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A. Majorana · S.A. Marano

On the Cauchy problem for spatially homogeneous semiconductor Boltzmann equations: existence and uniqueness

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Abstract. The non-linear semi-classical Boltzmann equation for an electron gas in a semiconductor is investigated in the framework of Lebesgue spaces by first providing a rigorous definition of the collision operator. The case of possibly unbounded collision frequencies is treated. Global existence and uniqueness of integrable, space independent solutions to the related Cauchy problem are established. Entropy inequalities, upper bounds for moments as well as mass conservation are also obtained.

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Key words. semiconductor Boltzmann equation – Dirac distribution – unbounded operators – existence and uniqueness

1. Introduction

The applications of semiconductor devices in modern instruments are widespread. Today, the size of devices is very small, so that refined models are required to give accurate numerical results. A well-accepted model for the charge carrier transport in semiconductors is the Boltzmann equation. It represents the starting point of analytical, asymptotic, and numerical investigations carried out in recent years [9, 25, 20, 3]. There are many differences between the semiconductor Boltzmann equation and the classical one for a perfect gas. They basically arise from the different physical nature of particles and become mathematically evident by analyzing the most characteristic term of the equation, namely the collision operator. Hence, the standard techniques, which are usually exploited to treat the classical Boltzmann equation, cannot be adapted in a simple way to the semiconductor case.

In this paper, we are interested in studying the Boltzmann equation where the unknown function f is the distribution function of free electrons inside the semiconductor device. The collision operator $Q(f)$ now describes the scattering processes between electrons and phonons. The physical rules of electron and phonon dynamics are enclosed in the scattering kernels. They contain also information about the

A. Majorana: Department of Mathematics, University of Catania, Viale A. Doria 6, 95125 Catania, Italy, e-mail: majorana@dmi.unict.it

S.A. Marano: Department P.A.U., University of Reggio Calabria, Salita Melissari, 89100 Reggio Calabria, Italy, e-mail: marano@dmi.unict.it

conservation laws, which hold during the collisions, and are mathematically expressed by means of Dirac distributions. Therefore, $Q(f)$ is not a classical integral operator because it exhibits terms like

$$I(f)(t, \mathbf{x}, \mathsf{k}) = \int_{\mathbb{R}^3} \mathcal{G}(\mathsf{k}, \mathsf{k}') f(t, \mathbf{x}, \mathsf{k}') \delta(\varepsilon(\mathsf{k}') - \varepsilon(\mathsf{k}) - u) \, d\mathsf{k}',$$

$(t, \mathbf{x}, \mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3 \times \mathbb{R}^3$, where $\mathcal{G}$ denotes a continuous function, ε represents the particle energy, u is a constant, while $\delta(\varepsilon(\mathsf{k}') - \varepsilon(\mathsf{k}) - u)$ means the composition of the function $(\mathsf{k}, \mathsf{k}') \mapsto \varepsilon(\mathsf{k}') - \varepsilon(\mathsf{k}) - u$ and the distribution δ. The presence of $\delta(\varepsilon(\mathsf{k}') - \varepsilon(\mathsf{k}) - u)$ enforces us to define I on the space $C^0(\mathbb{R}_0^+ \times \mathbb{R}^3 \times \mathbb{R}^3)$, thus requiring an appropriate functional framework for this equation. Moreover, it also makes the numerical treatment of the collision operator rather difficult. So, both in analytical and in numerical investigations, modified Boltzmann equations, where real functions replace Dirac distributions, have been introduced by some authors (see, for instance, [23,24,12]). Consequently, the integral operator I becomes compact and this makes the analytical or numerical treatment simpler.

The pioneering results of [24] are mathematically interesting. In fact, the existence of solutions to the coupled system of a modified Boltzmann transport equation and the Poisson equation is established. Moreover, the distribution function f also depends on the spatial coordinates spanning the three-dimensional physical space. Suitable boundary conditions at infinity allow us to explicitly solve the Poisson equation and so to insert the known electric field inside the Boltzmann equation. Thus, the problem is decoupled. Obviously, the case of devices having finite size requires appropriate boundary conditions, which prevent solving the Poisson equation. This makes the existence problem much more difficult and it is still completely open.

In the framework of numerical investigations (see [12]) trouble arises owing to the use of a smooth function having compact support instead of a Dirac distribution. In fact, a small support gives a good approximation of the δ distribution, but requires a lot of grid points to assure a satisfactory numerical discretization. On the other hand, a large support leads to a dual situation. Therefore, a careful and not trivial compromise was needed. From a physical point of view, such modified models may be unrealistic. In fact, the most popular and well-accepted numerical scheme describing transport phenomena in semiconductors is the Direct Simulation Monte Carlo (DSMC for short) method, which employs *true* singular kernels, where Dirac distributions appear. Actually, the DSMC method is the main powerful tool for practical calculations in charge transport. It is also used to assign numerical values to the physical parameters inside the kernel $\mathcal{G}$ of the operator I. So, the DSMC method and the Boltzmann equation are strictly related. Furthermore, in the case of a perfect classical gas, several analytical results established the "equivalence" between the two models. For an electron gas in a semiconductor, they are compared in a recent computational work [3], and an excellent agreement is found. To the best of our knowledge, rigorous mathematical results concerning the "equivalence" of the DSMC method and the Boltzmann equation do not exist in this case.

In a previous paper [19] the authors studied the relevant collision operator and established some of its properties under quite general assumptions on ε, which cover all the most common expressions considered in applications and simulations. Moreover, an existence and uniqueness theorem of continuous, space-independent solutions to a Cauchy problem was obtained in the case of bounded collision frequencies.

In the present paper, we again consider the same Cauchy problem without restriction on the collision frequency. This could be meaningful. In fact, for instance, it is always an unbounded function in the important cases of silicon and gallium-arsenide semiconductors [16,26]. From a mathematical point of view, several new non-trivial difficulties have to be overcome, since some bounded operators become unbounded. In Section 2, the extension of I to the space L^1_{loc} is thoroughly discussed and significant properties are pointed out. Section 3 treats the existence of space homogeneous integrable solutions to the Boltzmann equation without an electric field. We first find a sequence $\{f_n\}$ in $C^0(\mathbb{R}_0^+ \times \mathbb{R}^3)$ such that each f_n satisfies a modified equation, whose right-hand side is a bounded operator on $C^0(\mathbb{R}_0^+ \times \mathbb{R}^3)$. We then prove that the pointwise limit f of $\{f_n\}$ turns out to be a solution of the original problem and that $f(t, \cdot)$ lies in $L^1(\mathbb{R}^3)$ for any non-negative t. Besides entropy inequalities and moment estimates, a mass conservation result involving the function f is established in Section 4. Finally, Section 5 deals with the uniqueness of solutions that conserve the mass via a fixed-point theorem in ordered Banach spaces. To the best of our knowledge, this is the first time that fixed-point techniques in ordered spaces, usually adopted in proving existence, are exploited to get uniqueness. The continuity of solutions is also investigated.

Throughout the paper, *integrable* always means Lebesgue integrable.

2. Basic equations

When no electric field occurs and space-homogeneous solutions are considered, the Boltzmann equation for an electron gas in a semiconductor takes the form [16, 17,22]

$$\frac{\partial f}{\partial t} = \int_{\mathbb{R}^3} \left[S(\mathsf{k}', \mathsf{k}) f'(1-f) - S(\mathsf{k}, \mathsf{k}') f(1-f') \right] d\mathsf{k}' , \tag{1}$$

where the unknown function $(t, \mathsf{k}) \mapsto f(t, \mathsf{k})$ represents the existence probability of an electron with wave-vector $\mathsf{k} \in \mathbb{R}^3$, at time $t \in \mathbb{R}_0^+$. The term $(1 - f)$ arises from the Pauli exclusion principle. Since f is a probability function, one has $0 \leq f(t, \mathsf{k}) \leq 1$. The right-hand side of (1) is the collision operator $Q(f)$. Here, for any function ϕ we use the notation $\phi' = \phi(\mathsf{k}')$. The kernel S is defined by

$$S(\mathsf{k}, \mathsf{k}') = \sum_{i=1}^{p} \mathcal{G}_i(\mathsf{k}, \mathsf{k}') \left[(n_i + 1)\, \delta(\varepsilon' - \varepsilon + \hbar\omega_i) + n_i\, \delta(\varepsilon' - \varepsilon - \hbar\omega_i) \right], \tag{2}$$

where p is a fixed positive integer and, for $i = 1, 2, ..., p$, $\mathcal{G}_i : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_0^+$ denotes a given continuous function satisfying $\mathcal{G}_i(\mathsf{k}, \mathsf{k}') = \mathcal{G}_i(\mathsf{k}', \mathsf{k})$ for all k, $\mathsf{k}' \in \mathbb{R}^3$, ω_i is a non-negative constant (a phonon frequency), while

$$n_i = \left[\exp\left(\frac{\hbar\omega_i}{k_B T_L} \right) - 1 \right]^{-1} . \tag{3}$$

The parameters $\hbar$, k_B, and T_L are the Planck constant divided by 2π, the Boltzmann constant, and the lattice temperature, respectively. The particle energy ε is an assigned non-negative function of the wave-vector k. The symbol $\delta(\varepsilon' - \varepsilon \pm \hbar\omega_i)$ means the composition of the real-valued function $(\mathsf{k}, \mathsf{k}') \mapsto \varepsilon(\mathsf{k}') - \varepsilon(\mathsf{k}) \pm \hbar\omega_i$ with the Dirac distribution δ (see [13, Chap. III]) and, to guarantee that $\delta(\varepsilon' - \varepsilon \pm \hbar\omega_i)$ is also a distribution, suitable conditions on ε have to be imposed. Now, the expression of $\varepsilon(\mathsf{k})$ is determined by acceptable solutions of the Schrödinger equation for the actual crystal potential, which varies markedly from one solid to another. When, as in numerical applications, simple valuable expressions for ε are required, one resorts to some analytical model [16]. For instance, the simplest one, namely the parabolic band approximation, gives $\varepsilon(\mathsf{k}) = [\hbar^2/(2m^*)]|\mathsf{k}|^2$, where m^* is the effective electron mass. Following [19, 18], we make the general assumptions below on ε , which cover all the most common expressions considered in applications and simulations.

