that the 5-Veronese embedding $X := V_{2,5} \subset \mathbb{P}^{20}$ of $\mathbb{P}^2$ and its general i-tangential projections $X_i \subset \mathbb{P}^{20-3i}$, are smooth $\mathcal{O}\mathcal{A}^{k+1}_{k-1}$-surfaces, with $k = 6 - i$, for $0 \leqslant i \leqslant 3$. Notice that X_3 is nothing else than the general 3-internal projection of $V_{2,4} \subset \mathbb{P}^{14}$, the 4-Veronese embedding of $\mathbb{P}^2$.

We will proceed as in the previous examples and we will slightly modify and adapt to our needs a construction of Shepherd-Barron [55]. Let us first consider the case of $X = V_{2,5}$. Let us consider the incidence correspondence

$$F = \{(x, l) \in \mathbb{P}^2 \times \mathbb{P}^{2*} \ : \ x \in l\}.$$

Then F, as a divisor in $\mathbb{P}^2 \times \mathbb{P}^{2*}$ sits in $|\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2*}}(1,1)|$. Let p_1 and p_2 denote the projections of $\mathbb{P}^2 \times \mathbb{P}^{2*}$ to the two factors. We will use the same symbols to denote the restrictions of p_1 and p_2 to F. Let $\phi = \phi_{|\mathcal{O}_F(1,2)|} : F \hookrightarrow \mathbb{P}^{14}$. Since every fibre of $p_2 : F \to \mathbb{P}^{2*}$ is embedded as a line in $\mathbb{P}^{14}$, we get a morphism $\mathbb{P}^{2*} \to \mathbb{G}(1,14)$, which is $\mathrm{PGL}(3,\mathbb{C})$-equivariant by the obvious action of $\mathrm{PGL}(3,\mathbb{C})$ on $\mathbb{P}^2 \times \mathbb{P}^{2^*}$, on F, etc. (see [55]), and therefore it is an isomorphism to the image. By embedding $\mathbb{G}(1,14)$ into $\mathbb{P}^{104}$ via the Plücker embedding, one has a map $\psi : \mathbb{P}^{2*} \to \mathbb{P}^{104}$, which is an isomorphism to its image X.

Claim 5.9. *The image of ψ lands in a $\mathbb{P}^{20}$ and ψ is the* 5-*Veronese embedding of $\mathbb{P}^{2*}$ to $\mathbb{P}^{20}$.*

Proof of Claim 5.9. First of all we notice that ψ is given by a complete linear system, because it is clearly $\mathrm{PGL}(3,\mathbb{C})$-equivariant. Thus, to prove the claim, it suffices to show that $\deg(X) = 25$. This can be proved by a direct computation, which we leave to the reader, proving that ψ is defined by polynomials of degree 5. However, we indicate here a more conceptual argument (see [55, p. 74]).

Let us introduce the following Schubert cycles in $\mathbb{G} = \mathbb{G}(1, r)$:

$$A = \{l \in \mathbb{G} : l \quad \text{lies in a given hyperplane}\},$$

$$B = \{l \in \mathbb{G} : l \quad \text{meets a given linear space of codimension } 3\},$$

$$C = \{l \in \mathbb{G} : l \quad \text{meets a given linear space of codimension } 2\}.$$

Then C is a hyperplane section of $\mathbb{G}$ in its Plücker embedding and $C^2 \sim A + B$. Note that, in our case $r = 14$, we have $\deg(X) = X \cdot C^2 = X \cdot A + X \cdot B$.

Notice that:

$$X \cdot B = \deg(F) = (p_1^* \mathcal{O}_{\mathbb{P}^2}(1) + p_2^* \mathcal{O}_{\mathbb{P}^{2*}}(2))^3 = 18.$$

Let $H \subset \mathbb{P}^{14}$ be a general hyperplane and let $S = F \cap H$. Then S is the complete intersection of two divisors of type $(1, 1)$ and $(1, 2)$ on $\mathbb{P}^2 \times \mathbb{P}^{2*}$. By adjunction K_S is the restriction to S of a divisor of type $(-1, 0)$, hence $K_S^2 = 2$. Now, $X \cdot A$ is equal to the number of fibres of p_2 lying in H, i.e. the number of exceptional curves contracted by the birational morphism $p_2 : S \to \mathbb{P}^{2*}$. Then $X \cdot A = 9 - K_S^2 = 7$.

In conclusion $\deg(X) = 18 + 7 = 25$ proving Claim 5.9. $\square$

Let us recall now that given a vector space W of odd dimension $2k + 1$, there is a natural rational map $\psi : \mathbb{P}(\Lambda^2 W) - - \to \mathbb{P}(W^*)$, associating to a general alternating 2-form on W^* its kernel. Then the general fibre of ψ is a linear space and the map is defined by forms of degree k vanishing to the order al least $k - 1$ along $\mathbb{G}(1, 2k) \subset \mathbb{P}(\Lambda^2 W)$.

Now we are ready to prove the:

Claim 5.10. *$X := V_{2,5} \subset \mathbb{P}^{20}$ is a $\mathcal{O}\mathcal{A}_5^7$-surface.*

Proof of Claim 5.10. Apply the above remark to $W = H^0(\mathcal{O}_F(1, 2))$, in order to get a rational map $\psi : \mathbb{P}^{104} - - \to \mathbb{P}^{14}$. In [55, Lemma 12], it is shown that the locus of indetermination of ψ does not contain $S^6(X) = \langle X \rangle$ (as for the last equality see [14, Theorem 1.3] or Example 5.14 below). Thus one has a well-defined rational map $\tilde{\psi} : \langle X \rangle = \mathbb{P}^{20} - - \to \mathbb{P}^{14}$, and [55, Lemma 13] ensures that $\tilde{\psi}$ is dominant. Notice that this perfectly fits with the geometry of the situation. Indeed the closure of a general fibre of ψ is a $\mathbb{P}^{90}$, cutting $\langle X \rangle = \mathbb{P}^{20}$ in a linear space of dimension $90 + 20 - 104 = 6$, which is the general fibre of $\tilde{\psi}$. On the other hand, since $\tilde{\psi}$ is defined by forms of degree 7 vanishing to the order at least 6 along X, then $\tilde{\psi}$ contracts every 7-secant $\mathbb{P}^6$ to X. Thus a general 7-secant $\mathbb{P}^6$ to X is a general fibre of $\tilde{\psi}$, which implies $\nu_6(X) = 1$. $\square$

We can slightly modify the above construction to show that the general tangential projection X_i is a $\mathcal{OA}^{7-i}_{5-i}$-surface, for $i = 1, 2, 3$. We will sketch the case $i = 1$ only, since the others follow by iterating the same argument.

Let $p \in \mathbb{P}^{2*}$ be a general point. We consider the line $l := p_2^{-1}(p)$ of F. Notice that $p_1(l)$ is the line of $\mathbb{P}^2$ corresponding to p. Consider the projection $\pi_l : \mathbb{P}^{14} - - \to \mathbb{P}^{12}$ from l and set $F' := \pi_l(F)$. This is again a scroll in lines, and the family of lines of F' is parametrized by a surface $X' \subset \mathbb{G}(1, 12) \subset \mathbb{P}^{77}$.

