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On the asymptotic properties of IMEX Runge–Kutta schemes for hyperbolic balance laws

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ABSTRACT

Implicit–Explicit (IMEX) schemes are a powerful tool in the development of numerical methods for hyperbolic systems with stiff sources. Here we focus our attention on the asymptotic properties of such schemes, like the preservation of steady-states (well-balanced property) and the behavior in presence of small space–time scales (asymptotic preservation property). We analyze conditions under which the standard additive approach based on taking the fluxes explicitly and the sources implicitly yields a well-balanced behavior. In addition, we consider a partitioned strategy which possesses better well-balanced properties. The behavior of the additive and partitioned approaches under classical scaling limits is then studied in the context of asymptotic-preserving schemes. Additional order conditions that guarantee the correct behavior of the schemes in the Navier–Stokes regime are derived. Several examples illustrate these asymptotic behaviors and the performance of the new methods.

1. Introduction

We consider hyperbolic systems of conservation laws with sources in the form

$$U_t + F(U)_x = G(U), \tag{1}$$

where $U \in \mathbb{R}^N$, $F, G : \mathbb{R}^N \to \mathbb{R}^N$ and the Jacobian matrix $F'(U)$ has real eigenvalues and admits a basis of eigenvectors $\forall\, U \in \mathbb{R}^N$. For mathematical results concerning this kind of problems we refer to [1,2].

Implicit–Explicit (IMEX) Runge–Kutta schemes [3–19] represent a powerful tool for the time discretization of hyperbolic systems with sources of the form (1). The key idea of the approach is the assumption that the source term $G(U)$ operates on a much smaller time scale compared to the flux $F(U)_x$ and thus an implicit treatment is somewhat necessary to avoid too severe time step restrictions. On the other hand the flux term involves several other difficulties related to its nonlinear hyperbolic structure and an explicit treatment is almost mandatory in the construction of an effective numerical method. We will refer to this kind of methods based on taking the flux explicit and the source implicit as the *additive IMEX schemes*.

Compared to operator splitting methods the main advantage is the capability to achieve higher order accuracy even in presence of very stiff source terms [4–6,13,14,17,20]. Moreover since operator splitting is avoided IMEX Runge–Kutta schemes are good candidates for the development of well-balanced discretizations. It is well-known that a well-balanced approach is capable to preserve the steady state solutions U^* characterized by

$$F(U^*)_x = G(U^*). \tag{2}$$

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There is a large literature concerning the development of such methods, without being exhaustive we refer to [21–32] and the references therein. Most of the research activity has been focused on the development of suitable numerical fluxes that minimize the effect of numerical viscosity in (2) due to the space discretization.

Here we tackle the problem from a slightly different perspective since we will focus on the issue of time discretization and how this can influence the well-balancing features of a numerical method. To keep notations simple, along the manuscript we will use one-dimensional notation for the space derivatives even if our arguments are not limited by the dimension of the space. In contrast with standard Runge–Kutta methods the application of IMEX schemes may have a strong influence on the well-balancing properties of the resulting scheme. This is essentially due to the interplay of the different time levels in the stages of the explicit and the implicit solvers.

A particular type of hyperbolic system that we will also use to illustrate the subsequent theory is the following [33,1]

$$\begin{aligned} &u_t + f_1(u, v)_x = 0,\\ &v_t + f_2(u)_x = g(u, v). \end{aligned} \tag{3}$$

System (3) is a particular case of (1) for $U = (u, v)^T$, $u \in \mathbb{R}^M$, $v \in \mathbb{R}^{N-M}$, $M < N$, $F = (f_1, f_2)^T$ and $G = (0, g)^T$. For steady flow u^*, v^* the behavior of the solution to (3) is governed by

$$f_1(u^*, v^*)_x = 0, \qquad f_2(u^*)_x = g(u^*, v^*). \tag{4}$$

In this case, the system is naturally partitioned in two subsystems characterized by the two equations in (3) which can be solved with two different Runge–Kutta methods. The application of an explicit solver for the first equation combined with an implicit solver for the second equation yields what we will call a *partitioned IMEX scheme*. Note that since the flux in the second equation of (3) depends only on u which is explicitly computed from the first equation, in practice the scheme requires only the inversion of the source term as for the additive approach. Examples of schemes of this type have been considered in [33–35].

In this manuscript we will derive conditions under which the two above mentioned approaches yield a well-balanced time discretization. In addition, in the second part of the paper we will study how these IMEX schemes behave under the classical fluid and diffusive scalings in the context of asymptotic-preserving schemes [36]. In particular, for the fluid-scaling we will derive order conditions that guarantee $O(\varepsilon)$ accuracy of the numerical solution with respect to the scaling parameter ε. We emphasize that the order conditions here are computed directly from a simple prototype of hyperbolic system with relaxation through the classical Chapman–Enskog expansion. This approach differs from previous approaches in the literature [5,37] since it is based directly on a system of PDEs and corresponds to an asymptotic preserving property in the so-called Navier–Stokes regime.

The rest of the paper is organized as follows. The Section 2 is devoted to the study of the well-balanced properties of the additive and partitioned IMEX schemes. It is shown that the standard additive approach requires particular care in order to obtain a well-balanced discretization. On the contrary the partitioned strategy yields naturally a well-balanced scheme. In Section 3 we rescale our system accordingly to the standard fluid and diffusive limit and analyze the asymptotic preserving properties of the well-balanced methods. Additional order conditions that guarantee $O(\varepsilon)$ accuracy in the fluid-limit are also derived. We end the manuscript with several numerical examples illustrating the behavior of the schemes. Examples of second and third order schemes satisfying the additional order conditions are reported in separate Appendix.

2. Well balanced property

2.1. IMEX Runge–Kutta methods

First we introduce the general formulation of IMEX schemes (1) together with some preliminary definitions.

A general IMEX Runge–Kutta scheme applied to the differential system

$$y' = f(y) + g(y), \tag{5}$$

reads

$$Y^{(i)} = y^n + \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} f(Y^{(j)}) + \Delta t \sum_{j=1}^{\nu} a_{ij} g(Y^{(j)}) \tag{6}$$

$$y^{n+1} = y^n + \Delta t \sum_{i=1}^{\nu} \tilde{w}_i f(Y^{(i)}) + \Delta t \sum_{i=1}^{\nu} w_i g(Y^{(i)}). \tag{7}$$

The matrices $\tilde{A} = (\tilde{a}_{ij})$, $\tilde{a}_{ij} = 0$ for $j \geq i$ and $A = (a_{ij})$ are $\nu \times \nu$ matrices such that the resulting scheme is explicit in f, and implicit in g. In general, an IMEX Runge–Kutta scheme, is characterized by the above defined two matrices and the coefficient vectors $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_\nu)^T$, $w = (w_1, \ldots, w_\nu)^T$.