Let $S^2 = \{\mathsf{n} \in \mathbb{R}^3 \ : |\mathsf{n}| = 1\}$. We assume that:

(a_1) $\varepsilon : \mathbb{R}^3 \to \mathbb{R}_0^+$ *is continuous.*
(a_2) *There exists* $\mathsf{k}_0 \in \mathbb{R}^3$ *such that, if* $\eta(\rho, \mathsf{n}) = \varepsilon(\mathsf{k}_0 + \rho\mathsf{n})$, $\rho \geq 0$, $\mathsf{n} \in S^2$, *then:*
   (a_{21}) *the function* η *admits a continuous and positive partial derivative with respect to* ρ *for every* $(\rho, \mathsf{n}) \in]0, +\infty[\times S^2$;
   (a_{22}) $\displaystyle\lim_{\rho\to 0^+} \sup \left\{ \rho^2 \left[\frac{\partial\eta(\rho, \mathsf{n})}{\partial\rho} \right]^{-1} : \mathsf{n} \in S^2 \right\} = 0$;
   (a_{23}) *there is* $\rho_0 > 0$ *satisfying* $\inf \left\{ \dfrac{\partial\eta(\rho, \mathsf{n})}{\partial\rho} \ : \rho \geq \rho_0 \, , \mathsf{n} \in S^2 \right\} > 0.$

Thus, through the same arguments adopted in [19, Appendix B], one can give a precise meaning to the integral

$$\Lambda_0(\phi)(\mathsf{k}) = \int_{\mathbb{R}^3} \phi(\mathsf{k}')\, \delta(\varepsilon(\mathsf{k}') - \varepsilon(\mathsf{k}) - u)\, d\mathsf{k}', \quad u \in \mathbb{R}\,, \tag{4}$$

provided $\phi \in C^0(\mathbb{R}^3)$. Taking into account (2), we immediately see that the collision operator in (1) is a sum of terms of the above type. Consequently, it is well defined once $f(t, \cdot) \in C^0(\mathbb{R}^3)$, for every fixed $t \geq 0$. We next note that (4) actually gives a linear operator $\Lambda_0 : C^0(\mathbb{R}^3) \to C^0(\mathbb{R}^3)$, where the choice of the domain is due to the presence of a distribution while $\Lambda_0(C^0(\mathbb{R}^3)) \subseteq C^0(\mathbb{R}^3)$ by [19, Lemma B.4].

Continuous solutions to equation (1) have been previously obtained in [19] under suitable hypotheses. We expect that weaker conditions may not yield solutions with such a regularity. This assertion is based on the analogy between (1) and standard transport equations. For instance, it is well known [4–6, 11, 21] that the Boltzmann equation for a perfect gas is usually investigated in Lebesgue spaces. We further mention the interesting recent paper [10], where a review of the main results on spatially homogeneous solutions to the classical Boltzmann equation is provided.

If we want to investigate (1) in the framework of Lebesgue spaces, the first step is evidently the extension of the linear operator Λ_0 to the space $L^1_{\mathrm{loc}}(\mathbb{R}^3)$. This problem looks like the one that occurs when the collision operator for a perfect gas is considered. Really, there are meaningful differences. In order to clarify them and to give a reasonable extension of Λ_0, we recall briefly here the expression for the collision operator in the case of the rigid sphere model, which is the most popular and analyzed model for a perfect gas. In this setting the collision operator is given by [7,5,21,6]

$$Q(f,f)(\boldsymbol{\xi}) = \frac{\sigma^2}{m} \int_{\mathbb{R}^3} \int_{S_+} \left[f(\boldsymbol{\xi}') f(\boldsymbol{\xi}'_*) - f(\boldsymbol{\xi}) f(\boldsymbol{\xi}_*) \right] |\mathsf{V} \cdot \mathsf{n}| \, d\boldsymbol{\xi}_* \, d\mathsf{n} \,, \tag{5}$$

where

$$\boldsymbol{\xi}' = \boldsymbol{\xi} - (\mathsf{n} \cdot \mathsf{V})\mathsf{n} \,, \quad \boldsymbol{\xi}'_* = \boldsymbol{\xi}_* + (\mathsf{n} \cdot \mathsf{V})\mathsf{n} \,, \quad \mathsf{V} = \boldsymbol{\xi} - \boldsymbol{\xi}_* \,,$$

S_+ denotes the hemisphere of the unit sphere S^2 defined through the inequality $\mathsf{n} \cdot \mathsf{V} > 0$. The constants σ and m are the "diameter" and the mass of the molecule, respectively. The collision operator (5) is, for any admissible function f, a function of $\boldsymbol{\xi} \in \mathbb{R}^3$ (we are omitting time and space variables, since they play no role in the following discussion). In particular, it is clear that $Q(f,f)$ is well defined for any continuous function f having compact support in $\mathbb{R}^3$. This functional space is often used [11,21] to simplify the study of basic properties (as collision invariants, equilibrium distribution functions, H-theorem) of Q. The extension of Q to integrable functions is not trivial, and sometimes it is given without details. The difficulty in also defining Q for functions which are locally integrable only, arises when one considers the gain term of (5), i.e.

$$\frac{\sigma^2}{m} \int_{\mathbb{R}^3} \int_{S_+} f(\boldsymbol{\xi}') f(\boldsymbol{\xi}'_*) \, |\mathsf{V} \cdot \mathsf{n}| \, d\boldsymbol{\xi}_* \, d\mathsf{n} \,. \tag{6}$$

We will not examine in detail the mathematical framework of this operator, but simply emphasize a feature that makes (6) different from (4). To this end, write (6) as follows [6, p. 177]:

$$\frac{\sigma^2}{m} \int_{\mathbb{R}^6} f(\boldsymbol{\xi}') f(\boldsymbol{\xi}'_*) \, \delta[(\boldsymbol{\xi}' - \boldsymbol{\xi}) \cdot (\boldsymbol{\xi}'_* - \boldsymbol{\xi})] \, d\boldsymbol{\xi}' \, d\boldsymbol{\xi}'_* \,. \tag{7}$$

Now, letting

$$\boldsymbol{\xi}' = \boldsymbol{\xi} + \sqrt{r}\,\boldsymbol{\omega} + \sqrt{r_*}\,\boldsymbol{\omega}_* \,, \quad \boldsymbol{\xi}'_* = \boldsymbol{\xi} + \sqrt{r}\,\boldsymbol{\omega} - \sqrt{r_*}\,\boldsymbol{\omega}_* \,,$$

with $r \geq 0$ and $\boldsymbol{\omega}, \boldsymbol{\omega}_* \in S^2$, (7) becomes

$$\frac{2\,\sigma^2}{m} \int_{S^2} \int_{S^2} \int_0^{+\infty} f(\boldsymbol{\xi} + \sqrt{r}(\boldsymbol{\omega} + \boldsymbol{\omega}_*)) \; f(\boldsymbol{\xi} + \sqrt{r}(\boldsymbol{\omega} - \boldsymbol{\omega}_*)) \, r \, d\boldsymbol{\omega} \, d\boldsymbol{\omega}_* \, dr \,. \tag{8}$$

Since, for fixed $\boldsymbol{\xi}$, the functions

$$(r, \boldsymbol{\omega}, \boldsymbol{\omega}_*) \mapsto \boldsymbol{\xi} + \sqrt{r}(\boldsymbol{\omega} + \boldsymbol{\omega}_*) \,, \quad (r, \boldsymbol{\omega}, \boldsymbol{\omega}_*) \mapsto \boldsymbol{\xi} + \sqrt{r}(\boldsymbol{\omega} - \boldsymbol{\omega}_*)$$

map $[0, +\infty) \times S^2 \times S^2$ onto $\mathbb{R}^3$, the integral in (8) does not change every time one replaces f with a function g which is almost everywhere equal to f in $\mathbb{R}^3$.

Now consider (4). If we assume the parabolic band approximation for the electron energy ε, then setting

$$\mathsf{k}' = \frac{\sqrt{2m^*}}{\hbar}\sqrt{r'}\mathsf{n}', \quad \tilde{\phi}(r',\mathsf{n}') = \phi(\mathsf{k}'), \quad r' \geq 0, \; \mathsf{n}' \in S^2, \tag{9}$$

it takes the form

$$\begin{aligned} \Lambda_0(\phi)(\mathsf{k}) &= \sqrt{2}\left(\frac{\sqrt{m^*}}{\hbar}\right)^3 \int_0^{+\infty}\int_{S^2} \tilde{\phi}(r',\mathsf{n}')\delta(r'-\varepsilon(\mathsf{k})-u)\sqrt{r'}\,dr'\,d\mathsf{n}' \\ &= \sqrt{2}\left(\frac{\sqrt{m^*}}{\hbar}\right)^3 \sqrt{(\varepsilon(\mathsf{k})+u)^+}\int_{S^2} \tilde{\phi}(|\varepsilon(\mathsf{k})+u|,\mathsf{n}')\,d\mathsf{n}', \end{aligned}$$

where $(z)^+ = \max\{0, z\}$. Consequently, $\Lambda_0(\phi)(\mathsf{k})$ is determined only by the values of the function ϕ on the sphere centered at the origin and having radius equal to $|\varepsilon(\mathsf{k})+u|$. Of course, this set has measure zero in $\mathbb{R}^3$. Hence, the extension of Λ_0 to Lebesgue spaces cannot be carried out in a standard way as for the gain operator (6). Now, in order to justify the extension of Λ_0 made below, we first observe two facts. The presence of a distribution in Λ_0 naturally leads to defining this operator in the sense of distributions. On the other hand, if ϕ belongs to $L^1_{\text{loc}}(\mathbb{R}^3)$, then the integral

$$\sqrt{2}\left(\frac{\sqrt{m^*}}{\hbar}\right)^3 \sqrt{(r')^+}\int_{S^2} \tilde{\phi}(|r'|,\mathsf{n}')\,d\mathsf{n}',$$

makes sense for almost r', according to Fubini's Theorem.