Claim 5.11. *In the above situation, one has that X' is the tangential projection of $X = V_{2,5}$, the 5-Veronese embedding of $\mathbb{P}^{2*}$, from the point corresponding to p.*

Proof of Claim 5.11. Set $\mathbb{P}^2 \times \mathrm{Bl}_p(\mathbb{P}^{2*}) \supset \widetilde{F} = \mathrm{Bl}_l(F) \to \mathrm{Bl}_p(\mathbb{P}^{2*})$ and let $\phi : \widetilde{F} \to \mathbb{P}^{12}$ be the map given by the linear system $|p_1^*(\mathcal{O}_{\mathbb{P}^2}(1)) + p_2^*(\mathcal{O}_{\mathbb{P}^{2*}}(2)) - \widetilde{E}|$, where p_1 and p_2 are the projections of $\mathbb{P}^2 \times \mathrm{Bl}_p(\mathbb{P}^{2*})$ and $\widetilde{E}$ is the exceptional divisor of $\widetilde{F}$. Then $F' \simeq \phi(\widetilde{F})$ from which it follows that $X' \simeq \mathrm{Bl}_p(\mathbb{P}^{2*})$.

Now, the map $\pi_l : \mathbb{P}^{14} - - \to \mathbb{P}^{12}$ gives rise to a map $\widetilde{\pi}_l : \mathbb{G}(1, 14) - - \to \mathbb{G}(1, 12)$ which is nothing but the tangential projection of $\mathbb{G}(1, 14)$ from the point corresponding to l. This implies that the inclusion $X' \subset \mathbb{G}(1, 12) \subset \mathbb{P}^{77}$ is given by the pull-back on X' of a linear system of quintics of $\mathbb{P}^{2*}$ which are singular at p. To prove the claim it suffices to remark that the embedding $X' \subset \mathbb{G}(1, 12) \subset \mathbb{P}^{77}$ is given, as usual, by a complete linear system. Moreover one has $\deg(X') = 21$. To see this we have to make exactly the same calculation as for the computation of $\deg(X)$. In the present case one has that $X' \cdot B = \deg(F') = 15$ and $X' \cdot A = 6$ so that $\deg(X') = 21$.

Now we notice that $\langle X_1 \rangle = \mathbb{P}^{17} = S^5(X_1)$ (use Terracini's lemma or [14, Theorem 1.3] or Example 5.14 below). Arguing as for X, we have now a map $\psi : \mathbb{P}^{77} - - \to \mathbb{P}^{12}$ which is defined by forms of degree 6 vanishing to the order 5 along $\mathbb{G}(1, 12)$. One proves that $\langle X_1 \rangle$ does not lie in the indeterminacy locus of ψ so that one has a well defined rational map $\tilde{\psi} : \langle X_1 \rangle = \mathbb{P}^{17} - - \to \mathbb{P}^{12}$ and one shows that this map is dominant. The fibres of $\tilde{\psi}$ are the 6-secant $\mathbb{P}^5$'s to X_1, and therefore $v_5(X_1) = 1$. □

Example 5.12. *4-Veronese embedding of $\mathbb{P}^2$ and its internal projections*. In this example we note that $V_{2,4}$ is a $\mathcal{MA}^4_2$-surface. This can be proved by using the formulas in [23,41] to prove that $\deg(S^3(V_{2,4})) = 35$. By Theorem 4.2 (ii), we see that also that a general i-internal projection of $V_{2,4}$, $i = 1, 2$, has the same property.

Another interesting property of $V_{2,4}$ is that it is 4-defective and $S^4(V_{2,4})$ is a hypersurface in $\mathbb{P}^{14}$ (see [14, Theorem 1.3] or Example 5.14 below). One has $\deg(S^4(V_{2,4})) = 6$, hence $V_{2,4}$ is a $\mathcal{M}^4$-surface. This can be proved as follows. Look at $V_{2,4}$ as that 2-Veronese embedding of $V_{2,2} \subset \mathbb{P}^5$. Thus $S^4(V_{2,4}) \subseteq \langle V_{2,4} \rangle \cap S^4(V_{5,2})$, where $S^4(V_{5,2})$ is a hypersurface of degree 6. Notice that $\langle V_{2,4} \rangle$ is not contained in $S^4(V_{5,2})$. In fact, since $V_{2,2}$ is non-degenerate in $\mathbb{P}^5$, then given 6 general points of $V_{5,2}$ we can suppose that $V_{2,4}$ contains them. Thus, we may assume that $\langle V_{2,4} \rangle$ contains a general point of $S^5(V_{5,2}) = \mathbb{P}^{20}$ which can be chosen to be off $S^4(V_{5,2})$. Finally we know, by Theorem 4.2, that $\deg(S^4(V_{2,4})) \geqslant 6$. This implies that $S^4(V_{2,4})$ is the scheme-theoretic intersection of $\langle V_{2,4} \rangle$ and $S^4(V_{5,2})$ and that $\deg(S^4(V_{2,4})) = 6$.

Using this same line of argument, one can give a direct, more geometric proof that $\deg(S^3(V_{2,4})) = 35$. We leave the details to the reader.

Example 5.13. *The 3-Veronese embedding of the quadric surface in* $\mathbb{P}^3$. Let $X \subset \mathbb{P}^{15}$ be the 3-Veronese embedding of a smooth quadric surface $Q \subset \mathbb{P}^3$. Then X is a $\mathcal{MA}_3^5$-surface, i.e. $S^4(X) \subset \mathbb{P}^{15}$ is a hypersurface of degree 6. Indeed, the projection of X from a point on it is isomorphic to the 2-tangential projection of the 5-Veronese embedding of $\mathbb{P}^2$, which is a $\mathcal{OA}_3^5$-surface, see Example 5.8. The conclusion follows from Proposition 4.7.

By applying Proposition 4.8, one sees that also the 3-Veronese embedding of a quadric cone in $\mathbb{P}^3$ is a $\mathcal{MA}_3^5$-surface.

Example 5.14. *Defective surfaces*. The fact that $V_{2,4}$ is a $\mathcal{M}^4$-surface is a particular case of a more general family of examples of surfaces with minimal secant degree.

According to [14, Theorem 1.3], this is the list of k-defective surfaces $X \subset \mathbb{P}^r$:

(i) $r = 3k + 2$ and X is the 2-Veronese embedding of a surface of degree k in $\mathbb{P}^{k+1}$, and $\delta_k(X) = 1$;
(ii) X sits in a $(k + 1)$-dimensional cone over a curve.

We claim that the surfaces of type (i) are $\mathcal{M}^k$-surfaces. In fact such a X is contained in $V_{k+1,2}$ and therefore $S^k(X) \subseteq \langle X\rangle \cap S^k(V_{k+1,2})$. Here again we have that:

- $\langle X\rangle$ is not contained in $S^k(V_{k+1,2})$;
- $S^k(V_{k+1,2})$ is a hypersurface of degree $k + 2$, i.e. it is the set of singular quadrics in $\mathbb{P}^{k+1}$;
- $\deg(S^k(X)) \geqslant k + 2$, by Theorem 4.2.

These three facts together imply that the hypersurface $S^k(X)$ is the scheme-theoretic intersection of $\langle X\rangle$ and $S^k(V_{k+1,2})$ and that $\deg(S^k(X)) = k + 2$.

The first instance of this family of examples, obtained for $k = 1$, is the Veronese surface $V_{2,2}$ in $\mathbb{P}^5$, whose secant variety is a hypersurface of degree 3.

Example 5.15. *Weakly defective surfaces*. The previous example can be further extended.