Note that IMEX Runge–Kutta schemes can be represented by a double tableau in the usual Butcher notation,

$$\begin{array}{c|c} \tilde{c} & \tilde{A} \\ \hline & \tilde{w} \end{array} \qquad \begin{array}{c|c} c & A \\ \hline & w \end{array}. \tag{8}$$

Since computational efficiency in most cases is of paramount importance, usually IMEX Runge–Kutta schemes are restricted to diagonally implicit Runge–Kutta (DIRK) methods ($a_{ij} = 0$, for $j > i$). In fact, the use of a DIRK scheme is enough to assure that the term $f(y)$ is always evaluated explicitly. We also define the coefficients $\tilde{c}$ and c by the usual relation $\tilde{c} = \tilde{A}\mathbf{e}$, $c = A\mathbf{e}$, with $\mathbf{e} = (1, \ldots, 1) \in \mathbb{R}^{\nu}$.

We refer to [3,7,38,17] for details on the order conditions for IMEX schemes. Let us remark that IMEX schemes are a particular case of additive Runge–Kutta methods and so the order conditions can be derived as a generalization of the notion of Butcher tree [38]. Roughly speaking combined order conditions take into account standard order conditions for the two Runge–Kutta methods together with mixed conditions originated by all possible configurations of the vectors $c, \tilde{c}, w, \tilde{w}$ and the matrices A and $\tilde{A}$ in the standard order conditions. As a consequence, under the assumptions $\widetilde{c} = c$ and $\widetilde{w} = w$, mixed order conditions are automatically satisfied up to third order.

It is useful to characterize the different IMEX schemes we will consider in the sequel accordingly to the structure of the DIRK method. Following [33] we have

Definition 1. 1. We call an IMEX-RK method of *type I* or *type A* (see [17]) if the matrix $A \in \mathbb{R}^{\nu\times\nu}$ is invertible, or equivalently $a_{ii} \neq 0, i = 1, \ldots, \nu$.

2. We call an IMEX-RK method of *type II* or *type CK* (see [7]) if the matrix A can be written as

$$A = \begin{pmatrix} 0 & 0 \\ a & \hat{A} \end{pmatrix}, \tag{9}$$

with $a = (a_{21}, \ldots, a_{\nu 1})^T \in \mathbb{R}^{(\nu-1)}$ and the submatrix $\hat{A} \in \mathbb{R}^{(\nu-1)\times(\nu-1)}$ is invertible, or equivalently $a_{ii} \neq 0, i = 2, \ldots, \nu$. In the special case $a = 0$, $w_1 = 0$ the scheme is said to be of *type ARS* (see [3]) and the DIRK method is reducible to a method using $\nu - 1$ stages.

We will also make use of the following representation of the matrix $\tilde{A}$ in the explicit Runge–Kutta method

$$\tilde{A} = \begin{pmatrix} 0 & 0 \\ \tilde{a} & \hat{\tilde{A}} \end{pmatrix}, \tag{10}$$

where $\tilde{a} = (\tilde{a}_{21}, \ldots, \tilde{a}_{\nu 1})^T \in \mathbb{R}^{\nu-1}$ and $\hat{\tilde{A}} \in \mathbb{R}^{\nu-1\times\nu-1}$.

The following definition will be also useful to characterize the properties of the methods in the sequel.

Definition 2. We call an IMEX-RK method *implicitly stiffly accurate (ISA)* if the corresponding DIRK method is *stiffly accurate*, namely

$$a_{\nu i} = w_i, \quad i = 1, \ldots, \nu. \tag{11}$$

If in addition the explicit methods satisfy

$$\tilde{a}_{\nu i} = \tilde{w}_i, \quad i = 1, \ldots, \nu - 1 \tag{12}$$

the IMEX-RK method is said to be *globally stiffly accurate (GSA)*.

The above definitions follow naturally from the fact that ν-stage implicit Runge–Kutta methods for which $a_{\nu i} = w_i$ for $i = 1, \ldots, \nu$ are called *stiffly accurate* (see [38] for details) and ν-stage explicit Runge–Kutta methods for which $\tilde{a}_{\nu i} = \tilde{w}_i$ for $i = 1, \ldots \nu - 1$ are called FSAL (*First Same As Last*, see [39] for details). Note that FSAL methods have the advantage that they require only $\nu - 1$ function evaluations per time step, because the last stage of step n coincides with the first stage of the step $n + 1$.

Therefore, an IMEX-RK scheme is globally stiffly accurate if the implicit scheme is stiffly accurate and the explicit scheme is FSAL. Note that IMEX-RK schemes with the GSA property were already considered in [33,40].

Finally we introduce the notion of well-balanced for an IMEX method.

Definition 3. An IMEX scheme (6)–(7) is said *well-balanced* if $f(y^n) + g(y^n) = 0$ implies $y^{n+1} = y^n$.

Before entering the analysis of the schemes some remarks are in order.

Remark 1.
- For type I methods we have $a_{11} \neq 0$ and $\tilde{a}_{11} = 0$ thus $c_1 \neq \tilde{c}_1$ and we cannot assume the simplifying condition $c = \tilde{c}$.
- A special class of s-stage explicit Runge–Kutta schemes for which $\tilde{a}_{\nu i} = \tilde{w}_i$, for $i = 1, \ldots, \nu$ is called *First Same As Last* (FSAL). Such schemes have the advantage that they require only $\nu - 1$ function evaluations per time step, because the last stage of step n coincides with the first step of the step $n + 1$, (see [39] for details).
- The GSA property implies $\tilde{w}_\nu = 0$ and $w_\nu \neq 0$ thus we cannot assume the simplifying condition $w = \tilde{w}$. Note that for GSA schemes the numerical solution is the same as the last stage value, namely $y^{n+1} = Y^{(\nu)}$.

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2.2. The additive and partitioned approaches

We rewrite the IMEX Runge–Kutta method applied to system (1) using vector notations

$$\mathbf{U} = U^n\mathbf{e} - \Delta t\tilde{A}\,F(\mathbf{U})_x + \Delta t A G(\mathbf{U}) \tag{13}$$

$$U^{n+1} = U^n - \Delta t\tilde{w}^T F(\mathbf{U})_x + \Delta t w^T G(\mathbf{U}), \tag{14}$$

where $\mathbf{e} = (1, 1, \dots, 1)^T \in \mathbb{R}^{\nu\times N}$, $\mathbf{U} = (U^{(1)}, \dots, U^{(\nu)})^T$, $F(\mathbf{U}) = (F(U^{(1)}), \dots, F(U^{(\nu)}))^T$ and $G(\mathbf{U}) = (G(U^{(1)}), \dots, G(U^{(\nu)}))^T$.