The symbol $C^0_0(\mathbb{R}^3)$ denotes the space of all functions in $C^0(\mathbb{R}^3)$ that have compact support. Given $\phi \in L^1_{\text{loc}}(\mathbb{R}^3)$, we define the linear functional $\Lambda(\phi) : C^0_0(\mathbb{R}^3) \to \mathbb{R}$ as follows:

$$\langle \Lambda(\phi), \varphi\rangle = \int_{\mathbb{R}^3} \phi(\mathsf{k}')\left[\int_{\mathbb{R}^3} \varphi(\mathsf{k})\delta(\varepsilon(\mathsf{k}')-\varepsilon(\mathsf{k})-u))\,d\mathsf{k}\right]d\mathsf{k}', \tag{10}$$

for every $\varphi \in C^0_0(\mathbb{R}^3)$. It should be observed that (10) is a real number because by Lemma B.4 of [19] the function

$$\mathsf{k}' \mapsto \int_{\mathbb{R}^3} \varphi(\mathsf{k})\delta(\varepsilon(\mathsf{k}')-\varepsilon(\mathsf{k})-u)\,d\mathsf{k}$$

lies in $C^0_0(\mathbb{R}^3)$. This is also a consequence of assumption (a_{23}), which evidently implies $\lim_{|\mathsf{k}'|\to+\infty} \varepsilon(\mathsf{k}') = +\infty$. Moreover, if ϕ is continuous then [19, Lemma B.5]

$$\langle \Lambda(\phi), \varphi\rangle = \int_{\mathbb{R}^3} \Lambda_0(\phi)(\mathsf{k})\,\varphi(\mathsf{k})\,d\mathsf{k}.$$

Property 1. There exists a unique function $\lambda_\phi \in L^1_{\text{loc}}(\mathbb{R}^3)$ fulfilling

$$\langle \Lambda(\phi), \varphi\rangle = \int_{\mathbb{R}^3} \lambda_\phi(\mathsf{k})\,\varphi(\mathsf{k})\,d\mathsf{k} \quad \forall\varphi \in C^0_0(\mathbb{R}^3). \tag{11}$$

Proof. For the sake of simplicity, we examine the special case of the parabolic band approximation, i.e. $\varepsilon(\mathsf{k}) = [\hbar^2/(2m^*)]|\mathsf{k}|^2$, $\mathsf{k} \in \mathbb{R}^3$. The general case, namely for ε satisfying assumptions $(a_1) - (a_2)$, requires a cumbersome proof, which can be carried out through Lemma B.3 of [19]. Using (9), the spherical coordinates

$$\mathsf{k} = \frac{\sqrt{2m^*}}{\hbar}\sqrt{r}\mathsf{n}\,, \quad r \geq 0\,,\ \mathsf{n} \in S^2\,,$$

and writing $\tilde{\varphi}(r,\mathsf{n}) = \varphi(\mathsf{k})$, (10) takes the form

$$\begin{aligned}\langle \Lambda(\phi), \varphi\rangle &= \frac{2\,(m^*)^3}{\hbar^6}\int_0^{+\infty}\!dr'\!\int_{S^2}\! d\mathsf{n}'\tilde{\phi}(r',\mathsf{n}')\sqrt{r'}\!\int_0^{+\infty}\!dr\int_{S^2}\! d\mathsf{n}\,\tilde{\varphi}(r,\mathsf{n})\delta(r'-r-u)\sqrt{r}\\ &= \frac{2\,(m^*)^3}{\hbar^6}\int_0^{+\infty}\!dr'\int_{S^2}\! d\mathsf{n}'\tilde{\phi}(r',\mathsf{n}')\sqrt{r'}\int_{S^2}\! d\mathsf{n}\,\tilde{\varphi}(|r'-u|,\mathsf{n})\sqrt{(r'-u)^+}\\ &= \frac{2\,(m^*)^3}{\hbar^6}\int_0^{+\infty}\!ds\int_{S^2}\! d\mathsf{n}'\tilde{\phi}(|s+u|,\mathsf{n}')\sqrt{(s+u)^+}\int_{S^2}\! d\mathsf{n}\,\tilde{\varphi}(s,\mathsf{n})\sqrt{s}\\ &= \frac{\sqrt{2\,(m^*)^3}}{\hbar^3}\int_{\mathbb{R}^3}\left[\int_{S^2}\! d\mathsf{n}'\tilde{\phi}(|\varepsilon(\mathsf{k})+u|,\mathsf{n}')\sqrt{(\varepsilon(\mathsf{k})+u)^+}\right]\varphi(\mathsf{k})\,d\mathsf{k}\,.\end{aligned}$$

Thus, (11) is satisfied once we set

$$\lambda_\phi(\mathsf{k}) = \frac{\sqrt{2\,(m^*)^3}}{\hbar^3}\sqrt{(\varepsilon(\mathsf{k})+u)^+}\int_{S^2}\tilde{\phi}(|\varepsilon(\mathsf{k})+u|,\mathsf{n}')\,d\mathsf{n}',\ \mathsf{k}\in\mathbb{R}^3. \tag{12}$$

Moreover, λ_ϕ is unique up to sets of measure zero, as a standard computation shows. □

It is worth noting that, by (11), the functional $\Lambda(\phi)$ can be identified with the function λ_ϕ, which specifically means $\Lambda : L^1_{\mathrm{loc}}(\mathbb{R}^3) \to L^1_{\mathrm{loc}}(\mathbb{R}^3)$.

Two further simple properties of Λ will be useful in the next section.

Property 2. If $\phi(\mathsf{k}) \leq \phi_1(\mathsf{k})$ almost everywhere in $\mathbb{R}^3$ then $\lambda_\phi(\mathsf{k}) \leq \lambda_{\phi_1}(\mathsf{k})$ at almost all $\mathsf{k} \in \mathbb{R}^3$.

Proof. Taking account of (11) and (10), for any non-negative $\varphi \in C^0_0(\mathbb{R}^3)$ we have

$$\begin{aligned}&\int_{\mathbb{R}^3}\left[\lambda_\phi(\mathsf{k}) - \lambda_{\phi_1}(\mathsf{k})\right]\varphi(\mathsf{k})\,d\mathsf{k} = \langle\Lambda(\phi),\varphi\rangle - \langle\Lambda(\phi_1),\varphi\rangle\\ &= \int_{\mathbb{R}^3}\left[\phi(\mathsf{k}') - \phi_1(\mathsf{k}')\right]\left[\int_{\mathbb{R}^3}\varphi(\mathsf{k})\delta(\varepsilon(\mathsf{k}')-\varepsilon(\mathsf{k})-u))\,d\mathsf{k}\right]d\mathsf{k}' \leq 0\,,\end{aligned}$$

from which the conclusion follows at once. □

Property 3. If $\phi, \phi_1, \phi_2, \ldots$ belong to $L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and for almost every $\mathsf{k} \in \mathbb{R}^3$ one has

$$\phi_n(\mathsf{k}) \leq \phi_{n+1}(\mathsf{k})\ \forall n \in \mathbb{N}\,, \quad \lim_{n\to+\infty}\phi_n(\mathsf{k}) = \phi(\mathsf{k}), \tag{13}$$

then $\lim_{n\to+\infty}\lambda_{\phi_n}(\mathsf{k}) = \lambda_\phi(\mathsf{k})$ almost everywhere in $\mathbb{R}^3$.

Proof. Property 2, combined with (13), implies that the sequence $\{\lambda_{\phi_n}\}$ is monotonically increasing and

$$\lambda(\mathsf{k}) = \lim_{n\to+\infty} \lambda_{\phi_n}(\mathsf{k}) \leq \lambda_\phi(\mathsf{k}) \,.$$

Since, for $n \in \mathbb{N}$ and $\varphi \in C_0^0(\mathbb{R}^3)$ non-negative,

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^3} \left[\lambda_\phi(\mathsf{k}) - \lambda(\mathsf{k})\right] \varphi(\mathsf{k})\, d\mathsf{k} \leq \lim_{n\to+\infty} \int_{\mathbb{R}^3} \left[\lambda_\phi(\mathsf{k}) - \lambda_{\phi_n}(\mathsf{k})\right] \varphi(\mathsf{k})\, d\mathsf{k} \\ &= \lim_{n\to+\infty} \int_{\mathbb{R}^3} \left[\phi(\mathsf{k}') - \phi_n(\mathsf{k}')\right] \left[\int_{\mathbb{R}^3} \varphi(\mathsf{k})\delta(\varepsilon(\mathsf{k}') - \varepsilon(\mathsf{k}) - u))\, d\mathsf{k}\right] d\mathsf{k}' = 0 \end{aligned}$$

by the monotone convergence theorem, we immediately obtain $\lambda_\phi(\mathsf{k}) = \lambda(\mathsf{k})$ at almost all $\mathsf{k} \in \mathbb{R}^3$. □

In view of the above arguments, it makes sense to seek integrable solutions of the semiconductor Boltzmann equation (1). If $(t, \mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3$, we write

$$A(f)(t,\mathsf{k}) = \int_{\mathbb{R}^3} S(\mathsf{k}',\mathsf{k}) f(t,\mathsf{k}')\, d\mathsf{k}', \quad A^*(f)(t,\mathsf{k}) = \int_{\mathbb{R}^3} S(\mathsf{k},\mathsf{k}') f(t,\mathsf{k}')\, d\mathsf{k}',$$

for every function $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ fulfilling $f(t,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, $t \in \mathbb{R}_0^+$. Evidently, $A(f)(t,\cdot)$ and $A^*(f)(t,\cdot)$ belong to $L^1_{\mathrm{loc}}(\mathbb{R}^3)$. Setting

$$\nu(\mathsf{k}) = A(1)(t,\mathsf{k}) \,, \quad \nu^*(\mathsf{k}) = A^*(1)(t,\mathsf{k}) \quad \forall\, (t,\mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3 \,,$$

equation (1) becomes

$$\frac{\partial f}{\partial t} + (\nu + \nu^*) f = (1 - f) A(f) + f A^*(f) + \nu f \,. \tag{14}$$

The reason why the term νf appears in both sides of (14) will be clear later. The functions ν and ν^*, usually called collision frequencies, are non-negative, continuous, although possibly unbounded on $\mathbb{R}^3$.