According to [14, Theorem 1.3], this is the list of k-weakly defective, not k-defective, surfaces $X \subset \mathbb{P}^r$:

(i) $r = 9$, $k = 2$ and X is the 2-Veronese embedding of a surface of degree $d \geqslant 3$ in $\mathbb{P}^3$;
(ii) $r = 3k + 3$, and X is the cone over a k-defective surface of type (i) in Example 5.14;
(iii) $r = 3k + 3$, and X is the 2-Veronese embedding of a surface of degree $k + 1$ in $\mathbb{P}^{k+1}$;
(iv) X sits in a $(k + 2)$-dimensional cone over a curve C, with a vertex of dimension k.

We claim that the surfaces of types (i), (ii) and (iii) are $\mathcal{M}^k$-surfaces.

If X is a surface of type (i), one immediately sees that $S^2(X) = S^2(V_{3,2})$, hence $\deg(S^2(X)) = 4$ and X is therefore a $\mathcal{M}^2$-surface.

If X is a surface of type (ii), then $S^k(X)$ is the cone over the k-secant variety of a k-defective surface of type (i) in Example 5.14. Hence we have $\deg(S^k(X)) = k + 2$ and X is a $\mathcal{M}^k$-surface.

If X is a surface of type (iii), the same argument we made in Example 5.14 proves our claim. We leave the details to the reader.

Example 5.16. *Del Pezzo surfaces*. In this example we remark that smooth del Pezzo surfaces of degree r in $\mathbb{P}^r$, $r = 5, \dots, 9$, are $\mathcal{M}A_0^2$-surfaces. This can be easily seen by applying the double point formula. Proposition 4.8 implies that also singular del Pezzo surfaces are $\mathcal{M}A_0^2$-surfaces.

The Veronese surface $X := V_{2,3}$ is also an $\mathcal{M}A_1^3$-surface, as can be seen by applying Le Barz's formula [40]. However this is a classical result. Indeed $S^2(V_{2,3})$ is the hypersurface of $\mathbb{P}^9$ consisting of all cubics which are sums of three cubes of linear forms. These are the so-called *equihanarmonic* cubics, i.e. those characterized by the vanishing of the J-invariant. It is classically well known that there are four equihanarmonic cubics in a general pencil (see [25, p. 194]), i.e. $\deg(S^2(V_{2,3})) = 4$, which means that $V_{2,3}$ is a $\mathcal{M}A_1^3$-surface.

We can also give a more geometric proof of this fact by applying the ideas we have developed so far. Indeed, the general internal projection X^1 of X is the embedding of $\mathbb{F}_1$ in $\mathbb{P}^8$ via the linear system $|2E + 3F|$. This, according to Example 5.5, is a $\mathcal{O}\mathcal{A}_1^3$-surface. Thus X is a $\mathcal{M}A_1^3$-surface by Proposition 4.7.

Example 5.17. *Cones*. Let $X \subset \mathbb{P}^r \subset \mathbb{P}^{r+l+1}$, $l \geqslant 0$, be an irreducible variety of dimension n which is non-degenerate in $\mathbb{P}^r$. Let $L = \mathbb{P}^l \subset \mathbb{P}^{r+l+1}$ be such that $L \cap \mathbb{P}^r = \emptyset$. Let $Y = J(L, X)$ be the cone over X with vertex L. Then $\dim(Y) = n+l+1$. More generally for every $k \geqslant 1$ we have $S^k(Y) = S(L, S^k(X))$ so that $s^{(k)}(Y) = s^{(k)}(X)+l+1$. Therefore $h^{(k)}(Y) = r+l+1-s^{(k)}(Y) = r-s^{(k)}(X) = h^{(k)}(X)$. Moreover $\deg(S^k(Y)) = \deg(S^k(X))$ for every $k \geqslant 1$. In particular X has minimal k-secant degree if and only if Y has also minimal k-secant degree.

For instance, a rational normal scroll $X = S(a_1, \dots, a_n)$ is a variety of minimal k-secant degree if the least positive integer a_i is greater or equal than k (see Example 5.1).

The next example is a slight modification of the previous one. It shows that some of the hypotheses we will make in our classification theorems in Sections 8 and 9 are well motivated. The first instance of this example, i.e. the case $k = 1$, is due to A. Verra, who kindly communicated it to us. It could be easily generalized to higher dimensions and codimensions: we leave the details to the reader.

Example 5.18. Let $C \subset \mathbb{P}^{2k+1+h} \subset \mathbb{P}^{3k+2+h}$, $k \geqslant 1, h \geqslant 0$, be an irreducible curve, non-degenerate in $\mathbb{P}^{2k+1+h}$. Take $\Pi = \mathbb{P}^k \subset \mathbb{P}^{3k+2+h}$ such that $\Pi \cap \mathbb{P}^{2k+1+h} = \emptyset$ and a morphism $\phi : C \to C' \subset \mathbb{P}^k$ and take $X = \cup_{p \in C} \langle p, \phi(p) \rangle \subset \mathbb{P}^{3k+2+h}$. Then $v_k(X) = v_k(C)$. This is an exercise in projective geometry which we leave to the reader.

In particular, from Example 5.1 and from Theorem 6.1 below, we deduce that $v_k(X) = 1$ if and only if C is a rational normal curve. As soon as $k \geqslant 3$, one can take as ϕ a general projection of C and obtain examples of smooth surfaces $X \subset \mathbb{P}^{3k+2+h}$, which are not linearly normal. Let us remark that such a surface X is k-weakly defective, being contained in a cone of vertex a $\mathbb{P}^k$ over the curve C, see [14, Theorem 1.3 and Example 5.15].

6. Classification of curves with minimal secant degree

In this section we take care of the classification of curves with minimal k-secant degree.

Let $C \subset \mathbb{P}^r$ be an irreducible non-degenerate curve. Then C is never defective, so that $s^{(k)}(C) = \min\{2k+1, r\}$. This is classically well known and, by the way, follows also from the fact that C is not weakly defective (see [14]). The classification of curves with minimal k-secant degree is given by the following:

Theorem 6.1. *Let $C \subset \mathbb{P}^r$ be an irreducible non-degenerate curve. Let $k \geqslant 1$ be an integer such that $2k+1 \leqslant r$. Then C is an $\mathcal{MA}^{k+1}_{k-1}$ or an $\mathcal{OA}^{k+1}_{k-1}$-variety if and only if C is a rational normal curve.*

Proof. As we saw in Example 5.1, a rational normal curve is an $\mathcal{MA}^{k+1}_{k-1}$ or an $\mathcal{OA}^{k+1}_{k-1}$-variety.

Suppose, conversely, that C is an $\mathcal{MA}^{k+1}_{k-1}$ or an $\mathcal{OA}^{k+1}_{k-1}$-variety. In the latter case, i.e. if $r = 2k+1$, then the assertion is Theorem 3.4 of Catalano-Johnson [12]. In the former case, i.e. if $h = r - 2k - 1 > 0$, then (ii) of Corollary 4.5 tells us that C^h is an $\mathcal{OA}^{k+1}_{k-1}$-variety. Since, as we saw, C^h is a rational normal curve, then C itself is a rational normal curve, proving the assertion. □

Remark 6.2. Notice that, in the hypotheses of Theorem 6.1, the rationality of C follows by Corollary 4.5. If one adds the hypothesis that C is k-regular, then the assertion follows right away from Proposition 4.9.