We assume that the source term G is a smooth function such that uniqueness of the solution to (13)–(14) is guaranteed or equivalently that the equation

$$U^{(i)} - \Delta t a_{ii} G(U^{(i)}) = V,$$

for a given $V \in \mathbb{R}^N$, admits a unique solution $U^{(i)}$. We also assume that the IMEX scheme satisfies at least the first order conditions $\tilde{w}^T\mathbf{e} = w^T\mathbf{e} = 1$.

We can prove the following result

Theorem 1. *If $\tilde{c} = c$ then the IMEX scheme (13)–(14) is well-balanced in the sense that $F(U^n)_x = G(U^n)$ implies $U^{n+1} = U^n$.*

Proof. From (14) we have $U^{n+1} = U^n$ if

$$\tilde{w}^T F(\mathbf{U})_x + w^T G(\mathbf{U}) = 0. \tag{15}$$

This is guaranteed if $\mathbf{U} = U^n\mathbf{e}$ since in this case we have $F(\mathbf{U})_x = F(U^n)_x\mathbf{e}$, $G(\mathbf{U}) = G(U^n)\mathbf{e}$ and from the first order conditions $\tilde{w}^T\mathbf{e} = w^T\mathbf{e} = 1$ we get

$$\tilde{w}^T F(U^n)_x\mathbf{e} + w^T G(U^n)\mathbf{e} = F(U^n)_x + G(U^n) = 0.$$

Let us now verify that $\mathbf{U} = U^n\mathbf{e}$ is a solution to (13) under the assumption $F(U^n)_x = G(U^n)$. In this case we need to satisfy

$$-\tilde{A}\,F(\mathbf{U})_x + AG(\mathbf{U}) = 0. \tag{16}$$

Substituting $F(\mathbf{U})_x = F(U^n)_x\mathbf{e}$ and $G(\mathbf{U}) = G(U^n)\mathbf{e}$ we obtain

$$-\tilde{A}\,F(U^n)_x\mathbf{e} + AG(U^n)\mathbf{e} = -\tilde{c}\,F(U^n)_x + c\,G(U^n) = 0,$$

since $\tilde{c} = \tilde{A}\mathbf{e}$, $c = A\mathbf{e}$ and by hypothesis $\tilde{c} = c$. $\square$

Remark 2. As a consequence type I IMEX schemes in additive form are not well-balanced since the first stage

$$U^{(1)} = U^n + ha_{11}G(U^{(1)}),$$

does not preserve the steady state manifold $F(U^n)_x - G(U^n) = 0$ unless $a_{11} = 0$. Analogous conclusions can be derived as a consequence of the local error estimates for IMEX Runge–Kutta methods in [10].

The partitioned IMEX Runge–Kutta method applied to system (3) reads

$$u^{(i)} = u^n - \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} f_1(u^{(j)}, v^{(j)})_x$$

$$v^{(i)} = v^n - \Delta t \sum_{j=1}^{i} a_{ij} \left(f_2(u^{(j)})_x - g(u^{(j)}, v^{(j)})\right) \tag{17}$$

$$u^{n+1} = u^n - \Delta t \sum_{i=1}^{\nu} \tilde{w}_i f_1(u^{(i)}, v^{(i)})_x$$

$$v^{n+1} = v^n - \Delta t \sum_{i=1}^{\nu} w_i \left(f_2(u^{(i)})_x - g(u^{(i)}, v^{(i)})\right), \tag{18}$$

or equivalently in vector form

$$\mathbf{u} = u^n\mathbf{e}_1 - \Delta t\tilde{A} f_1(\mathbf{u}, \mathbf{v})_x$$

$$\mathbf{v} = v^n\mathbf{e}_2 - \Delta t A\,(f_2(\mathbf{u})_x - g(\mathbf{u}, \mathbf{v})) \tag{19}$$

$$u^{n+1} = u^n - \Delta t\tilde{w}^T f_1(\mathbf{u}, \mathbf{v})_x,$$

$$v^{n+1} = v^n - \Delta t w^T\,(f_2(\mathbf{u})_x - g(\mathbf{u}, \mathbf{v})), \tag{20}$$

with $\mathbf{u}, f_1(\mathbf{u}, \mathbf{v}), \mathbf{e}_1 \in \mathbb{R}^{\nu\times M}$ and $\mathbf{v}, f_2(\mathbf{u}), g(\mathbf{u}, \mathbf{v}), \mathbf{e}_2 \in \mathbb{R}^{\nu\times(N-M)}$.

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It is immediate to show the following

Theorem 2. *The IMEX scheme (19)–(20) is well-balanced in the sense that* $f_1(u^n, v^n)_x = 0$, $f_2(u^n)_x = g(u^n, v^n)$ *implies* $u^{n+1} = u^n$, $v^{n+1} = v^n$.

Proof. It is enough to observe that taking $\mathbf{u} = u^n\mathbf{e}_1$ and $\mathbf{v} = v^n\mathbf{e}_2$ we get the system

$$\tilde{A} f_1(\mathbf{u}, \mathbf{v})_x = 0,$$
$$A\ (f_2(\mathbf{u})_x - g(\mathbf{u}, \mathbf{v})) = 0,$$

which under the assumption $f_1(u^n, v^n)_x = 0, f_2(u^n)_x = g(u^n, v^n)$ admits the unique solution $f_1(\mathbf{u}, \mathbf{v})_x = 0, f_2(\mathbf{u})_x = g(\mathbf{u}, \mathbf{v})$. This permits to conclude that $u^{n+1} = u^n$, $v^{n+1} = v^n$. □

3. Asymptotic preservation properties

Asymptotic preservation properties of IMEX Runge–Kutta methods have been analyzed and studied in several papers [33,41,42,13,17]. Here we quickly recall some results and derive conditions for uniform accuracy in the case of a prototype hyperbolic problem in the Chapman–Enskog expansion.