A function $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to [0, 1]$ is said to be a solution of (14) whenever $f(\cdot,\mathsf{k}) \in AC([0, T])$ for all $T > 0$, $f(t,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, and (14) holds almost everywhere in $\mathbb{R}_0^+ \times \mathbb{R}^3$. We will look for solutions to (14) satisfying the initial condition

$$f(0,\mathsf{k}) = \Phi(\mathsf{k}) \,, \tag{15}$$

where $\Phi \in C^0(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ besides $0 \leq \Phi(\mathsf{k}) \leq 1$ in $\mathbb{R}^3$.

3. Existence of integrable solutions

In order to get a solution of (14) fulfilling (15) we first study a modified equation, with the right-hand side of (14) replaced by a bounded operator in $C^0(\mathbb{R}_0^+ \times \mathbb{R}^3)$.

Define, for every $n \in \mathbb{N}$ and $\mathsf{k}, \mathsf{k}' \in \mathbb{R}^3$,

$$\bar{\psi}_n(\mathsf{k}, \mathsf{k}') = \psi_n\left(\frac{\varepsilon(\mathsf{k})}{k_B T_L}\right)\psi_n\left(\frac{\varepsilon(\mathsf{k}')}{k_B T_L}\right),$$

where

$$\psi_n(z) = \begin{cases} 1 & \text{if} \quad |z| \le n\,, \\ 1+n-|z| & \text{if} \quad n < |z| < n+1\,, \\ 0 & \text{if} \quad |z| \ge n+1\,. \end{cases}$$

Hypothesis (a_{23}) guarantees that the set $\left\{\mathsf{k}' \in \mathbb{R}^3 \ : \varepsilon(\mathsf{k}') \le k_B T_L(n+1)\right\}$ is compact. Hence, the function $\mathsf{k}' \mapsto \psi_n\left(\varepsilon(\mathsf{k}')/(k_B T_L)\right)$, $\mathsf{k}' \in \mathbb{R}^3$, has a compact support too and we can put, for any $f \in C^0(\mathbb{R}_0^+ \times \mathbb{R}^3)$,

$$A_n(f)(t, \mathsf{k}) = \int_{\mathbb{R}^3} S(\mathsf{k}', \mathsf{k})\bar{\psi}_n(\mathsf{k}, \mathsf{k}') f(t, \mathsf{k}')\, d\mathsf{k}' \tag{16}$$

$$A_n^*(f)(t, \mathsf{k}) = \int_{\mathbb{R}^3} S(\mathsf{k}, \mathsf{k}')\bar{\psi}_n(\mathsf{k}, \mathsf{k}') f(t, \mathsf{k}')\, d\mathsf{k}' \tag{17}$$

$$\nu_n(\mathsf{k}) = A_n(1)(t, \mathsf{k})\,, \qquad \nu_n^*(\mathsf{k}) = A_n^*(1)(t, \mathsf{k})\,, \tag{18}$$

as well as

$$N_n(f) = (1-f)A_n(f) + fA_n^*(f) + \nu_n f\,.$$

Likewise, if

$$\bar{S}(\mathsf{k}, \mathsf{k}') = \sum_{i=1}^{p} \mathcal{G}_i(\mathsf{k}, \mathsf{k}')(n_i+1)\left[\delta(\varepsilon' - \varepsilon + \hbar\omega_i) + \delta(\varepsilon' - \varepsilon - \hbar\omega_i)\right], \tag{19}$$

then we define

$$\bar{A}_n(f)(t, \mathsf{k}) = \int_{\mathbb{R}^3} \bar{S}(\mathsf{k}, \mathsf{k}')\bar{\psi}_n(\mathsf{k}, \mathsf{k}') f(t, \mathsf{k}')\, d\mathsf{k}'\,, \qquad \bar{\nu}_n(\mathsf{k}) = \bar{A}_n(1)(t, \mathsf{k})\,.$$

Obviously, for non-negative f one has

$$A_n(f)(t, \mathsf{k}) \le \bar{A}_n(f)(t, \mathsf{k}) \quad \text{and} \quad A_n^*(f)(t, \mathsf{k}) \le \bar{A}_n(f)(t, \mathsf{k})\,. \tag{20}$$

Some elementary properties of the operator N_n on the space

$$Z = \left\{f \in C^0\left(\mathbb{R}_0^+ \times \mathbb{R}^3\right) \ : 0 \le f(t, \mathsf{k}) \le 1 \ \forall\, (t, \mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3\right\}$$

are collected in the lemma below. Henceforth, given f, $g \in C^0(\mathbb{R}_0^+ \times \mathbb{R}^3)$, the symbol $f \le g$ means, as usual, $f(t, \mathsf{k}) \le g(t, \mathsf{k})$ for every $(t, \mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3$.

Lemma 1. (i) $N_n(f) \geq 0, \quad \forall f \in Z$.
(ii) $N_n(f) \leq N_n(g)$, *for all* $f, g \in Z$ *such that* $f \leq g$.
(iii) $N_n(f) \leq N_{n+1}(f), \quad \forall f \in Z$.

Proof. Assertion (i) is evident. Let us verify (ii). For any $f, g \in Z$ one has

$$N_n(f) - N_n(g) = (1-f)A_n(f-g) + fA_n^*(f-g) + (f-g)\left[A_n(1-g) + A_n^*(g)\right], \tag{21}$$

as an elementary computation shows. Now, if $f \leq g$, then (21) implies $N_n(f) \leq N_n(g)$. Finally, to prove (iii) we simply note that $\psi_n(z) \leq \psi_{n+1}(z)$, for all $z \in \mathbb{R}$. Hence, $A_n(f) \leq A_{n+1}(f)$, $A_n^*(f) \leq A_{n+1}^*(f)$, and $\nu_n \leq \nu_{n+1}$. This immediately yields $N_n(f) \leq N_{n+1}(f)$. □

Now, consider the Cauchy problem

$$\begin{cases} \dfrac{\partial f}{\partial t} + (\nu + \nu^*)f = N_n(f) & \text{in } \mathbb{R}_0^+ \times \mathbb{R}^3, \\ f(0, \mathsf{k}) = \Phi_n(\mathsf{k}) & \text{on } \mathbb{R}^3, \end{cases} \tag{22}$$

where $\Phi_n(\mathsf{k}) = \Phi(\mathsf{k})\psi_n(\varepsilon(\mathsf{k})/(k_B T_L))$, $\mathsf{k} \in \mathbb{R}^3$. Clearly, if $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to [0, 1]$ is a solution to (22) then $\operatorname{supp}(f(t, \cdot)) \subseteq \left\{\mathsf{k} \in \mathbb{R}^3 : \varepsilon(\mathsf{k}) \leq k_B T_L(n+1)\right\}$, $t \in \mathbb{R}_0^+$, and thus it is compact.

Theorem 1. *Problem* (22) *possesses a unique solution* $f_n \in Z$.

Proof. Let $\alpha > 0$ and let X be the space of all continuous functions $g : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ such that

$$\|g\|_\alpha = \sup\left\{e^{-\alpha t}|g(t, \mathsf{k})| : (t, \mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3\right\} < +\infty .$$

A standard argument (see for instance [8, pp. 2–3]) ensures that $(X, \|\cdot\|_\alpha)$ is a real Banach space. Moreover, Z is a closed subset of X. For every $g \in Z$ we define

$$T_n(g)(t, \mathsf{k}) = e^{-[\nu(\mathsf{k})+\nu^*(\mathsf{k})]t}\Phi_n(\mathsf{k}) + \int_0^t e^{[\nu(\mathsf{k})+\nu^*(\mathsf{k})](r-t)} N_n(g)(r, \mathsf{k})\, dr, \quad (t, \mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3. \tag{23}$$

Owing to (i) and (ii) in Lemma 1 it results in

$$0 \leq T_n(g)(t, \mathsf{k}) \leq e^{-[\nu(\mathsf{k})+\nu^*(\mathsf{k})]t}\left\{\Phi_n(\mathsf{k}) + \int_0^t e^{[\nu(\mathsf{k})+\nu^*(\mathsf{k})]r}[\nu(\mathsf{k}) + \nu^*(\mathsf{k})]\, dr\right\}$$
$$= e^{-[\nu(\mathsf{k})+\nu^*(\mathsf{k})]t}\left\{\Phi_n(\mathsf{k}) - 1 + e^{[\nu(\mathsf{k})+\nu^*(\mathsf{k})]t}\right\} \leq 1, \quad \forall (t, \mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3, g \in Z .$$

Thus, $T_n(Z) \subseteq Z$. Since a function $f_n \in Z$ is a solution to (22) if and only if it is a fixed point of the operator $T_n : Z \to Z$, the proof is accomplished by showing that T_n is a contraction with respect to the norm $\|\cdot\|_\alpha$, for some $\alpha > 0$. Pick $g, h \in Z$ and observe that, in view of (21) and (20),

$$\begin{aligned} &|N_n(g) - N_n(h)| \\ &\quad \le (1-g)\bar{A}_n(|g-h|) + g\bar{A}_n(|g-h|) + |g-h|\big[\bar{A}_n(1-h) + \bar{A}_n(h)\big] \\ &\quad = \bar{A}_n(|g-h|) + \bar{\nu}_n|g-h| \,. \end{aligned}$$