7. On a theorem of Castelnuovo–Enriques

The next sections will be devoted to the classification of $\mathcal{OA}^{k+1}_{k-1}$-surfaces and $\mathcal{M}^k$-surfaces. For this we will need some preliminaries, which we believe to be of independent interest, concerning linear systems of curves on a surface. Indeed the present section is devoted to review, and improve on, a classical theorem of Enriques, which in turn generalizes to arbitrary surfaces an earlier result proved by Castelnuovo for rational surfaces, see [8,24]. The expert reader will find relations between the results of this section and the ones in [33,49]. We will freely use here the notation introduced in Sections 1.11 and 1.12.

The basic tool in this section is Proposition 7.1 below. This result essentially goes back to Iitaka [36] and Dicks [20, Theorem 3.1], though under the stronger assumption that D is an irreducible smooth curve. The case D ample is also well known in the literature, e.g. see [38]. The short proof below, based on Mori's theory, is essentially the same as in [20], and we included it here for the reader's convenience.

Proposition 7.1. *Let X be a smooth, irreducible, projective surface. Let D be a nef divisor on X. Set $d := D^2$, $g := p_a(D)$. Assume the pair (X, D) is minimal, not a h-scroll with $h \leqslant 1$ and it is not a m-Veronese pair with $m \leqslant 2$. Then $K + D$ is nef and therefore*:

(i) $d \leqslant 4(g-1) + K^2$;
(ii) *$g \geqslant 1$ and equality holds if and only if K and D are numerically dependent and either $d = 0$ or (X, D) is a del Pezzo pair.*

Proof. Let C be a curve on X such that $C \cdot (K + D) < 0$. Since D is nef, one has $K \cdot C < 0$. By Mori's cone Theorem (see [45, Theorem 1.4]), the curve C is a linear combination of extremal rays. More precisely, there are extremal rays $E_1, \ldots, E_h$ such that $C \sim \sum_{i=1}^{h} m_i E_i$, with $m_1, \ldots, m_h$ positive real numbers. Thus there is one of the extremal rays $E_1, \ldots, E_h$, e.g. $E := E_1$ such that $E \cdot (K + D) < 0$. Now one concludes by separately discussing the various possibilities for E (cf. [45, Theorem 2.1]):

- if E is a (-1)-curve, one has $K \cdot E = -1$ and therefore $D \cdot E = 0$, against the minimality of (X, D);
- if $E \simeq \mathbb{P}^1$ and $E^2 = 0$, one has $K \cdot E = -2$ and therefore $D \cdot E \leqslant 1$, against the fact that (X, D) is not a h-scroll for $h \leqslant 1$;
- if $E \simeq \mathbb{P}^1$ e $E^2 = 1$, one has $K \cdot E = -3$ and therefore $1 \leqslant D \cdot E \leqslant 2$, against the fact that (X, D) is not a m-Veronese with $m \leqslant 2$.

Now notice that:

$$(K + D)^2 = K^2 + 4(g-1) - d \tag{7.1}$$

Since $K + D$ is nef, one has $(K + D)^2 \geqslant 0$, so that

$$d \leqslant 4(g-1) + K^2, \tag{7.2}$$

proving (i).

Similarly, since $K + D$ is nef, one has $2g - 2 = (K + D) \cdot D \geqslant 0$, proving the first assertion of (ii). If $g = 1$, one has $(K + D) \cdot D = 0$. Then the Hodge index theorem implies that $K + D$ and D are numerically dependent, thus $K \sim lD$, for some rational number l. If $d > 0$ then $0 = (K + D) \cdot D = (l + 1)d$ implies $l = -1$ and (X, D) is a del Pezzo pair. Conversely if (X, D) is a del Pezzo pair then $g = 1$. Similarly, if $d = 0$ and K and D are numerically dependent, one has $g = 1$. $\square$

Corollary 7.2. *Let X be a smooth, irreducible, projective surface. Let D be a nef divisor on X. Assume the pair (X, D) is not a h-scroll with $h \leqslant 1$. Set $g := p_a(D)$. Then $g \geqslant 0$ and $g = 0$ if and only if (X, D) is obtained by a m-Veronese with $m \leqslant 2$ with a sequence of blowing-ups.*

Proof. By iterated contractions of (-1)-curves E such that $E \cdot D = 0$, we arrive to a minimal pair (X', D') such that (X, D) is obtained from (X', D') with a sequence of blowing-ups. Moreover $g' := p_a(D') = g$. Notice that (X', D'), as well as (X, D), is not a h-scroll with $h \leqslant 1$. Then the assertion follows by the second part of Proposition 7.1. □

As a consequence we have the following result, essentially due to Castelnuovo [8] and Enriques [24]. The bound (7.3) was also obtained by Hartshorne, [33, Corollary 2.4 and Theorem 3.5], under the assumption D smooth irreducible curve. Hartshorne does not consider the classification of the extremal cases, as in [8], but he remarks that the bound is sharp looking at the cases (i) and (iv) with $a = 0$, Example in [33, p. 121]. All the results of Hartshorne are now straightforward consequences of Proposition 7.1.

Theorem 7.3. *Let X be a smooth, irreducible, projective surface. Let D be an irreducible curve on X. Set $d := D^2$, $g := p_a(D)$, $r := \dim(|D|)$. Assume $d \geqslant 0$ and the pair (X, D) is not a h-scroll with $h \leqslant 1$. Then*:

$$d \leqslant 4g + 4 + \varepsilon, \tag{7.3}$$

where $\varepsilon = 1$ if $g = 1$ and $\varepsilon = 0$ if $g \neq 1$. Consequently one has

$$r \leqslant 3g + 5 + \varepsilon \tag{7.4}$$

and the equality holds in (7.3) if and only if it holds in (7.4).

If, in addition, the pair (X, D) is minimal, then the equality holds in (7.3), or equivalently in (7.4), if and only if one of the following happens:

(i) $g = 0$, $r = 5$, *and (X, D) is a 2-Veronese pair*;
(ii) $g = 1$, $r = 9$, *and (X, D) is a 3-Veronese pair*;
(iii) $g = 3$, $r = 14$, *and (X, D) is a 4-Veronese pair*;
(iv) *(X, D) is a $(2, a + g + 1)$-pair on $X \simeq \mathbb{F}_a$, $a \geqslant 0$.*

Proof. By arguing as in the proof of Corollary 7.2 we may, and will, assume that the pair (X, D) is minimal. Then note that if (X, D) is a m-Veronese with $m \leqslant 2$, both (7.3) and (7.4) hold. So we may assume (X, D) is not a m-Veronese with $m \leqslant 2$.

Let us now prove (7.3). The divisor D is nef so that bound (7.2) holds.

Assume that $d > 4g+4+\varepsilon$. Then $K \cdot D = 2g-2-d \leqslant -2g-6-\varepsilon < 0$. Therefore $\kappa(X) = -\infty$. Moreover:

$$4g+4+\varepsilon < d \leqslant 4g-4+K^2$$

yields $K^2 \geqslant 9+\varepsilon$. Therefore $\varepsilon = 0$, i.e. $g \neq 1$, $K_X^2 = 9$ and $X \simeq \mathbb{P}^2$. Hence $D \in |\mathcal{O}_{\mathbb{P}^2}(m)|$, with $m \geqslant 4$, since (X, D) is not a Veronese pair with $m \leqslant 2$ and $g \neq 1$. For such a D one has $m^2 = d \leqslant 4g+4 = 2m^2 - 6m + 8$. This contradiction proves (7.3).

Next we remark that (7.3) implies (7.4). Indeed, since the general curve $D \in |D|$ is irreducible, by Riemann–Roch theorem we have $r \leqslant \max\{d-g+1, g\}$, which implies (7.4).

