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SMOOTH EXTENSIONS OF LIPSCHITZIAN REAL FUNCTIONS

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Dedicated to Professor F. Guglielmino, with my deepest admiration, on his sixtieth birthday

ABSTRACT. In this short note we point out that any Lipschitzian real function f defined in a subset K of a Banach space E, with $\overline{\operatorname{span}}(K) \neq E$, can be extended to a surjective, open and Lipschitzian real function g on E in such a way that, for every $r \in \mathbf{R}$, the set $g^{-1}(r)$ is arcwise connected. In fact, a more refined result is proved.

The aim of this short note is to point out Theorem 1 below. Before giving its statement, we recall that a topological space Y is said to be an absolute extensor for paracompact spaces (see [**3**]) if, for every paracompact topological space T, every closed set $A \subseteq T$ and every continuous function $\psi: A \to Y$, there exists a continuous function $\varphi: T \to Y$ such that $\varphi|_A = \psi$. We also recall that if U, V are two nonempty sets in a normed space $(E, ||\cdot||)$, their Hausdorff distance, $d_H(U,V)$, is defined by putting

$$d_H(U,V) = \max\left\{\sup_{u\in U}\inf_{v\in V}||u-v||, \sup_{v\in V}\inf_{u\in U}||u-v||\right\}.$$

Finally, if E is as above, any set $S \subseteq E$ will be regarded as a topological space endowed with the relative norm topology.

THEOREM 1. *Let $(E, ||\cdot||)$ be a real Banach space, W a closed linear subspace of E, with $W \neq E$, K a nonempty subset of W and f a Lipschitzian real function on K, with Lipschitz constant L. Then, for every $M > 2L$, there exists a real function g on E with the following properties*:

(a) *$g(x) = f(x)$ for all $x \in K$*;

(b) *g is Lipschitzian, with Lipschitz constant $(M + 2L)/2$*;

(c) *for every $r \in \mathbf{R}$, the set $g^{-1}(r)$ is a nonempty absolute extensor for paracompact spaces*;

(d) *for every $s, t \in \mathbf{R}$, one has*

$$d_H(g^{-1}(s), g^{-1}(t)) \leq (2/(M-2L))|s-t|.$$

PROOF. By Theorem I of [**1**], we can extend f to a Lipschitzian real function Ψ on E, with Lipschitz constant L. Moreover, by a corollary of the Hahn-Banach theorem, there exists a non-null continuous linear functional Λ on E such that $\Lambda(x) = 0$ for all $x \in W$. Now, for every $x \in E$, put $\Phi(x) = (M/2||\Lambda||_{E^*})\Lambda(x)$ as

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well as $g(x) = \Phi(x) + \Psi(x)$. The real function g, of course, satisfies (a) and (b). Let us show that g satisfies also (c) and (d). To this end, observe that, for every $u, v \in \mathbf{R}$, one has

$$d_H(\Phi^{-1}(u), \Phi^{-1}(v)) = (2/M)|u - v|. \tag{1}$$

Indeed, (1) follows easily from a classical result by G. Ascoli (see Lemma 1.2 of [**5**]), taking into account that $\|\Phi\|_{E^*} = M/2$. For every $r \in \mathbf{R}$ and $x \in E$, put $F_r(x) = \Phi^{-1}(r - \Psi(x))$ as well as $\mathrm{Fix}(F_r) = \{y \in E : y \in F_r(y)\}$. Fix $r \in \mathbf{R}$. Plainly, we have

$$g^{-1}(r) = \mathrm{Fix}(F_r). \tag{2}$$

By (1), for every $x, y \in E$, we obtain

$$d_H(F_r(x), F_r(y)) = (2/M)|\Psi(x) - \Psi(y)| \leq (2L/M)\|x - y\|. \tag{3}$$

Thus, since $2L/M < 1$, we infer that $x \to F_r(x)$ is a multivalued contraction from E into E, with closed convex values. Now, (c) is a direct consequence of (2) and of Theorem 1 of [**4**]. Finally, fix $s, t \in \mathbf{R}$. Taking into account (1), (2), (3) and Lemma 1 of [**2**], we then obtain

$$\begin{aligned} d_H(g^{-1}(s), g^{-1}(t)) &= d_H(\mathrm{Fix}(F_s), \mathrm{Fix}(F_t)) \\ &\leq \frac{M}{M - 2L} \sup_{x \in E} d_H(F_s(x), F_t(x)) = \frac{2}{M - 2L}|s - t|. \end{aligned}$$

This completes the proof.

REMARK. It is worth noting that, in Theorem 1, (d) implies that the real function g is open. Furthermore, (c) implies that each set $g^{-1}(r)$ is arcwise connected. Lemma 3.4 of [**6**] then ensures that the set $g^{-1}(A)$ is connected for every interval A of $\mathbf{R}$.

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