Using the obvious inequality $|g-h| \le e^{\alpha t}\|g-h\|_\alpha$ yields

$$\bar{A}_n(|g-h|) \le e^{\alpha t}\|g-h\|_\alpha \bar{\nu}_n \,,$$

which implies

$$|N_n(g) - N_n(h)| \le 2e^{\alpha t}\|g-h\|_\alpha \bar{\nu}_n \,.$$

Therefore,

$$\begin{aligned} &|T_n(g)(t,\mathsf{k}) - T_n(h)(t,\mathsf{k})| \\ &\le \int_0^t e^{[\nu(\mathsf{k})+\nu^*(\mathsf{k})](r-t)}|N_n(g)(r,\mathsf{k}) - N_n(h)(r,\mathsf{k})|\,dr \\ &\le 2\|g-h\|_\alpha\,\bar{\nu}_n(\mathsf{k})\,e^{-[\nu(\mathsf{k})+\nu^*(\mathsf{k})]t}\int_0^t e^{[\nu(\mathsf{k})+\nu^*(\mathsf{k})+\alpha]r}\,dr \\ &\le e^{\alpha t}\frac{2\,\bar{\nu}_n(\mathsf{k})}{\nu(\mathsf{k})+\nu^*(\mathsf{k})+\alpha}\,\|g-h\|_\alpha, \end{aligned} \tag{24}$$

for all $(t,\mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3$. Now, let $\alpha > 0$ be such that

$$L_\alpha = \sup\left\{\frac{2\,\bar{\nu}_n(\mathsf{k})}{\nu(\mathsf{k})+\nu^*(\mathsf{k})+\alpha} \;:\mathsf{k}\in\mathbb{R}^3\right\} < 1 \,.$$

Of course, α depends on n. Since (24) immediately leads to

$$\|T_n(g) - T_n(h)\|_\alpha \le L_\alpha\|g-h\|_\alpha \quad \forall g,h \in Z$$

and $L_\alpha < 1$, the conclusion follows. □

Remark 1. The function f_n satisfies the integral equation

$$f_n(t,\mathsf{k}) = e^{-[\nu(\mathsf{k})+\nu^*(\mathsf{k})]t}\left\{\Phi_n(\mathsf{k}) + \int_0^t e^{[\nu(\mathsf{k})+\nu^*(\mathsf{k})]r}N_n(f_n)(r,\mathsf{k})\,dr\right\} \tag{25}$$

pointwise in $\mathbb{R}_0^+ \times \mathbb{R}^3$.

Remark 2. Taking account of the contraction principle, we also get

$$f_n(t,\mathsf{k}) = \lim_{i\to+\infty} f_{n,i}(t,\mathsf{k}) \qquad \text{for every } (t,\mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3 \,,$$

where $f_{n,0} \equiv 0$ while $f_{n,i} = T_n(f_{n,i-1})$, $i \in \mathbb{N}$, with T_n given by (23). Of course, every $f_{n,i}$ lies in Z.

Basic properties of the sequence $\{f_n\}$ are the following:

Theorem 2. (i) $0 \le f_n \le f_{n+1} \le 1$*, for all* $n \in \mathbb{N}$.
(ii) *Setting* $f(t,\mathsf{k}) = \lim_{n\to+\infty} f_n(t,\mathsf{k})$, $(t,\mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3$*, one has* $0 \le f \le 1$ *as well as* $f(t,\cdot) \in L^1(\mathbb{R}^3)$ *for any non-negative* t.

Proof. To verify (i) we simply note that $0 \le f_n \le 1$, because $f_n \in Z$, and $f_{n,i} \le f_{n+1,i}$, $i \in \mathbb{N}$. In fact, $f_{n,0} = f_{n+1,0} = 0$. Assume the inequality holds for some positive integer i; we will prove it for $i+1$. By (iii) and (ii) of Lemma 1 it results in

$$f_{n,i+1} = T_n(f_{n,i}) \le T_{n+1}(f_{n,i}) \le T_{n+1}(f_{n+1,i}) = f_{n+1,i+1},$$

which yields the conclusion. Let us next show (ii). Assertion (i) immediately leads to $0 \le f \le 1$. Fix $n \in \mathbb{N}$. Since

$$\frac{\partial f_n}{\partial t} = N_n(f_n) - (\nu + \nu^*) f_n \le N_n(f_n) - (\nu_n + \nu_n^*) f_n \,,$$

integrating with respect to the first variable gives

$$f_n(t,\mathsf{k}) - \Phi_n(\mathsf{k}) \le \int_0^t Q_n(f_n)(r,\mathsf{k})\, dr, \tag{26}$$

for all $t \in \mathbb{R}_0^+$ and every $\mathsf{k} \in \mathbb{R}^3$, where

$$Q_n(g) = (1-g)A_n(g) - (\nu_n^* - A_n^*(g))g, \quad g \in Z\,. \tag{27}$$

Now, observe that the functions ν_n, ν_n^*, and $f_n(t,\cdot)$ have a compact support. Hence, the same arguments adopted in the proof of [19, Proposition 3.2] here give

$$\int_{\mathbb{R}^3} Q_n(f_n)(t,\mathsf{k})\, d\mathsf{k} = 0 \quad \forall t \in \mathbb{R}_0^+\,. \tag{28}$$

Owing to (26) we thus obtain

$$\int_{\mathbb{R}^3} f_n(t,\mathsf{k})\, d\mathsf{k} \le \int_{\mathbb{R}^3} \Phi_n(\mathsf{k})\, d\mathsf{k} \le \int_{\mathbb{R}^3} \Phi(\mathsf{k})\, d\mathsf{k}\,.$$

As $\Phi \in L^1(\mathbb{R}^3)$ while n was arbitrary, the monotone convergence theorem yields $f(t,\cdot) \in L^1(\mathbb{R}^3)$ for any non-negative t. □

Remark 3. Since $\{f_n\} \subseteq Z$, conclusion (ii) of the above result shows that $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to [0,1]$ is a Baire class one function [14, Chap. 10].

We finally come to the original Cauchy problem (14)–(15).
If $g : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ and $g(t,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, $t \in \mathbb{R}_0^+$, define

$$N(g) = (1-g)A(g) + gA^*(g) + \nu g\,. \tag{29}$$

Theorem 3. *The function* $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to [0,1]$ *given by* (ii) *of Theorem* 2 *is a solution to Problem* (14)–(15).

Proof. One clearly has $f_n(0,\mathsf{k}) = \Phi_n(\mathsf{k})$, for all $n \in \mathbb{N}$. Hence,

$$f(0,\mathsf{k}) = \lim_{n\to+\infty} f_n(0,\mathsf{k}) = \lim_{n\to+\infty} \Phi_n(\mathsf{k}) = \Phi(\mathsf{k}) \,, \quad \mathsf{k} \in \mathbb{R}^3,$$

namely f fulfils the initial condition (15). Now, let $t > 0$. Theorem 2 ensures that $f(\cdot,\mathsf{k}')$ is measurable for almost every $\mathsf{k}' \in \mathbb{R}^3$ while $f(r,\cdot) \in L^1(\mathbb{R}^3)$, for any $r \in [0,t]$. So, the function $r \mapsto N(f)(r,\mathsf{k})$ belongs to $L^1([0,t])$, as a simple argument shows. The conclusion will be achieved once we verify that

$$f(t,\mathsf{k}) = e^{-[\nu(\mathsf{k})+\nu^*(\mathsf{k})]t} \left\{ \Phi(\mathsf{k}) + \int_0^t e^{[\nu(\mathsf{k})+\nu^*(\mathsf{k})]r} N(f)(r,\mathsf{k})\, dr \right\}, \tag{30}$$

for almost all $\mathsf{k} \in \mathbb{R}^3$. Using the properties of $\{f_n\}$ and Lemma 1 gives

$$N_n(f_n) \le N_{n+1}(f_{n+1}) \quad \text{besides} \quad N_n(f_n) \le \nu + \nu^* \,, \quad n \in \mathbb{N}\,.$$

Since

$$A_n(f_n)(t,\mathsf{k}) = \psi_n\left(\frac{\varepsilon(\mathsf{k})}{k_B T_L}\right) A\left(\psi_n\left(\frac{\varepsilon(\cdot)}{k_B T_L}\right) f_n\right)(t,\mathsf{k}) \,,$$

by Property 3 we obtain $\lim_{n\to+\infty} A_n(f_n) = A(f)$. An analogous reasoning applies to the sequence $\{A_n^*(f_n)\}$. Therefore,

$$\lim_{n\to+\infty} N_n(f_n) = N(f) \quad \text{a.e. in } [0,t]\times\mathbb{R}^3.$$

At this point, (30) immediately follows from (25) and the monotone convergence theorem. □

4. Entropy inequalities, moment estimates, and mass conservation

Keep the same notation of the preceding section and define

$$S_{n,i}^{\pm}(\mathsf{k},\mathsf{k}') = \mathcal{G}_{n,i}(\mathsf{k},\mathsf{k}')\delta(\varepsilon(\mathsf{k}') - \varepsilon(\mathsf{k}) \pm \hbar\omega_i) \quad \forall (\mathsf{k},\mathsf{k}') \in \mathbb{R}^3 \times \mathbb{R}^3,$$

with

$$\mathcal{G}_{n,i}(\mathsf{k},\mathsf{k}') = \mathcal{G}_i(\mathsf{k},\mathsf{k}')\bar{\psi}_n(\mathsf{k},\mathsf{k}') \,.$$

Evidently, $\mathcal{G}_{n,i}$ has a compact support in $\mathbb{R}^3 \times \mathbb{R}^3$. To shorten formulae, we write $\phi' = \phi(t,\mathsf{k}')$ besides $\phi = \phi(t,\mathsf{k})$ when no confusion can arise.