The fluid scaling of system (1) is obtained introducing a small nonnegative parameter ε and by rescaling space and time in a symmetric way accordingly to $x' \to \varepsilon x$, $t' \to \varepsilon t$. By omitting the primes we can write the scaled system

$$U_t + F(U)_x = \frac{1}{\varepsilon} G(U). \tag{21}$$

To characterize the behavior for small values of ε it is natural to consider a hyperbolic system with source of relaxation type. Namely we assume that $G(U)$ is a dissipative relaxation operator [1]. Such operator is endowed with a $M \times N$ matrix Q of rank $M < N$ such that $QG(U) = 0, \forall\ U$. This gives a vector of M conserved quantities $u = QU$. Solutions U which belong to the kernel of the operator G, namely $G(U) = 0$, are uniquely determined by the conserved quantities u, in the form $U = E(u)$ where $E(u)$ is called local equilibrium.

Thus system (21) is associated with the M-dimensional set of conservation laws

$$QU_t + QF(U)_x = 0. \tag{22}$$

As $\varepsilon \to 0$ we obtain $G(U) = 0$ and so $U = E(u)$. The above system reduces to the equilibrium conservation laws

$$u_t + f(u)_x = 0, \tag{23}$$

where $f(u) = QF(E(u))$.

Schemes which are capable to capture numerically the asymptotic process just described are referred to as *asymptotic preserving* in the fluid-limit $\varepsilon \to 0$. We refer the reader to [41,42] for more precise definitions of the terminology here used and applications to kinetic equations.

Since system (3) is a particular case of (1) the above arguments apply in the same way. Note that in this latter case the matrix Q is of the form $Q = (I, 0)$ with I the M-dimensional identity matrix.

Here we omit the obvious details of the application of the standard additive and the partitioned approaches which correspond to the rescaled methods (13)–(14) and (19)–(20).

3.1. The limit behavior

First we recall the main result [17] which applies both to the additive and the partitioned approach.

Theorem 3. *If the IMEX method applied to (21), in additive or partitioned form, is of type I then in the limit* $\varepsilon \to 0$ *it becomes the explicit RK scheme characterized by the pair* $(\tilde{A}, \tilde{w})$ *applied to the limit equilibrium system (23).*

Proof. First let us remark that both approaches are associated with the following explicit Runge–Kutta scheme for (22)

$$Q\mathbf{U} = QU^n\mathbf{e} - \Delta t\tilde{A}\, QF(\mathbf{U})_x \tag{24}$$
$$QU^{n+1} = QU^n - \Delta t\,\tilde{w}^T QF(\mathbf{U})_x, \tag{25}$$

where we used the notation abuse $Q\mathbf{U}$ to denote $(QU^{(1)}, \dots, QU^{(\nu)})^T$ and similarly for $QF(\mathbf{U})_x$.

It is easy to verify that in the limit $\varepsilon \to 0$ the IMEX schemes collapse to the system

$$A\, G(\mathbf{U}) = 0. \tag{26}$$

The above system, being A invertible, corresponds to the set of algebraic equations $G(\mathbf{U}) = 0$, which admits $\mathbf{U} = E(\mathbf{u})$ as unique solution, with $\mathbf{u} = Q\mathbf{U}$. By direct substitution into (24)–(25) we obtain the desired scheme for (23). □

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If we furthermore ask that the limiting equilibrium solution lies on the equilibrium manifold, namely $G(U^{n+1}) = 0$, this is guaranteed if the IMEX scheme is GSA since in this latter case we have $U^{n+1} = U^{(\nu)}$. Note however that for type I IMEX schemes the lack of GSA property does not imply any loss of accuracy on the equilibrium system (23).

For IMEX schemes of type II the situation is different and it is useful to introduce the notion of initial data consistent with the limit problem.

Definition 4. The initial data for Eq. (21) are said *consistent* or *well prepared* if

$$U_0(x) = E(u_0) + O(\varepsilon). \tag{27}$$

We can now show the following [41].

Theorem 4. *If the IMEX method applied to (21), in additive or partitioned form, is of type II and GSA then for consistent initial data in the limit $\varepsilon \to 0$ it becomes the explicit RK scheme characterized by the pair $(\tilde{A}, \tilde{w})$ applied to the limit equilibrium system (23).*

Proof. To prove this result it is enough to observe that as $\varepsilon \to 0$ we get the system

$$a\,G(U^{(1)}) + \hat{A}\,G(\hat{\mathbf{U}}) = 0, \tag{28}$$

where we used notations (9) and set $\mathbf{U} = (U^{(1)}, \hat{\mathbf{U}})^T$, $G(\mathbf{U}) = (G(U^{(1)}), G(\hat{\mathbf{U}}))^T$. If the initial data are well prepared we have $G(U^{(1)}) = 0$ since $U^{(1)} = E(u^{(1)})$ and from the invertibility of $\hat{A}$ we obtain $\hat{\mathbf{U}} = E(\hat{\mathbf{u}})$ with $\mathbf{u} = (u^{(1)}, \hat{\mathbf{u}})^T$. This is enough to guarantee that the very first time step coincides with the explicit Runge–Kutta method applied to (1). Similarly the GSA property is required to preserve this feature in the numerical solution for the next time step. □

3.2. The $O(\varepsilon)$ behavior

A natural question that arises from the above analysis concerns the behavior of the schemes for small but non zero values of ε. In this case, which physically corresponds to the compressible Navier–Stokes limit, degradation of accuracy is expected unless additional conditions are satisfied by the IMEX method (see [4,5,37,17]). To illustrate this we consider the rescaled problem (3) for $N = 2, M = 1$ in the linear case $f_1(u, v) = v, f_2(u) = \alpha^2 u$ and $g(u, v) = \beta u - v$, where α, β are nonnegative constants

$$\begin{aligned} &u_t + v_x = 0,\\ &v_t + \alpha^2 u_x = -\frac{1}{\varepsilon}(v - \beta u). \end{aligned} \tag{29}$$

Following [1] we derive the following equation for the $O(\varepsilon)$ solutions to (29)

$$u_t + \beta u_x = \varepsilon(\alpha^2 - \beta^2)u_{xx}. \tag{30}$$

Note that the above equation requires the condition $\alpha^2 > \beta^2$ to be well-posed. This implies that the characteristic speed β of the limiting equation (30) is bounded by the characteristic speeds $\pm\alpha$ of the hyperbolic system (29). This condition is usually referred to as *sub-characteristic condition* [1] and represents an important concept regarding the relationship between the wave-dynamics of the relaxation system and the local equilibrium approximation.