Let us prove now that equality holds in (7.3) if and only if equality holds in (7.4). The above argument shows that if equality holds in (7.3) then it holds in (7.4). Conversely, if equality holds in (7.4) then Riemann–Roch theorem implies that $d-g+1 \geqslant r$ and equality holds in (7.3).

Finally, suppose equality holds in (7.3). Then reasoning as above we deduce $\kappa(X) = -\infty$ and $K^2 \geqslant 8+\varepsilon$. Therefore if $g = 1$ one has $K^2 = 9$, (X, D) is a del Pezzo pair and we are in case (ii). We can thus suppose $\varepsilon = 0$ in (7.3) and hence $K^2 \geqslant 8$.

If $K^2 = 9$, then $X \simeq \mathbb{P}^2$, $D \in |\mathcal{O}_{\mathbb{P}^2}(m)|$, with $m \geqslant 1$. The equality $d = 4g+4$ is translated into $m^2 = 2m^2 - 6m + 8$, so that $m = 2$ or 4 and we get cases (i) and (iii).

Assume that $K^2 = 8$. Thus $X \simeq \mathbb{F}_a$, $a \geqslant 0$. Furthermore (7.1) shows that $(K+D)^2 = 0$ holds. One has:

$$D \sim \alpha E + \beta F,$$

where E is a $(-a)$-curve and F a fibre of the ruling of $\mathbb{F}_a$, with $\beta \geqslant a\alpha$ because $D \cdot E \geqslant 0$, and $\alpha \geqslant 2$ since the pair (X, D) is not a scroll. On the other hand:

$$K \sim -2E - (a+2)F$$

and therefore

$$K + D \sim (\alpha - 2)E + (\beta - a - 2)F.$$

If $\alpha = 2$ then adjunction formula implies

$$\beta = a + g + 1$$

i.e. the assertion. Now

$$(K+D)^2 = (\alpha - 2)(2\beta - a\alpha - 4).$$

If $a=0$, $(K+D)^2=0$ implies either $\alpha=2$ or $\beta=2$, and we are done. If $a=1$, the minimality condition yields $\beta \geqslant \alpha+1$. Therefore $(K+D)^2=0$ implies $\alpha=2$, and we are done again. If $a \geqslant 2$, one has $2\beta - a\alpha - 4 \geqslant a\alpha - 4 = 2(\alpha-2)$. Then $(K+D)^2=0$ implies $\alpha=2$, and we conclude as above. □

Remark 7.4. Proposition 7.1 can be improved. Indeed, we can prove that if one adds the hypothesis that D is effective and big, then $K+D$ is also effective. This can be seen as a wide extension of the results in [3, pp. 196–200]. Following the ideas in [9] one can even give suitable, interesting lower bounds for $(K+D)^2$.

It is also possible to partly extend Proposition 7.1 to higher dimensional varieties.

The hypothesis D effective and irreducible in Theorem 7.3 is essentially used to prove that (7.3) implies (7.4) and it is too strong. Indeed, we can prove that it suffices to assume that either $g \neq 1$ or $d > 0$. However the proof, based on the aforementioned extensions of Proposition 7.1 as indicated in [9], is rather long and we decided not to put it here. We plan to come back to this and to other extensions of Proposition 7.1 and Theorem 7.3 in the future.

Definition 7.5. If the pair (X, D) is as in (iv) of Theorem 7.3, we will say that it is a (a, g)-*Castelnuovo pair* and the corresponding surface $\phi_{|D|}(X) \subset \mathbb{P}^{3g+5}$ of degree $d = 4g+4$, with hyperelliptic hyperplane sections, will be called an (a, g)-*Castelnuovo surface* and denoted by $X_{a,g}$. The motivation for this definition resides in the fact that Castelnuovo first considered these pairs in his paper [8]. In general, a pair like in (i)–(iii) or (iv) of Theorem 7.3, will be called a *Castelnuovo extremal pair*.

We notice that pairs (X, D) as in (ii), (iii) or (iv) can be characterized as those with D effective, irreducible and nef for which the hypotheses of Proposition 7.1 are met, so that $K+D$ is nef, but $K+D$ is not big.

Remark 7.6. An (a, k)-Castelnuovo surface $X_{a,k}$ is $(k+1)$-defective as soon as $a + 1 + k \equiv 0 \pmod 2$ (see case (i) of Theorem 1.3 of [14] and Example 5.14). In this case the Castelnuovo surface will be said to be *even*. Instead $X_{a,k}$ is an $\mathcal{O}A_k^{k+2}$ surface if $a+1+k \equiv 1 \pmod 2$, and then the Castelnuovo surface will be said to be *odd*. In fact in this case $X_{a,k}$ is one of the surfaces described in Example 5.5.

Note that an (a, k)-Castelnuovo surface $X_{a,k}$ is smooth unless $k = a-1$, in which case the Castelnuovo surface is even and it is the 2-Veronese embedding of a cone over a rational normal curve of degree a.

It is useful to point out the following immediate corollaries, whose easy proofs can be left to the reader:

Corollary 7.7. *Let X be a smooth, irreducible, projective surface. Let $\mathcal{L}$ be a linear system of dimension $r > 0$ whose general divisor D is irreducible with geometric genus g. Let $(X', \mathcal{L}')$ be the resolution on $(X, \mathcal{L})$. Suppose $(X', \mathcal{L}')$ is not a scroll. Then* (7.4) *of Theorem* 7.3 *holds. If, in addition, $(X', \mathcal{L}')$ is minimal and equality holds in* (7.4), *then $(X, \mathcal{L}) = (X', \mathcal{L}')$ and $\mathcal{L}$ is base point free, complete and the pair (X, D) is as in* (i)–(iii) *or* (iv) *of Theorem* 7.3.

Corollary 7.8. *Let $X \subset \mathbb{P}^r$, $r \geqslant 3g+5$, $g \geqslant 2$, be an irreducible, non-degenerate surface which is not a scroll and having general hyperplane section D of geometric genus g. Then $r = 3g+5$, the surface X is linearly normal, of degree $4g+4$ and it is one of the following*:

(i) $g = 3, r = 14$ *and* $X = V_{2,4}$ *is the* 4*-Veronese embedding of* $\mathbb{P}^2$ *in* $\mathbb{P}^{14}$;
(ii) $X = X_{a,g}$ *is a smooth* (a, g)*-Castelnuovo surface, with* $0 \leqslant a \leqslant g$;
(iii) X *has only one singular point and it is the* 2*-Veronese embedding of a cone over a rational normal curve of degree* a, $a \geqslant 3$ *and* $g = a - 1$, *i.e.* $X = X_{g+1,g}$ *is a* $(g+1, g)$*-Castelnuovo surface.*

We finish this section by proving a slight extension of the above results, which will be essential in our subsequent classification theorems. Further generalizations, in the spirit of [8] or [49], can be obtained, but we will not consider them here, since we will not use them now. Similarly, we refrain from formulating the next result in its maximal generality, i.e. for big and nef, but not necessarily ample, pairs, since we will not need such a generality here.