Lemma 2. *For every $g, \eta \in C^0(\mathbb{R}_0^+ \times \mathbb{R}^3)$ one has*

$$\int_{\mathbb{R}^3} Q_n(g)(t,\mathsf{k})\, \eta(t,\mathsf{k})\, d\mathsf{k}$$
$$= \sum_{i=1}^p \int_{\mathbb{R}^3\times\mathbb{R}^3} S_{n,i}^-(\mathsf{k},\mathsf{k}')[(n_i+1)g'(1-g) - n_i g(1-g')](\eta - \eta')\, d\mathsf{k}' d\mathsf{k}\,. \tag{31}$$

So, in particular,

$$\int_{\mathbb{R}^3} Q_n(g)(t,\mathsf{k})\, d\mathsf{k} = 0 \quad \forall t \in \mathbb{R}_0^+\,. \tag{32}$$

Proof. Fix $g, \eta \in C^0(\mathbb{R}_0^+ \times \mathbb{R}^3)$. By (27) and (16)–(18) we get

$$\int_{\mathbb{R}^3} Q_n(g)(t,\mathsf{k})\,\eta(t,\mathsf{k})\,d\mathsf{k}$$
$$= \int_{\mathbb{R}^3\times\mathbb{R}^3} \left[S(\mathsf{k}',\mathsf{k})g'(1-g) - S(\mathsf{k},\mathsf{k}')g(1-g')\right]\bar{\psi}_n(\mathsf{k},\mathsf{k}')\eta\,d\mathsf{k}'d\mathsf{k}\,.$$

Exchanging k with k' in the first term of the above sum yields

$$\int_{\mathbb{R}^3} Q_n(g)(t,\mathsf{k})\,\eta(t,\mathsf{k})\,d\mathsf{k}$$
$$= \int_{\mathbb{R}^3\times\mathbb{R}^3} S(\mathsf{k},\mathsf{k}')g(1-g')\left(\eta'-\eta\right)\bar{\psi}_n(\mathsf{k},\mathsf{k}')\,d\mathsf{k}'d\mathsf{k}$$
$$= \sum_{i=1}^{p}\int_{\mathbb{R}^3\times\mathbb{R}^3}\left[(n_i+1)S^+_{n,i}(\mathsf{k},\mathsf{k}') + n_i S^-_{n,i}(\mathsf{k},\mathsf{k}')\right]g(1-g')\left(\eta'-\eta\right)d\mathsf{k}'d\mathsf{k}\,.$$

Since $S^+_{n,i}(\mathsf{k}',\mathsf{k}) = S^-_{n,i}(\mathsf{k},\mathsf{k}')$, through the same argument we achieve (31). Finally, (32) immediately follows from (31) by choosing $\eta \equiv 1$. □

Lemma 3. *Let* $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *be monotonically increasing and continuous, let* $M(\mathsf{k}) = \exp\left(-\dfrac{\varepsilon(\mathsf{k})}{k_B T_L}\right)$, $\mathsf{k} \in \mathbb{R}^3$, *and let* $\beta > 0$. *Then*

$$\int_{\mathbb{R}^3} Q_n(g)(t,\mathsf{k})\,\psi\left(\frac{g(t,\mathsf{k})}{M(\mathsf{k})(1-g(t,\mathsf{k}))+\beta g(t,\mathsf{k})}\right)d\mathsf{k} \le 0, \tag{33}$$

for all $g \in Z$, $t \in \mathbb{R}_0^+$.

Proof. Pick $g \in Z$ and define

$$\eta(t,\mathsf{k}) = \psi\left(\frac{g(t,\mathsf{k})}{M(\mathsf{k})(1-g(t,\mathsf{k}))+\beta g(t,\mathsf{k})}\right),\ (t,\mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3.$$

Owing to (31) it results in

$$\int_{\mathbb{R}^3} Q_n(g)(t,\mathsf{k})\,\eta(t,\mathsf{k})\,d\mathsf{k}$$
$$= \sum_{i=1}^{p} n_i \int_{\mathbb{R}^3\times\mathbb{R}^3} S^-_{n,i}(\mathsf{k},\mathsf{k}')\left[a_i g'(1-g) - g(1-g')\right]\left(\eta-\eta'\right)d\mathsf{k}'d\mathsf{k}\,,$$

where $a_i = (n_i+1)/n_i = \exp(\hbar\omega_i/(k_B T_L))$; see (3). In view of the obvious identity

$$\int_{\mathbb{R}^3}\delta(\varepsilon(\mathsf{k}')-\varepsilon(\mathsf{k})-\hbar\omega_i)\frac{1}{M(\mathsf{k}')}\,d\mathsf{k}' = \frac{a_i}{M(\mathsf{k})}\int_{\mathbb{R}^3}\delta(\varepsilon(\mathsf{k}')-\varepsilon(\mathsf{k})-\hbar\omega_i)\,d\mathsf{k}',$$

we have

$$\int_{\mathbb{R}^3} Q_n(g)(t,\mathsf{k})\,\eta(t,\mathsf{k})\,d\mathsf{k}$$
$$= \sum_{i=1}^{p} n_i \int_{\mathbb{R}^3\times\mathbb{R}^3} S_{n,i}^{-}(\mathsf{k},\mathsf{k}')\,M\left[\frac{g'}{M'}(1-g) - \frac{g}{M}(1-g')\right]\left(\eta-\eta'\right)d\mathsf{k}'d\mathsf{k}$$
$$= \sum_{i=1}^{p} n_i \int_{\mathbb{R}^3\times\mathbb{R}^3} S_{n,i}^{-}(\mathsf{k},\mathsf{k}')\,M\left(\eta-\eta'\right)$$
$$\times\left[\frac{g'}{M'}\left(1-g+\beta\,\frac{g}{M}\right) - \frac{g}{M}\left(1-g'+\beta\,\frac{g'}{M'}\right)\right]d\mathsf{k}'d\mathsf{k}$$
$$= \sum_{i=1}^{p} n_i \int_{\mathbb{R}^3\times\mathbb{R}^3} S_{n,i}^{-}(\mathsf{k},\mathsf{k}')\,M\left(\eta-\eta'\right)\left(1-g+\beta\,\frac{g}{M}\right)\left(1-g'+\beta\,\frac{g'}{M'}\right)$$
$$\times\left[\frac{g'}{M'(1-g')+\beta\,g'} - \frac{g}{M(1-g)+\beta\,g}\right]d\mathsf{k}'d\mathsf{k}\,.$$

Bearing in mind that $g \in Z$ while ψ is monotonically increasing, the conclusion follows. □

Lemma 4. *Suppose $\mu \in [1,+\infty)$ and, moreover,*

$$b = \int_{\mathbb{R}^3} d\mathsf{k} \int_0^{\Phi(\mathsf{k})} \left[\log\left(1+\frac{x}{M(\mathsf{k})(1-x)}\right)\right]^{\mu} dx < +\infty\,. \tag{34}$$

Then there exists a constant $c>0$ such that

$$\int_{\mathbb{R}^3} f_n(t,\mathsf{k})[\varepsilon(\mathsf{k})]^{\mu} d\mathsf{k} \le c\,, \qquad t \in \mathbb{R}_0^+,$$

where f_n is given by Theorem 1, for any $n \in \mathbb{N}$.

Proof. The function f_n belongs to Z and one has

$$\frac{\partial f_n}{\partial t} = Q_n(f_n) + [(\nu_n+\nu_n^*) - (\nu+\nu^*)]f_n \le Q_n(f_n) \tag{35}$$

because $\nu_n \le \nu$, $\nu_n^* \le \nu^*$. Thanks to Lemma 3 we thus get

$$\int_{\mathbb{R}^3} \frac{\partial f_n(t,\mathsf{k})}{\partial t}\,\psi\left(\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})(1-f_n(t,\mathsf{k}))+\beta f_n(t,\mathsf{k})}\right)d\mathsf{k}$$
$$\le \int_{\mathbb{R}^3} Q_n(f_n)(t,\mathsf{k})\,\psi\left(\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})(1-f_n(t,\mathsf{k}))+\beta f_n(t,\mathsf{k})}\right)d\mathsf{k} \le 0,$$

with ψ, M, and β as in that result. Exploiting the absolute continuity of $f_n(\cdot,\mathsf{k})$ yields

$$\int_{\mathbb{R}^3}\left[\frac{\partial}{\partial t}\int_0^{f_n(t,\mathsf{k})} \psi\left(\frac{x}{M(\mathsf{k})(1-x)+\beta\,x}\right)dx\right]d\mathsf{k}$$
$$= \int_{\mathbb{R}^3} \frac{\partial f_n(t,\mathsf{k})}{\partial t}\,\psi\left(\frac{f_n(t,\mathsf{k})}{M(\mathsf{k})(1-f_n(t,\mathsf{k}))+\beta f_n(t,\mathsf{k})}\right)d\mathsf{k} \le 0\,,$$

which implies

$$\int_{\mathbb{R}^3} d\mathsf{k} \int_0^{f_n(t,\mathsf{k})} \psi\left(\frac{x}{M(\mathsf{k})(1-x)+\beta\, x}\right) dx \le \int_{\mathbb{R}^3} d\mathsf{k} \int_0^{\Phi_n(\mathsf{k})} \psi\left(\frac{x}{M(\mathsf{k})(1-x)+\beta\, x}\right) dx \quad \forall t \in \mathbb{R}_0^+. \tag{36}$$

Next, fix $t \geq 0$ and set

$$\Omega_t = \left\{\mathsf{k} \in \mathbb{R}^3 : \; f_n(t,\mathsf{k}) \geq \frac{\sqrt{M(k)}}{1+\sqrt{M(k)}}\right\}.$$

Since

$$\int_{\mathbb{R}^3} d\mathsf{k} \int_0^{f_n(t,\mathsf{k})} \psi\left(\frac{x}{M(\mathsf{k})(1-x)+\beta\, x}\right) dx \geq \int_{\Omega_t} d\mathsf{k} \int_{\sqrt{M(k)}/(1+\sqrt{M(k)})}^{f_n(t,\mathsf{k})} \psi\left(\frac{x}{M(\mathsf{k})(1-x)+\beta\, x}\right) dx \geq \int_{\Omega_t} \psi\left(\frac{1}{\sqrt{M(\mathsf{k})}+\beta}\right)\left[f_n(t,\mathsf{k}) - \frac{\sqrt{M(k)}}{1+\sqrt{M(k)}}\right] d\mathsf{k},$$

from (36) we infer

$$0 \leq \int_{\Omega_t} \psi\left(\frac{1}{\sqrt{M(\mathsf{k})}+\beta}\right)\left[f_n(t,\mathsf{k}) - \frac{\sqrt{M(k)}}{1+\sqrt{M(k)}}\right] d\mathsf{k} \leq \int_{\mathbb{R}^3} d\mathsf{k} \int_0^{\Phi_n(\mathsf{k})} \psi\left(\frac{x}{M(\mathsf{k})(1-x)+\beta\, x}\right) dx\,. \tag{37}$$