In vector form a general IMEX scheme for (29) can be written as

$$\begin{aligned} &\mathbf{u} = u^n\mathbf{e} - \Delta t\tilde{A}\,\mathbf{v}_x\\ &\mathbf{v} = v^n\mathbf{e} - \Delta t A_*\alpha^2\mathbf{u}_x - \Delta t A\frac{1}{\varepsilon}(\mathbf{v} - \beta\mathbf{u}) \end{aligned} \tag{31}$$

$$\begin{aligned} &u^{n+1} = u^n - \Delta t\tilde{w}^T\mathbf{v}_x,\\ &v^{n+1} = v^n - \Delta t w_*^T\alpha^2\mathbf{u}_x - \Delta t w^T\frac{1}{\varepsilon}(\mathbf{v} - \beta\mathbf{u}), \end{aligned} \tag{32}$$

where $A_* = \tilde{A}$, $w_* = \tilde{w}$ for the additive approach and $A_* = A$, $w_* = w$ for the partitioned case.

Similarly to [1] we now consider the expansion

$$\mathbf{v} = \beta\mathbf{u} + \varepsilon\mathbf{v}_1,$$

which from (31) gives

$$A\mathbf{v}_1 = -\frac{\beta\mathbf{u} - v^n\mathbf{e}}{\Delta t} - A_*\alpha^2\mathbf{u}_x - \frac{\varepsilon}{\Delta t}\mathbf{v}_1.$$

If we assume the initial data to be well-prepared

$$v^n = \beta u^n + \varepsilon v_1^n,$$

we obtain

$$\begin{aligned} A\mathbf{v}_1 &= -\beta\frac{\mathbf{u}-u^n\mathbf{e}}{\Delta t} - A_*\alpha^2\mathbf{u}_x - \varepsilon\frac{\mathbf{v}_1 - v_1^n\mathbf{e}}{\Delta t} \\ &= \beta\tilde{A}\mathbf{v}_x - A_*\alpha^2\mathbf{u}_x - \varepsilon\frac{\mathbf{v}_1 - v_1^n\mathbf{e}}{\Delta t} \\ &= \beta^2\tilde{A}\mathbf{u}_x - A_*\alpha^2\mathbf{u}_x - \varepsilon\left(\frac{\mathbf{v}_1 - v_1^n\mathbf{e}}{\Delta t} - \beta\tilde{A}(\mathbf{v}_1)_x\right). \end{aligned}$$

Thus for type I IMEX schemes we can write

$$\mathbf{v}_1 = A^{-1}\left(\tilde{A}\beta^2 - A_*\alpha^2\right)\mathbf{u}_x + O(\varepsilon), \tag{33}$$

where we assumed $\mathbf{v}_1$ to have bounded first derivatives in space and time. We finally obtain the following Runge–Kutta scheme for Eq. (30)

$$\begin{aligned} \mathbf{u} &= u^n\mathbf{e} - \Delta t\tilde{A}\,\beta\mathbf{u}_x + \varepsilon\Delta t\tilde{A}A^{-1}\left(A_*\alpha^2 - \tilde{A}\beta^2\right)\mathbf{u}_{xx} \\ u^{n+1} &= u^n - \Delta t\tilde{w}^T\beta\mathbf{u}_x + \varepsilon\Delta t\tilde{w}^T A^{-1}\left(A_*\alpha^2 - \tilde{A}\beta^2\right)\mathbf{u}_{xx}. \end{aligned} \tag{34}$$

Note that, independently on the choice of A_*, the above scheme can be interpreted as an additive Runge–Kutta method for (30) based on the coefficient matrices $\tilde{A}$, $\tilde{A}A^{-1}\tilde{A}$ and the weights $\tilde{w}^T$, $\tilde{w}^T A^{-1}\tilde{A}$. Thus we must require additionally that this combined Runge–Kutta method satisfies suitable order conditions.

Setting

$$B = \tilde{A}A^{-1}\tilde{A}, \qquad v^T = \tilde{w}^T A^{-1}\tilde{A}, \tag{35}$$

we have the additional conditions, up to third order, reported in Table 1 where $d = B\mathbf{e}$ and we use the notation d^2 to denote the vector $(d_1^2, \dots, d_\nu^2)^T$.

We have the following

Proposition 1. *If an IMEX scheme of type I is GSA then*

$$v^{n+1} - \beta u^{n+1} = \mathcal{O}(\varepsilon). \tag{36}$$

Proof. In fact, in a straightforward manner we obtain from (32)

$$v^{n+1} - \beta u^{n+1} = \Delta t\left((\tilde{w}^T - w^T A^{-1}\tilde{A})\beta^2 + (w^T A^{-1}A_* - w_*^T)\alpha^2\right)\mathbf{u}_x + \mathcal{O}(\varepsilon). \tag{37}$$

If the method is GSA, i.e. ISA and FSAL, we have $\tilde{w}^T - w^T A^{-1}\tilde{A} = 0$ and since $w^T A^{-1}A_* - w_*^T = e_s^T A_* - w_*^T$, then we get in additive or in partitioned approach $e_s^T A_* - w_*^T = 0$ and hence (36). □

A similar analysis can be carried on using type II IMEX methods. In this latter case, using notations (9)–(10) and setting $\mathbf{u}^T = (u^{(1)}, \hat{\mathbf{u}}^T)$, $\tilde{w}^T = (\tilde{w}_1, \hat{\tilde{w}}^T)$, $\mathbf{e}^T = (1, \hat{\mathbf{e}}^T)$, we get for (30) the additive Runge–Kutta scheme

$$u^{(1)} = u^n$$

$$\hat{\mathbf{u}} = u^n\hat{\mathbf{e}} - \Delta t\tilde{a}\,\beta u_x^n - \Delta t\hat{\tilde{A}}\,\beta\hat{\mathbf{u}}_x + \varepsilon\Delta t\hat{\tilde{A}}\hat{A}^{-1}\left(\hat{A}_*\alpha^2 - \hat{\tilde{A}}\beta^2\right)\hat{\mathbf{u}}_{xx} \tag{38}$$

$$+\ \varepsilon\Delta t\hat{\tilde{A}}\hat{A}^{-1}\left(a_*\alpha^2 - \tilde{a}\beta^2\right)u_{xx}^n + \varepsilon\Delta t\left(\hat{\tilde{A}}\hat{A}^{-1}a - \tilde{a}\right)(v_1^n)_x \tag{39}$$

$$u^{n+1} = u^n - \Delta t\tilde{w}^T\beta\mathbf{u}_x + \varepsilon\Delta t\hat{\tilde{w}}^T\hat{A}^{-1}\left(\hat{A}_*\alpha^2 - \hat{\tilde{A}}\beta^2\right)\hat{\mathbf{u}}_{xx}$$

$$+\ \varepsilon\Delta t\hat{\tilde{w}}^T\hat{A}^{-1}\left(a_*\alpha^2 - \tilde{a}\beta^2\right)u_{xx}^n + \varepsilon\Delta t\left(\hat{\tilde{w}}^T\hat{A}^{-1}a - \tilde{w}_1\right)(v_1^n)_x. \tag{40}$$