Theorem 7.9. *Let X be a smooth, irreducible, projective surface. Let D be an effective ample divisor on X. Set $d := D^2$, $g := p_a(D)$, $r := \dim(|D|)$. Assume that $g \geqslant 2$ and that the pair (X, D) is minimal, not a scroll and suppose that $r = 3g + 5 - s$, with $1 \leqslant s \leqslant 3$. Then X is rational, D is very ample, and one of the following cases occurs*:

(i) (X, D) *is a projection of a* 4*-Veronese pair from* $i = 1, 2, 3$ *points. One has* $g = 3$, $d = 16 - s$ *and* $s = i$;
(ii) (X, D) *is a projection of an* (a, g)*-Castelnuovo pair, with* $0 \leqslant a \leqslant g$, *from* $i = 1, 2, 3$ *points. One has* $d = 4g + 4 - s$, $s = i$;
(iii) $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ *and* D *is of type* $(3, 3)$ *on* X. *One has* $g = 4$, $d = 18$ *and* $s = 2$;
(iv) (X, D) *is the tangential projection of a* 5*-Veronese pair from* $i = 0, 1, 2$ *points. One has* $g = 6 - i$, $d = 25 - 4i$, $s = 3$.

Proof. By the theorem of Riemann–Roch we have $d - g + 1 \geqslant r \geqslant 3g + 5 - s$, hence $d \geqslant 4g + 4 - s$. Moreover, by (7.2), $d \leqslant 4g - 4 + K^2$, so that $K^2 \geqslant 8 - s \geqslant 5$ and X is rational since $K \cdot D = 2g - 2 - d \leqslant -2g - 1 < 0$. By (7.1), we have

$$(K + D)^2 = K^2 - 8 + s. \tag{7.5}$$

Notice that $D^2 = d \geqslant 4g + 1 \geqslant 9$ implies, by Reider's theorem (see [5]) and the hypotheses D ample and (X, D) not a scroll, that $|K + D|$ is base point free. So either $(K+D)^2 = 0$ and $|K+D|$ is composite with a base point free pencil $|M|$, or the general curve $C \in |K + D|$ is smooth and irreducible. Note also that $\dim(|K + D|) = g - 1$. Hence if $g = 2$, then $|K + D|$ is a base point free pencil and therefore $(K + D)^2 = 0$.

Assume that $K^2 = 9$, i.e. $X \simeq \mathbb{P}^2$. Then (7.5) implies that $(K + D)^2 = 1 + s$. So the only possibility is $s = 3$ and (X, D) is a 5-Veronese pair.

From now on we will assume $K^2 \leqslant 8$ and therefore $0 \leqslant (K + D)^2 \leqslant s \leqslant 3$ by (7.5). We examine separately the various cases.

If $(K+D)^2=0$ and $|K+D|$ is composite with a base point free pencil $|M|$, the general curve in $|D|$ is hyperelliptic and therefore $D\cdot M=2$. Since $M\cdot(K+D)=0$, we have $K\cdot M=-D\cdot M=-2$, and $M^2=0$ yields that the general curve in $|M|$ is rational. By (7.5) we have $K^2=8-s$, so we have s reducible curves in $|M|$, which are formed by pairs of (-1)-curves meeting transversally at one point and both meeting D at one point. By contracting s disjoint of these (-1)-curves, we have a morphism $p: X\to\mathbb{F}_a$, for some $a\geqslant 0$. Let $D'=p_*(D)$. Then $p_a(D')=g$ and $D'^2=d+s=4g+4$. Then, by Theorem 7.3 and Corollary 7.8, we conclude we are in case (ii).

If $(K+D)^2=1$, then $\phi_{|K+D|}$ is a birational morphism of X to $\mathbb{P}^2$, hence X is the blow-up of $\mathbb{P}^2$ at $9-K^2=s$ points $x_1,\ldots,x_s$. If E is a (-1)-curve contracted by $|K+D|$, then one has $E\cdot(K+D)=0$, hence $E\cdot D=-E\cdot K=1$, which means that the image of $|D|$ in $\mathbb{P}^2$ has simple base points at $x_1,\ldots,x_s$. Furthermore $g-1=\dim(|K+D|)=2$, hence $g=3$. We are thus in case (i).

If $(K+D)^2=2$, then the series cut out by $|K+D|$ on its general curve C is a complete g_2^{g-2}, which implies $g\leqslant 4$.

If $g=4$, then C is rational and $\phi_{|K+D|}$ is a birational morphism of X to a quadric in $\mathbb{P}^3$. Thus X is the blow-up of $\mathbb{F}_a$, $a=0,2$, at $8-K^2=s-2$ points. Note that the ampleness hypothesis on D rules out the case $a=2$. Then $s-2\geqslant 0$, namely $2\leqslant s\leqslant 3$. If $s=2$, then we clearly are in case (iii), whereas, if $s=3$, we are in case (iv), $i=2$.

Suppose $g=3$. Let C be the general curve in $|K+D|$. One computes $(K+C)\cdot C=0$ and $(K+C)^2=(2K+D)^2=8-s>0$. This contradicts the Hodge index theorem.

If $(K+D)^2=3$, then the series cut out by $|K+D|$ on its general curve C is a complete g_3^{g-2}, which implies $g\leqslant 5$. On the other hand (7.5) implies that $s=3$, $K^2=8$, i.e. X is a surface $\mathbb{F}_a$, for some $a\geqslant 0$.

If $g=5$, then C is rational and $\phi_{|K+D|}$ is then an isomorphism of X to $\mathbb{F}_1$ embedded in $\mathbb{P}^4$ as a rational normal cubic scroll. It is then clear that we are in case (iv), $i=1$.

If $g\leqslant 4$, one computes $(K+C)\cdot C=8-2g$ and $(K+C)^2=21-4g$, which contradicts the Hodge index theorem.

The proof is thus completed. □

The pairs listed in (i)–(iv) of Theorem 7.9 above will be called *almost extremal Castelnuovo pairs*. The corresponding surfaces $\phi_{|D|}(X)$ will be called *almost extremal Castelnuovo surfaces*.

8. The classification of $\mathcal{OA}_{k-1}^{k+1}$-surfaces

In this section we give the classification of surfaces $X\subset\mathbb{P}^{3k+2}$, $k\geqslant 2$ with $v_k(X)=1$. Recall that the case $k=1$ was classically considered by Severi [54] and proved by Russo [51] (see also [18]). We notice that this classification was in part divined by Bronowski in [7], where however the argument he gives relies on the unproved conjecture stated in Remark 4.6.

Theorem 8.1. *Let $X \subset \mathbb{P}^{3k+2}$, $k \geqslant 2$, be a smooth, projective, surface which is linearly normal, and such that $v_k(X) = 1$. We let d be the degree and g be the sectional genus of X. Then X is one of the following*:

(i) *a rational normal scroll $S(a_1, a_2)$ with $k \leqslant a_1 \leqslant a_2$, $d = a_1 + a_2 = 3k + 1$ and sectional genus $g = 0$ (see Example 5.1)*;
(ii) *an odd Castelnuovo surface $X_{a,k-1}$, with $0 \leqslant a \leqslant k-1$ and $a + k \equiv 1 \pmod 2$ (see Example 5.5 and Remark 7.6). In this case $d = 4k$, $g = k-1$ and the hyperplane sections of X are hyperelliptic curves*;
(iii) *the internal projection from three distinct points of a Castelnuovo surface $X_{a,k} \subset \mathbb{P}^{3k+5}$ with $0 \leqslant a \leqslant k$. In this case $d = 4k + 1$ and $g = k$ and the hyperplane sections are hyperelliptic curves (see Example 5.5)*;
(iv) *the tangential projection of a 5-Veronese surface $V_{2,5}$ from $i = 0, 1, 2, 3$ points (see Example 5.8). Here $d = 25 - 4i$, $g = k = 6 - i$.*

Proof. From the classification of weakly defective surfaces (see [14, Theorem 1.3 and Example 5.15] above), we see that X, being not k-defective and spanning a $\mathbb{P}^{3k+2}$, is also not k-weakly defective. We can, and will, therefore apply Proposition 1.6. Let $p_1, \ldots, p_k \in X$ be general points and let $\mathcal{L}$ be the linear system of hyperplane sections of X tangent at $p_1, \ldots, p_k$. Since X is not $(k-1)$-defective, we have $\dim(\mathcal{L}) = 2$. Moreover $\mathcal{L} = F + \mathcal{M}$, where F is the fixed part and $\mathcal{M}$ the movable part, as described in Proposition 1.6. The relevant information is that, by Theorem 2.7, $\tau_{X,k} : X - - \to \mathbb{P}^2$ is birational, hence X is rational and the general curve $M \in \mathcal{M}$ is rational and $\mathcal{M}$ determines a birational map of X to $\mathbb{P}^2$. In particular, $\mathcal{M}$ is base point free off $p_1, \ldots, p_k$ (see [18, Proposition 6.3]).