Now, if $\psi(z) = \left[\log(1+z)\right]^\mu$, $z \in \mathbb{R}_0^+$, then

$$\int_0^{\Phi_n(\mathsf{k})} \psi\left(\frac{x}{M(\mathsf{k})(1-x)+\beta\, x}\right) dx \leq \int_0^{\Phi(\mathsf{k})} \psi\left(\frac{x}{M(\mathsf{k})(1-x)}\right) dx \leq \int_0^1 \left[\log\left(1+\frac{x}{M(\mathsf{k})(1-x)}\right)\right]^\mu dx < +\infty\,.$$

Consequently, by (34) besides (37),

$$0 \leq \int_{\Omega_t} \left[\log\left(1+\frac{1}{\sqrt{M(\mathsf{k})}+\beta}\right)\right]^\mu \left[f_n(t,\mathsf{k}) - \frac{\sqrt{M(k)}}{1+\sqrt{M(k)}}\right] d\mathsf{k} \leq b,$$

for any $\beta > 0$. Letting $\beta \to 0^+$ and using the inequality

$$\log\left(1+\frac{1}{\sqrt{M(\mathsf{k})}}\right) \geq \frac{\varepsilon(\mathsf{k})}{2\, k_B T_L}$$

provides

$$\int_{\Omega_t} f_n(t,\mathsf{k})[\varepsilon(\mathsf{k})]^\mu \, d\mathsf{k} \leq b(2\, k_B T_L)^\mu + \int_{\Omega_t} \frac{\sqrt{M(k)}}{1+\sqrt{M(k)}}[\varepsilon(\mathsf{k})]^\mu \, d\mathsf{k}\,.$$

Finally, as

$$\int_{\mathbb{R}^3\setminus\Omega_t} f_n(t,\mathsf{k})[\varepsilon(\mathsf{k})]^\mu\, d\mathsf{k} \le \int_{\mathbb{R}^3\setminus\Omega_t} \frac{\sqrt{M(k)}}{1+\sqrt{M(k)}}[\varepsilon(\mathsf{k})]^\mu\, d\mathsf{k}\,,$$

we actually have

$$\int_{\mathbb{R}^3} f_n(t,\mathsf{k})[\varepsilon(\mathsf{k})]^\mu\, d\mathsf{k} \le b(2\,k_B T_L)^\mu + \int_{\mathbb{R}^3} \frac{\sqrt{M(k)}}{1+\sqrt{M(k)}}[\varepsilon(\mathsf{k})]^\mu\, d\mathsf{k}, \quad \forall t \in \mathbb{R}_0^+.$$

This represents the desired conclusion, with

$$c = b(2\,k_B T_L)^\mu + \int_{\mathbb{R}^3} \frac{\sqrt{M(k)}}{1+\sqrt{M(k)}}[\varepsilon(\mathsf{k})]^\mu\, d\mathsf{k}\,. \tag{38}$$

It should be noted that the function

$$\mathsf{k} \mapsto \frac{\sqrt{M(k)}}{1+\sqrt{M(k)}}[\varepsilon(\mathsf{k})]^\mu$$

belongs to $L^1(\mathbb{R}^3)$. In fact, an elementary computation gives a positive constant d satisfying

$$\frac{\sqrt{M(k)}}{1+\sqrt{M(k)}}[\varepsilon(\mathsf{k})]^\mu \le \sqrt{M(k)}[\varepsilon(\mathsf{k})]^\mu \le d\,\sqrt[3]{M(k)}, \quad \forall \mathsf{k} \in \mathbb{R}^3,$$

while $\mathsf{k} \mapsto e^{-\varepsilon(\mathsf{k})/(3\,k_B T_L)}$ is integrable on $\mathbb{R}^3$ because of (a_{23}). □

Remark 4. If the function $\mathsf{k} \mapsto \Phi(\mathsf{k})\,[\varepsilon(\mathsf{k})]^\mu$ lies in $L^1(\mathbb{R}^3)$ then (34) comes true. To verify this, we first observe that

$$\begin{aligned}\left[\log\left(1+\frac{x}{M(\mathsf{k})(1-x)}\right)\right]^\mu &\le \left[\frac{\varepsilon(\mathsf{k})}{k_B T_L} - \log(1-x)\right]^\mu \\ &\le 2^{\mu-1}\left\{\left[\frac{\varepsilon(\mathsf{k})}{k_B T_L}\right]^\mu + \left[-\log(1-x)\right]^\mu\right\},\end{aligned} \tag{39}$$

for all $x \in [0,1[$ and any $\mathsf{k} \in \mathbb{R}^3$. Moreover, as a simple argument shows,

$$\int_0^y \left[-\log(1-x)\right]^\mu dx \le y \int_0^1 \left[-\log(1-x)\right]^\mu dx = \Gamma(\mu+1)\,y\,, \tag{40}$$

$y \in [0,1]$, where Γ denotes the gamma function. Now, gathering (39) and (40) together leads to

$$\begin{aligned}\int_0^{\Phi(\mathsf{k})} \left[\log\left(1+\frac{x}{M(\mathsf{k})(1-x)}\right)\right]^\mu dx &\le \frac{1}{2}\left(\frac{2}{k_B T_L}\right)^\mu \Phi(\mathsf{k})\,[\varepsilon(\mathsf{k})]^\mu \\ &\quad + \Gamma(\mu+1)\,2^{\mu-1}\Phi(\mathsf{k})\,, \quad \mathsf{k} \in \mathbb{R}^3\,,\end{aligned}$$

which proves the assertion.

We are in a position now to establish the following mass conservation result for the solution f of problem (14)–(15) obtained in Section 3:

Theorem 4. *If* $\mu \in [1, +\infty)$*, condition* (34) *holds, and moreover*

$$\limsup_{|\mathsf{k}| \to +\infty} \frac{\nu(\mathsf{k}) + \nu^*(\mathsf{k})}{\varepsilon(\mathsf{k})^\mu} < +\infty, \tag{41}$$

then

$$\int_{\mathbb{R}^3} f(t, \mathsf{k})\, d\mathsf{k} = \int_{\mathbb{R}^3} \Phi(\mathsf{k})\, d\mathsf{k} \quad \textit{for all non-negative } t\,.$$

Proof. Through Lemma 4, Theorem 2, and the monotone convergence theorem, we have

$$\int_{\mathbb{R}^3} f(t, \mathsf{k})\, \varepsilon(\mathsf{k})^\mu\, d\mathsf{k} \le c\,, \qquad t \in \mathbb{R}_0^+\,. \tag{42}$$

Fix any $t > 0$. Combining the absolute continuity of $f_n(\cdot, \mathsf{k})$ with (35) yields

$$\int_{\mathbb{R}^3} f_n(t, \mathsf{k})\, d\mathsf{k} = \int_{\mathbb{R}^3} \Phi_n(\mathsf{k})\, d\mathsf{k}$$
$$+ \int_{\mathbb{R}^3} d\mathsf{k} \int_0^t \left\{ Q_n(f_n)(r, \mathsf{k}) + \left[(\nu_n(\mathsf{k}) + \nu_n^*(\mathsf{k})) - (\nu(\mathsf{k}) + \nu^*(\mathsf{k})) \right] f_n(r, \mathsf{k}) \right\} dr.$$

Due to Fubini's Theorem and (28), it implies that

$$\int_{\mathbb{R}^3} f_n(t, \mathsf{k})\, d\mathsf{k} = \int_{\mathbb{R}^3} \Phi_n(\mathsf{k})\, d\mathsf{k}$$
$$- \int_0^t dr \int_{\mathbb{R}^3} \left[(\nu(\mathsf{k}) + \nu^*(\mathsf{k})) - (\nu_n(\mathsf{k}) + \nu_n^*(\mathsf{k})) \right] f_n(r, \mathsf{k})\, d\mathsf{k} \quad \forall n \in \mathbb{N}. \tag{43}$$

Next, observe that

$$0 \le \left[(\nu + \nu^*) - (\nu_n + \nu_n^*) \right] f_n \le (\nu + \nu^*) f \quad \forall n \in \mathbb{N}\,, \tag{44}$$

$$\lim_{n \to +\infty} \left[(\nu + \nu^*) - (\nu_n + \nu_n^*) \right] f_n = 0 \tag{45}$$

pointwise in $\mathbb{R}_0^+ \times \mathbb{R}^3$. Since, by (41), there exists $\ell, r > 0$ fulfilling

$$\left[\nu(\mathsf{k}) + \nu^*(\mathsf{k}) \right] f(r, \mathsf{k}) \le \ell f(r, \mathsf{k}) \varepsilon(\mathsf{k})^\mu \quad \forall (r, \mathsf{k}) \in [0, t] \times (\mathbb{R}^3 \setminus B_r)\,, \tag{46}$$

where $B_r = \left\{ \mathsf{k} \in \mathbb{R}^3 \ : |\mathsf{k}| \le r \right\}$, while the function $(\nu + \nu^*) f|_{[0,t] \times B_r}$ belongs to the space $L^1([0, t] \times B_r)$, taking account of (44), (46), (42), besides (45), the dominated convergence theorem can be applied, and we obtain

$$\lim_{n \to +\infty} \int_0^t dr \int_{\mathbb{R}^3} \left[(\nu(\mathsf{k}) + \nu^*(\mathsf{k})) - (\nu_n(\mathsf{k}) + \nu_n^*(\mathsf{k})) \right] f_n(r, \mathsf{k})\, d\mathsf{k} = 0\,.$$

Now, the conclusion follows from (43), Theorem 2, and the monotone convergence theorem. □

5. Uniqueness and continuity

This final section deals with both the uniqueness and the continuity of solutions to problem (14)–(15). In [19], the same hypotheses employed to get existence also yield these properties. It might be no longer true without boundedness conditions on the collision frequencies. In fact, for instance, the existence of multiple solutions to the linear Boltzmann equation obtained from (1) by replacing the factor $(1 - f)$ with 1 has been proved in [1].