From the above expression, since v_1^n is arbitrary, the following conditions must be satisfied by type II IMEX schemes

$$\hat{\tilde{A}}\hat{A}^{-1}a = \tilde{a}, \qquad \hat{\tilde{w}}^T\hat{A}^{-1}a = \tilde{w}_1. \tag{41}$$

Concerning these additional conditions we have the following proposition

Proposition 2. *If an IMEX scheme of type II is GSA the condition $\hat{\tilde{A}}\hat{A}^{-1}a = \tilde{a}$ implies $\hat{\tilde{w}}^T\hat{A}^{-1}a = \tilde{w}_1$.*

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Table 1
Additional order conditions up to order three for $O(\varepsilon)$ accuracy. Here B and v^T are given by (35) for type I IMEX schemes and by (42) for type II IMEX schemes. For type II schemes relations (41) must also be satisfied.

Combined order	Additional $O(\varepsilon)$ conditions
First order	$v^T\mathbf{e} = 1$
Second order	$\tilde{w}^T d = 1/2$, $v^T\tilde{c} = 1/2$
Third order	$v^T\tilde{c}^2 = 1/3$, $\tilde{w}^T\tilde{c}d = 1/3$, $\tilde{w}^T d\tilde{c} = 1/3$
	$\tilde{w}^T B\tilde{c} = 1/6$, $\tilde{w}^T\tilde{A}d = 1/6$,
	$v^T\tilde{A}\tilde{c} = 1/6$, $v^T B\tilde{c} = 1/6$

Proof. In fact, we obtain from FSAL property $\hat{\tilde{w}}^T\hat{A}^{-1}a - \tilde{w}_1 = e_{s-1}^T\hat{\tilde{A}}\hat{A}^{-1}a - \tilde{w}_1 = e_{s-1}^T\tilde{a} - \tilde{w}_1 = 0$. □

Then if (41) are satisfied, independently on the choice of $A_* = (a_*, \hat{A}_*)$, we obtain the same conditions of Table 1 but where now

$$B = \begin{pmatrix} 0 & 0 \\ b & \hat{B} \end{pmatrix}, \qquad v^T = (v_1, \hat{v}^T), \tag{42}$$

with

$$b = \hat{\tilde{A}}\hat{A}^{-1}\tilde{a}, \qquad \hat{B} = \hat{\tilde{A}}\hat{A}^{-1}\hat{\tilde{A}}, \qquad v_1 = \hat{\tilde{w}}^T\hat{A}^{-1}\tilde{a}, \qquad \hat{v}^T = \hat{\tilde{w}}^T\hat{A}^{-1}\hat{\tilde{A}}. \tag{43}$$

We can summarize our findings in the following

Theorem 5. *An IMEX method applied to system (29), in additive or partitioned form, for small values of ε and consistent initial data, yields an explicit additive Runge–Kutta method for the $O(\varepsilon)$ limit (30) characterized by the pairs $(\tilde{A}, \tilde{w})$ and (B, v) given by (35) for type I schemes and by (42) for type II schemes. In addition for type II schemes conditions (41) must be satisfied.*

Some remarks are in order.

Remark 3.
- We observe that there are no second order GSA IMEX-RK methods of type I with $\nu = 3$, i.e. with three stages. For details we refer to Appendix or to Ref. [33].
- It is easy to show that is not possible to have a second order IMEX scheme of type II with three internal stages, satisfying the additional second order conditions in Table 1, in fact, their evaluations are exactly zero and not $1/2$.

4. Numerical examples

Test 1. Well-balanced property

To emphasize the effect of the well-balanced property we consider a simple system of ODEs

$$U' = f(U) + g(U), \tag{44}$$

where $U = (u, v)^T$, $f(U) = (v, -u)^T$ and $g(U) = (0, 1 - v)^T$. This choice guarantees that the eigenvalues of the system are imaginary. The unique equilibrium point is $v_* = 0$, $u_* = 1$.

We apply to this simple problem some standard first and second order IMEX schemes in additive and partitioned forms to illustrate the results of the last two paragraphs. We shall use the notation NAME(s, σ, p), where the triplet (s, σ, p) characterizes the number s of function evaluations of the implicit scheme, the number σ of function evaluations of the explicit scheme and the order p of the IMEX scheme. We report the results in Figs. 1 and 2, where we used the simplest IMEX scheme with $\tilde{c} = c$, namely the type II GSA explicit–implicit Euler method EI(1, 1, 1) with $\nu = 2$, $\tilde{c} = c = (0, 1)^T$, $\tilde{w} = (1, 0)^T$ and $w = (0, 1)^T$. As a comparison we consider also its counterpart with $\tilde{c} \neq c$, namely the type I implicit–explicit Euler method SP(1, 1, 1) with $\nu = 1$, $\tilde{c} = 0$, $c = 1$, $\tilde{w} = w = 1$. In the same figures we also report the numerical solutions obtained with the second order schemes ARS(2, 2, 2) [3] (type II IMEX scheme with $\tilde{c} = c$) and PR(2, 2, 2) [17] (type I IMEX scheme with $\tilde{c} \neq c$).

The results clearly show how only the schemes with $\tilde{c} = c$ are able to capture the correct stationary state of both components of the solution in the additive formulation, whereas in the partitioned case all schemes give a correct description of the long time behavior of the system.