We will separately discuss the various cases according to Proposition 1.6:

(1) F is empty;
(2) F is not empty and irreducible;
(3) F consists of k irreducible curves Γ_i with $p_i \in \Gamma_i$.

In case (1) the curve M is rational with k nodes at $p_1, \ldots, p_k$ and no other singularity. Then $g = k$ and $d = 4k + 1$ and therefore X is an almost extremal Castelnuovo surface with $\varepsilon = 3$. By Theorem 7.9, we are either in case (iii) or in case (iv).

In case (2), the curve F is smooth and rational. Look at the linear system $|F|$ on X. Since X is linearly normal and there is a unique curve F containing the general points $p_1, \ldots, p_k$, then we have $\dim(|F|) = k$, hence $F^2 = k - 1$. Moreover M is also rational and smooth. Look at the system $|M|$. Since there is a 2-dimensional linear system of curves in $|M|$ containing $p_1, \ldots, p_k$, we have $\dim(|M|) = k + 2$, thus $M^2 = k + 1$. Moreover $M \cdot F = k$ by Proposition 1.6. This implies that:

$$d = M^2 + 2M \cdot F + F^2 = 4k, \quad g = p_a(M) + p_a(F) + M \cdot F - 1 = k - 1$$

hence X is an extremal Castelnuovo surface. By Corollary 7.8, we are in case (ii), because the Veronese surface $V_{2,4}$ is 4-defective (see Remark 7.6).

In case (3), the curves Γ_i are rational and linearly equivalent, and $\Gamma_i^2 = 0$, for $i = 1, \ldots, k$. This implies that we are in case (i). $\square$

Remark 8.2. The assumption that X be linearly normal is essential to have a *finite* classification in Theorem 8.1 above, as shown in Example 5.18. We do not know whether there are more examples of non-linearly normal $\mathcal{OA}_{k-1}^{k+1}$-surfaces other than the ones exhibited in Example 5.18.

According to Proposition 4.9, k-regularity implies linear normality. So one could be tempted to replace the linear normality hypothesis in Theorem 8.1 by the k-regularity assumption, which seems to be, in this context, a right generalization of the concept of smoothness. However, the k-regularity hypothesis is almost never verified by the surfaces in the list (i)–(iv) of Theorem 8.1. This suggests that k-regularity is too rigid. It would be interesting to find a weaker concept which, in this context, could play the right role.

9. The classification of $\mathcal{M}^k$-surfaces

In this section we consider the classification of $\mathcal{M}^k$-surfaces (see also [7]). The case of k-defective and k-weakly defective surfaces has been already considered in Examples 5.14, 5.15 and 5.18, see also [37]. We summarize the result in the following:

Theorem 9.1. *Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, surface. If X is k-defective, then it is an $\mathcal{M}^k$-surface if and only if one if the following happens*:

(i) $r = 3k + 2$ *and X is the* 2*-Veronese embedding of a surface of degree k in* $\mathbb{P}^{k+1}$;
(ii) *X sits in a $(k + 1)$-dimensional cone, with a vertex of dimension $k - 1$, over a rational normal curve C of degree $d \geqslant 2k + 3$.*

If X is k-weakly defective, but not k-defective, then it is an $\mathcal{M}^k$-surface if and only if one if the following happens:

(iii) $r = 9$, $k = 2$ *and X is the* 2*-Veronese embedding of a surface of degree $d \geqslant 3$ in* $\mathbb{P}^3$;
(iv) $r = 3k + 3$ *and X is the cone over a k-defective surface of type* (i);
(v) $r = 3k + 3$ *and X is the* 2*-Veronese embedding of a surface of degree $k + 1$ in* $\mathbb{P}^{k+1}$;
(vi) *X sits in a $(k+2)$-dimensional cone, with a vertex of dimension k, over a rational normal curve C of degree $d \geqslant 2k + 2$.*

The main result of this section is the classification theorem for $\mathcal{MA}_{k-1}^{k+1}$-surfaces, which concludes the classification of $\mathcal{M}^k$-surfaces:

Theorem 9.2. *Let $X \subset \mathbb{P}^{3k+2+h}$, with $k, h \geqslant 1$, be a smooth, irreducible, non-degenerate, $\mathcal{M}^k$-surface which is linearly normal and not k-weakly defective. Let d*

be the degree and g *the sectional genus of* X. *Then* X *is one of the following*:

(i) *a rational normal scroll* $S(a_1, a_2)$ *of degree* $d = 3k+1+h$ *and type* (a_1, a_2) *with* $k \leqslant a_1 \leqslant a_2$ (*see Example* 5.1);

(ii) *a del Pezzo surface of degree* $d = 5+h$ *and* $g = 1$, *with* $1 \leqslant h \leqslant 4$ *and* $k = 1$ (*see Example* 5.16);

(iii) *the internal projection from* $3-h$, *with* $1 \leqslant h \leqslant 3$, *distinct points of an odd Castelnuovo surface* $X_{a,k} \subset \mathbb{P}^{3k+5}$ *with* $0 \leqslant a \leqslant k$ *and* $a+k \equiv 0 \pmod 2$. *In this case* $d = 4k+1+h$, $g = k$ *and the hyperplane sections are hyperelliptic curves* (*see Example* 5.5);

(iv) *the internal projection from* $3-h$ *points*, *with* $1 \leqslant h \leqslant 2$, *of the Veronese surface* $V_{2,4}$. *In this case* $d = 13+h$, $g = 3$, $k = 3$ (*see Example* 5.12);

(v) *the* 3-*Veronese embedding in* $\mathbb{P}^{15}$ *of a smooth quadric in* $\mathbb{P}^3$. *Here* $d = 18$, $g = 4$, $k = 4$, $h = 1$ (*see Example* 5.13);

(vi) *the* 3-*Veronese embedding* $V_{2,3}$ *of* $\mathbb{P}^2$. *In this case* $d = 9$, $g = 1$, $k = 2$, $h = 1$ (*see Example* 5.16).

Proof. Since X is not k-weakly defective, we can apply again Proposition 1.6. Let $p_1, \ldots, p_k \in X$ be general points and, as in the proof of Theorem 8.1, we let $\mathcal{L}$ be the linear system of hyperplane sections of X tangent at $p_1, \ldots, p_k$. Since X is not $(k-1)$-defective, we have $\dim(\mathcal{L}) = 2+h$. Moreover $\mathcal{L} = F + \mathcal{M}$, where F is the fixed part and $\mathcal{M}$ the movable part, as described in Proposition 1.6. By Corollary 4.5, $\tau_{X,k} : X \dashrightarrow X_k \subset \mathbb{P}^{h+2}$ is birational and X_k is a surface of minimal degree $h+1$, hence X is rational and the general curve $M \in \mathcal{M}$ is also rational.