Let us first consider the uniqueness problem. Under additional assumptions, a uniqueness result is established through a fixed point theorem in ordered Banach spaces. We adopt the same notation as above and write Z_Φ for the set of measurable functions $g : \mathbb{R}_0^+ \times \mathbb{R}^3 \to [0, 1]$ such that

$$g(t, \cdot) \in L^1(\mathbb{R}^3) \quad \text{and} \quad \int_{\mathbb{R}^3} g(t, \mathsf{k})\, d\mathsf{k} = \int_{\mathbb{R}^3} \Phi(\mathsf{k})\, d\mathsf{k}, \quad \forall t \in \mathbb{R}_0^+.$$

Theorem 5. *Suppose the function f given by Theorem* 2 *lies in Z_Φ. Then, for every $g \in Z_\Phi$ which satisfies* (30)*, one has $f(t, \mathsf{k}) = g(t, \mathsf{k})$ at almost all* $(t, \mathsf{k}) \in \mathbb{R}_0^+ \times \mathbb{R}^3$.

Proof. Pick $g \in Z_\Phi$ as well as $t > 0$, and consider the real Banach space $L^1([0, t] \times \mathbb{R}^3)$ with the partial ordering

$$h_1 \leq h_2 \text{ if and only if } h_1(r, \mathsf{k}) \leq h_2(r, \mathsf{k}) \text{ a.e. in } [0, t] \times \mathbb{R}^3.$$

The same arguments exploited in the proof of Lemma 1 ensure here that $T_n(h_1) \leq T_n(h_2)$ provided $h_1, h_2 \in [0, g]$ and $h_1 \leq h_2$, where T_n is defined by (23) while $[0, g]$ denotes the order interval

$$[0, g] = \left\{h \in L^1([0, t] \times \mathbb{R}^3) \ : 0 \leq h \leq g\right\}.$$

Since g fulfils (30), we also have

$$0 \leq T_n(h) \leq T_n(g) \leq g \quad \forall h \in [0, g],$$

namely $T_n([0, g]) \subseteq [0, g]$. Thus, thanks to [2, Theorem 2.1] (see besides [15, Theorem I]), the operator T_n possesses a fixed point $h \in [0, g]$. Bearing in mind Remark 2, from $f_n = T_n(f_n)$ in $\mathbb{R}_0^+ \times \mathbb{R}^3$ and $h = T_n(h)$ in $[0, t] \times \mathbb{R}^3$, it immediately follows that

$$f_n(r, \mathsf{k}) \leq h(r, \mathsf{k}) \leq g(r, \mathsf{k}) \quad \text{at almost all } (r, \mathsf{k}) \in [0, t] \times \mathbb{R}^3.$$

As n was arbitrary, Theorem 2 leads to

$$f(r, \mathsf{k}) \leq g(r, \mathsf{k}) \quad \text{a.e. in } [0, t] \times \mathbb{R}^3.$$

Hence, taking account of the assumption $f, g \in Z_\Phi$, we obtain

$$0 \leq \int_0^t dr \int_{\mathbb{R}^3} [g(r, \mathsf{k}) - f(r, \mathsf{k})]\, d\mathsf{k} = 0,$$

which implies $f = g$ in $[0, t] \times \mathbb{R}^3$. □

An easy consequence of Theorems 4 and 5, besides (42), is the following:

Corollary 1. *If conditions* (34) *and* (41) *hold for some* $\mu \in [1, +\infty)$ *then problem* (14)–(15) *has a unique solution* $f \in Z_{\Phi}$. *Moreover,*

$$\int_{\mathbb{R}^3} f(t, \mathsf{k})\,[\varepsilon(\mathsf{k})]^{\mu} d\mathsf{k} \le c \quad \forall t \in \mathbb{R}_0^+,$$

where c is given by (38).

Let us next treat the continuity problem. We will make the following assumption concerning the collision frequencies:

(a_3) *There exist* $\tau > 0$, $a \ge 0$, $q \ge 0$ *such that*

$$L_{\gamma} = \sup\left\{ \frac{\bar{\nu}(\mathsf{k})\left(e^{a\hbar\bar{\omega}\tau} + 1\right)}{\nu(\mathsf{k}) + \nu^*(\mathsf{k}) + \gamma(\mathsf{k})} \ : \mathsf{k} \in \mathbb{R}^3 \right\} < 1\,,$$

where $\bar{\omega} = \max\{\omega_i \ : 1 \le i \le p\}$ *and* $\gamma(\mathsf{k}) = a\ \varepsilon(\mathsf{k}) + q$, $\mathsf{k} \in \mathbb{R}^3$.

For instance, (a_3) is satisfied whenever $\bar{\nu}(\mathsf{k}) = o(\varepsilon(\mathsf{k}))$, as $|\mathsf{k}|$ goes to infinity.

Theorem 6. *If* (a_3) *holds, then problem* (14)–(15) *admits a solution belonging to* $C^0(\mathbb{R}_0^+ \times \mathbb{R}^3)$.

Proof. The proof is patterned after that of Theorem 1. So, we only give the main ideas. Let $(X_{\tau}, \|\cdot\|_{\gamma})$ be the real Banach space of all continuous functions $g : [0, \tau] \times \mathbb{R}^3 \to \mathbb{R}$ such that

$$\|g\|_{\gamma} = \sup\left\{ e^{-\gamma(\mathsf{k})t} |g(t, \mathsf{k})| \ : (t, \mathsf{k}) \in [0, \tau] \times \mathbb{R}^3 \right\} < +\infty\,.$$

Define

$$Z_{\tau} = \left\{ g \in C^0([0, \tau] \times \mathbb{R}^3) \ : 0 \le g(t, \mathsf{k}) \le 1 \ \forall (t, \mathsf{k}) \in [0, \tau] \times \mathbb{R}^3 \right\}.$$

For every $g \in Z_{\tau}$, $(t, \mathsf{k}) \in [0, \tau] \times \mathbb{R}^3$, we set

$$T(g)(t, \mathsf{k}) = e^{-[\nu(\mathsf{k}) + \nu^*(\mathsf{k})]t} \Phi(\mathsf{k}) + \int_0^t e^{[\nu(\mathsf{k}) + \nu^*(\mathsf{k})](r - t)} N(g)(r, \mathsf{k})\, dr,$$

where $N(g)$ is given by (29). One clearly has $T(g) \in C^0([0, \tau] \times \mathbb{R}^3)$ because, owing to [19, Lemma B.4], the function $N(g)$ is continuous. A simple computation then shows that $T(Z_{\tau}) \subseteq Z_{\tau}$. Now, let

$$\bar{A}(g)(t, \mathsf{k}) = \int_{\mathbb{R}^3} \bar{S}(\mathsf{k}, \mathsf{k}') g(t, \mathsf{k}')\, d\mathsf{k}', \ \ \bar{\nu}(\mathsf{k}) = \bar{A}(1)(t, \mathsf{k}), \ \ (t, \mathsf{k}) \in [0, \tau] \times \mathbb{R}^3.$$

Arguing as in the proof of Theorem 1 we obtain

$$|N(g) - N(h)| \le \bar{A}(|g - h|) + \bar{\nu}|g - h| \quad \forall g, h \in Z_{\tau}\,. \tag{47}$$

Since, on account of (19),

$$\begin{aligned}\bar{A}(|g-h|)(t,\mathsf{k}) &\le \|g-h\|_\gamma \int_{\mathbb{R}^3} \bar{S}(\mathsf{k},\mathsf{k}')\mathrm{e}^{\gamma(\mathsf{k}')t} d\mathsf{k}' \\ &= \mathrm{e}^{\gamma(\mathsf{k})t}\,\|g-h\|_\gamma \int_{\mathbb{R}^3} \sum_{i=1}^{p} \mathcal{G}_i(\mathsf{k},\mathsf{k}')(n_i+1) \\ &\quad \times \left[\mathrm{e}^{-a\hbar\omega_i t}\,\delta(\varepsilon'-\varepsilon+\hbar\omega_i) + \mathrm{e}^{a\hbar\omega_i t}\,\delta(\varepsilon'-\varepsilon-\hbar\omega_i)\right] d\mathsf{k}' \\ &\le \mathrm{e}^{\gamma(\mathsf{k})t}\,\|g-h\|_\gamma\,\mathrm{e}^{a\hbar\bar{\omega}t}\,\bar{\nu}(\mathsf{k})\,,\end{aligned}$$

inequality (47) immediately leads to

$$|N(g)(t,\mathsf{k}) - N(h)(t,\mathsf{k})| \le \mathrm{e}^{\gamma(\mathsf{k})t}\left(\mathrm{e}^{a\hbar\bar{\omega}t}+1\right)\|g-h\|_\gamma\,\bar{\nu}(\mathsf{k})\,.$$

Therefore, as in the proof of Theorem 1,

$$\|T(g)-T(h)\|_\gamma \le L_\gamma \|g-h\|_\gamma \quad \forall g\,,h \in Z_\tau\,.$$

By hypothesis (a_3), the Banach–Caccioppoli fixed-point principle can be applied, and we get a function $g_1 \in Z_\tau$ fulfilling (14)–(15) in $[0,\tau]\times\mathbb{R}^3$. Now, we note that the time interval $[0,\tau]$ does not depend on the initial condition and that the variable t does not explicitly appear in (14). This allows us to find a global solution in $\mathbb{R}_0^+$ by adopting the standard continuation technique employed for autonomous systems of ordinary differential equations. In fact, through the above arguments we can prove the existence of a unique solution $g_2 \in C^0([\tau,2\,\tau]\times\mathbb{R}^3)$ to (14) satisfying the initial condition $g_2(\tau,\mathsf{k}) = g_1(\tau,\mathsf{k})$, for all $\mathsf{k}\in\mathbb{R}^3$. Repeating the process, we finally arrive at the desired conclusion. □

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