Similar results are observed in the case of the following PDE example. Let us consider the hyperbolic system

$$\begin{aligned} f_t + f_x &= \nu(g - f), \\ g_t - g_x &= \nu(f - g). \end{aligned} \tag{45}$$

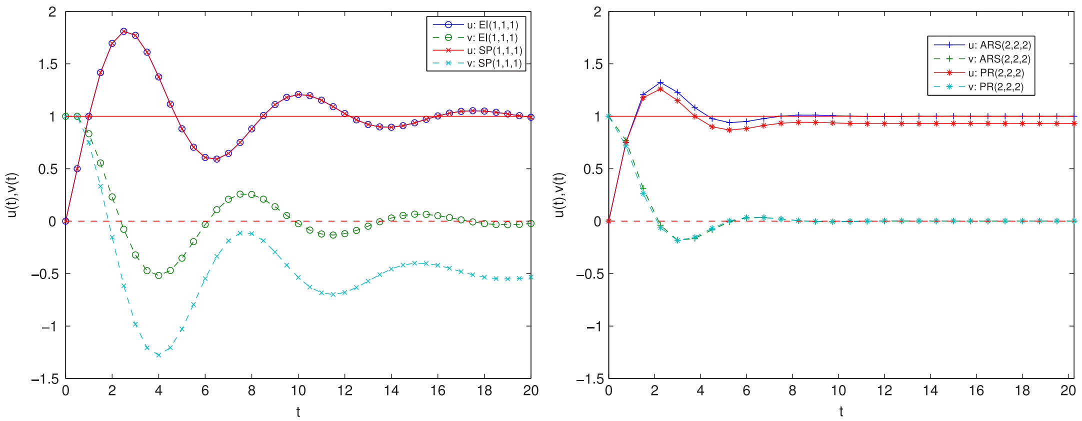

Fig. 1. Additive approach: Solution of the simple test problem (44) using some first and second order IMEX schemes. Left: first order schemes for $\Delta t = 0.5$. Right: second order schemes for $\Delta t = 0.75$.

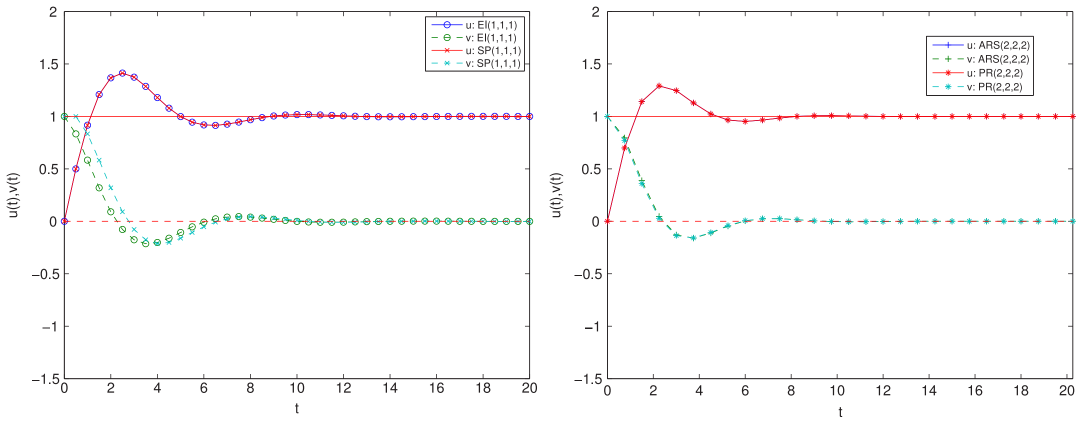

Fig. 2. Partitioned approach: Solution of the simple test problem (44) using some first and second order IMEX schemes. Left: first order schemes for $\Delta t = 0.5$. Right: second order schemes for $\Delta t = 0.75$.

The above linear system is also known as the Goldstein–Taylor model of the Boltzmann equation and describes the time evolution of two particle densities $f(x, t)$ and $g(x, t)$ moving with velocities ± 1, respectively, and at the same time changing velocities at a rate $\nu > 0$. We additionally consider the problem for $x \in [-L, L]$ with the boundary conditions

$$f(-L) = f_l, \qquad g(L) = g_r. \tag{46}$$

Introducing the macroscopic (fluid) variables mass density u and flux v

$$u = f + g, \qquad v = f - g$$

the system can be written in the form (3) with $f_1(u, v) = v, f_2(u) = u$ and $g(u, v) = -2\nu v$

$$\begin{aligned} &u_t + v_x = 0, \\ &v_t - u_x = -2\nu v. \end{aligned} \tag{47}$$

The equilibrium solutions for (47) correspond to

$$\begin{aligned} &\frac{\partial v^*}{\partial x} = 0 \Rightarrow v^* = C_1, \\ &\frac{\partial u^*}{\partial x} = -2\nu v^* \Rightarrow u^* = -2\nu C_1 x + C_2, \end{aligned} \tag{48}$$

with C_1, C_2 suitable constants depending on the boundary conditions. More precisely we have

$$f_l = \frac{1}{2}(C_1(1 + 2\nu L) + C_2), \qquad g_r = \frac{1}{2}(-C_1(1 + 2\nu L) + C_2),$$

which gives

$$C_1 = \frac{f_l - g_r}{1 + 2\nu L}, \qquad C_2 = f_l + g_r.$$

Table 2
L_∞-error for the steady state solution to problem (47).

Schemes	Additive : u^*	Additive : v^*	Partitioned : u^*	Partitioned : v^*
EI(1,1,1)	2.1697e−14	7.4801e−14	2.2716e−14	9.6784e−14
SP(1,1,1)	7.5826e−03	9.0991e−02	2.4570e−14	8.7014e−14
ARS(2,2,2)	1.7637e−14	7.2359e−14	2.0195e−14	8.7014e−14
PR(2,2,2)	4.0091e−04	5.5857e−03	4.5149e−14	1.6517e−13

Table 3
Convergence rate for the second order IMEX-RK schemes: (A.2)–(A.5) in L_∞-norm for the u-component in the nonlinear case $f(u) = u^2$ for problem (49).

Schemes	$\varepsilon = 1$	$\varepsilon = $ 1e−1	$\varepsilon = $ 1e−2	$\varepsilon = $ 1e−3	$\varepsilon = $ 1e−4	$\varepsilon = $ 1e−5	$\varepsilon = $ 1e−6
IMEX-I-GSA2	2.00	2.00	1.95	1.83	2.21	2.00	2.00
IMEX-I-ISA2	2.00	2.00	2.00	1.78	1.96	2.00	2.00
IMEX-II-GSA2	2.00	2.01	2.04	2.07	2.07	1.98	2.00
IMEX-II-ISA2	2.00	2.00	2.00	2.21	2.02	1.98	1.98

Table 4
Convergence rate for the third order IMEX-RK schemes (A.6) and (A.7) in L_∞-norm for the u-component in the linear case $f(u) = bu$ for problem (49).