Again, as in the proof of Theorem 8.1, one has to separately discuss the various cases according to Proposition 1.6.

If F is empty, then $g = k$ and $d = 4k+h+1$. If $k = 1$ we are in case (ii). If $k > 1$, by applying Corollary 7.8 and Theorem 7.9, we see that we have cases (iii), (iv) and (v).

If F is not empty and irreducible, then $g = k-1$ and $d = 4k+h$. By Theorem 7.3, the only possible case is $h = 1$, $g = 1$, which implies $k = 2$ and we are in case (vi).

If F consists of k irreducible curves we are in case (i). $\square$

We can now state our result concerning the generalized Bronowski's conjecture for surfaces (see Remark 4.6):

Corollary 9.3. *The generalized Bronowsi's conjecture holds for smooth surfaces.*

Proof. Let $X \subset \mathbb{P}^{3k+2+h}$, $h := \mathrm{codim}(S^k(X))$, be a smooth, irreducible, projective, not k-defective surface and assume that the general k-tangential projection $\tau_{X,k} : X \dashrightarrow X_k \subset \mathbb{P}^{h+2}$ birationally maps X to a surface of minimal degree $h+1$ in $\mathbb{P}^{h+2}$. The same argument we made in the proofs of Theorems 8.1 and 9.1 proves that X is either or minimal degree or Castelnuovo extremal or Castelnuovo almost extremal. As we saw in Section 5, these are $\mathcal{MA}_{k-1}^{k+1}$ or $\mathcal{OA}_{k-1}^{k+1}$-surfaces, according to whether $h > 0$ or $h = 0$. $\square$

10. A generalization of a theorem of Severi

Terracini's Lemma 1.1 implies that a defective variety is swept out by very degenerate subvarieties. As a consequence, one has a famous theorem of Severi [54] (see also [52]), which says that the Veronese surface $V_{2,2}$ in $\mathbb{P}^5$ is the only irreducible non-degenerate, projective surface in $\mathbb{P}^r$, $r \geqslant 5$, not a cone, such that $\dim(S(X)) = 4$. This result can be restated as follows: *the Veronese surface in* $\mathbb{P}^5$ *is the only 1-defective, not 0-weakly defective, irreducible non-degenerate, projective surface in* $\mathbb{P}^r$, $r \geqslant 5$ (cf. Remark 1.2).

This section is devoted to point out an extension of Severi's theorem, namely Theorem 10.1 below. This result yields a projective characterization of extremal Castelnuovo surfaces, in particular it stresses a distinction between odd and even (a, k)-Castelnuovo surfaces, as suggested by Bronowski in [7].

Theorem 10.1 could also be deduced by the classification of weakly defective surfaces (see [14, Examples 5.14 and 5.15]). However, the proof in [14] requires a subtle analysis involving involutions on irreducible varieties and a generalization of the Castelnuovo–Humbert theorem to higher dimensional varieties. It seems interesting to us to present here an easy argument based on the ideas developed in this paper.

Theorem 10.1. *Let* $X \subset \mathbb{P}^r$, $r \geqslant 3k+2$ *and* $k \geqslant 1$, *be a smooth, irreducible, non-degenerate surface. Suppose that X is k-defective but not* $(k-1)$*-weakly defective. Then* $r = 3k+2$ *and X is the* 2*-Veronese embedding of a smooth surface of degree k in* $\mathbb{P}^{k+1}$, *i.e. it is one of the following*:

(i) $X = V_{2,2}$ *is the Veronese surface in* $\mathbb{P}^5$, *then* $k = 1$ *and* $\deg(S(X)) = 3$;
(ii) $X = V_{2,4}$ *is the* 4*-Veronese embedding of* $\mathbb{P}^2$ *in* $\mathbb{P}^{14}$, *then* $k = 4$ *and* $\deg(S^4(X)) = 6$;
(iii) *X is a smooth even Castelnuovo surface* $X_{a,k-1}$, *with* $0 \leqslant a \leqslant k-1$, *which is the* 2*-Veronese embedding of a smooth rational normal scroll of degree k in* $\mathbb{P}^k$.

In particular a k-defective, not $(k-1)$*-weakly defective, surface in* $\mathbb{P}^r$, $r \geqslant 3k+2$, *is an* $\mathcal{M}^k$*-surface in* $\mathbb{P}^{3k+2}$.

Proof. Let $p_0, \ldots, p_k \in X$ be general points. Since X is not $(k-1)$-defective, one has $\dim(T_{X,p_1,\ldots,p_k}) = 3k-1$. Since X is not degenerate in $\mathbb{P}^r$, $r \geqslant 3k+2$, the projection of X from $T_{X,p_1,\ldots,p_k}$ cannot be a point. Hence $s^{(k)}(X) = \dim(T_{X,p_0,\ldots,p_k}) = 3k+1$.

We can suppose $k \geqslant 2$ by Severi' theorem [54]. Also we may assume that $X \subset \mathbb{P}^r$ is linearly normal. Since X is not $(k-1)$-weakly defective we may apply Lemma 1.4 to deduce that $\tau_{X,k-1} : X - - \to X_{k-1} \subset \mathbb{P}^{r-3k+3}$ is birational to its image. Then $r - 3k + 3 \geqslant 5$ and $X_{k-1} \subset \mathbb{P}^{r-3k+3}$ is an irreducible non-degenerate surface. By Terracini's Lemma $\dim(S(X_{k-1})) = 4$ and moreover X_{k-1} is not 0-weakly defective because $X \subset \mathbb{P}^r$ is not $(k-1)$-weakly defective. Thus Severi's theorem applies and yields that X_{k-1} is the Veronese surface in $\mathbb{P}^5$ and that $r = 3k+2$. Note that X cannot be a scroll, since $X_{k-1} = V_{2,2}$ does not contain lines.

The rest of the proof is analogous to the one in Theorem 8.1. Since X is not $(k-2)$-weakly defective we can apply Proposition 1.6. Let $p_1, \ldots, p_{k-1} \in X$ be general points and, as in the proof of Theorem 8.1, we let $\mathcal{L}$ be the linear system of hyperplane sections of X tangent at $p_1, \ldots, p_{k-1}$. The general curve $M \in \mathcal{M}$ is rational being birational to a hyperplane section of the Veronese surface $X_{k-1} \subset \mathbb{P}^5$ and we have $\dim(\mathcal{L}) = 5$. Moreover $\mathcal{L} = F + \mathcal{M}$, where F is the fixed part and $\mathcal{M}$ the movable part, as described in Proposition 1.6.

Again, one has to separately discuss the various cases according to Proposition 1.6.

If F is empty, then $g = k - 1$ and $d = 4k$. In the case $k = 2$, then X is a del Pezzo surface of degree 8 and we are in case (iii) (see Example 5.16). If $k \geqslant 3$, by applying Corollary 7.8, we have cases (ii) and (iii).

If F is not empty and irreducible, then $g = k - 2$ and $d = 4k - 1$. We can suppose that $k \geqslant 3$ since X is not a scroll. Note also that $3(k-2) + 5 = 3k - 1$. Since X is not a scroll, then Corollary 7.8 implies that this case does not exist.

If F consists of $k - 1$ irreducible curves, then they belong to a pencil of lines, a contradiction, since X is not a scroll. □

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