Schemes	$\varepsilon = 1$	$\varepsilon = $ 1e−1	$\varepsilon = $ 1e−2	$\varepsilon = $ 1e−3	$\varepsilon = $ 1e−4	$\varepsilon = $ 1e−5	$\varepsilon = $ 1e−6
IMEX-II-GSA3	3.02	3.02	2.59	2.24	1.20	3.09	3.14
IMEX-II-ISA3	3.19	3.05	2.98	2.78	2.96	2.94	3.35

We report in Table 2 the maximum norm of the relative error obtained using the previous schemes. The results confirm the loss of accuracy of IMEX schemes in additive form when $c \neq \tilde{c}$. In the numerical results, for the additive approach we used first order upwind, whereas for the partitioned approach we used central differences. The time is integrated up to $t = 1500$.

Test 2. Uniform accuracy

In this numerical test we investigate numerically the convergence rate of the new second and third order ISA and GSA IMEX-RK schemes introduced in the Appendix that satisfy the additional order conditions given in Table 1. To this aim we apply the schemes to the simple prototype hyperbolic system with stiff relaxation (3), in the case $f_1(u, v) = u, f_2(u, v) = v$ and $g(u, v) = f(u) - v$ in the fluid-scaling

$$\begin{aligned} &u_t + v_x = 0, \\ &v_t + u_x = -\frac{1}{\varepsilon}(v - f(u)), \quad x \in [0, 2],\ t \in [0, T], \end{aligned} \tag{49}$$

with $\varepsilon > 0$. We show that these schemes are able to handle efficiently the stiffness of the system (49) in the whole range of the relaxation time, including the $O(\varepsilon)$ regime.

In our numerical test, we take a periodic smooth solution with well-prepared initial data $u(x, 0) = \sin(2\pi x)$ and $v(x, 0) = f(u(x, 0)) + \varepsilon v_1(x, 0)$, where $v_1(x, 0) = (f'(u(x, 0))^2 - 1)\partial_x u(x, 0)$. We consider a nonlinear test case characterized by $f(u) = u^2$ and a linear test case for $f(u) = bu$. The final time is $T = 0.01$ and the system has been integrated for $x \in [0, 2]$. For the spatial discretization of the domain a third order WENO scheme has been adopted [43].

The numerical convergence rate is calculated by the formula

$$p = \log_2(\|E_{\Delta t_1}\|_\infty / \|E_{\Delta t_2}\|_\infty),$$

where $E_{\Delta t_1}$ and $E_{\Delta t_2}$ are the global errors with step Δt_1 and $\Delta t_2 = \Delta t_1/2$. The value $\Delta t = \lambda \Delta x$ has been used with $\lambda = 0.5$. In this numerical example Δx decreases with the step size Δt. The observed temporal order of convergence has been measured by using $N = 100$ doubled up to $N = 400$. The tables summarize the convergence rates, as a function of ε, for different schemes (Type I and Type II) of second and third order, both GSA or ISA with $w = \tilde{w}$, using different values of ε ranging from 10^{-6} to 1. Here we show the results obtained with the new schemes implemented in additive form. Analogous results are obtained with a partitioned strategy, which therefore are omitted.

More precisely, in Table 3 we report the convergence rate for the u-component for different second order IMEX RK schemes of types I and II (see Appendix, (A.2)–(A.5)) in the nonlinear case, and in Tables 4 and 5, the convergence rates for two third order IMEX-RK scheme of type II (see Appendix (A.6) and (A.7)) in the linear and in the nonlinear case, respectively. We point out that no effort has been done to try to optimize the coefficient of the schemes in terms of stability regions. For second order methods the error behaves uniformly even in the nonlinear case, whereas for third order schemes some deterioration of accuracy is observed for the GSA scheme and in the nonlinear case.

Table 5
Convergence rate for the third order IMEX-RK schemes (A.6) and (A.7) in L_∞-norm for the u-component in the nonlinear case $f(u) = u^2$ for problem (49).

Schemes	$\varepsilon = 1$	$\varepsilon =$ 1e−1	$\varepsilon =$ 1e−2	$\varepsilon =$ 1e−3	$\varepsilon =$ 1e−4	$\varepsilon =$ 1e−5	$\varepsilon =$ 1e−6
IMEX-II-GSA3	2.95	4.40	2.88	2.12	2.11	3.05	3.00
IMEX-II-ISA3	3.05	0.91	2.96	2.76	3.54	3.14	2.98

5. Conclusions

In this paper we discussed the asymptotic properties of IMEX Runge–Kutta schemes both in additive and partitioned forms when applied to hyperbolic systems of balance laws. Even though the additive form is the one commonly used, the partitioned form presents better well balanced properties when the description of the stationary solution of the system is important. We also derived, for linear systems, general order conditions that guarantee the $O(\varepsilon)$ accuracy (Navier–Stokes regime) of the method in the fluid scaling. Examples of schemes that satisfy the additional order conditions up to order three are derived. The new schemes, even if they require several additional stages in order to satisfy the new $O(\varepsilon)$ conditions, are able to achieve the desired accuracy for a wide range of the stiffness parameter even for nonlinear problems. Further testing is necessary to understand the efficiency of the schemes when compared to IMEX schemes with less stages and adaptive time stepping.

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Appendix. IMEX schemes satisfying conditions in Table 1

In the Appendix we present the derivation of the GSA and ISA IMEX-RK schemes of types I and II, used in *Test 2*, that satisfy the additional order conditions of Table 1. Below, these schemes are represented as usual by the double Butcher tableau (8).

We start to design a first order GSA IMEX-RK scheme of type I. Such a scheme requires $\nu = 3$ stages in order to satisfy the first order conditions $v^T\mathbf{e} = 1$ in Table 1. Therefore, to obtain the scheme we consider the following double Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \tilde{a}_{21} & \tilde{a}_{21} & 0 & 0 \\ 1 & \tilde{w}_1 & \tilde{w}_2 & 0 \\ \hline & \tilde{w}_1 & \tilde{w}_2 & 0 \end{array} \qquad \begin{array}{c|ccc} 0 & \gamma & 0 & 0 \\ c_2 & a_{21} & \gamma & 0 \\ 1 & w_1 & w_2 & \gamma \\ \hline & w_1 & w_2 & \gamma \end{array}. \tag{A.1}$$

The condition $v^T\mathbf{e} = 1$ is reduced to $\tilde{a}_{21} = \gamma/\tilde{w}_2$ and by setting $w_2 = 0$, $a_{21} = 0$, $\tilde{w}_1 = 1 - \tilde{w}_2$, and $w_1 = 1 - \gamma$ with $\gamma > 0$, we get the first order GSA RK scheme of type I.

Next, we design a first order GSA IMEX-RK scheme of type II satisfying the first order condition $v^T\mathbf{e} = 1$ and conditions (41). By direct computation we have the following result.

Proposition 3. *The pair (B, v) defined in (35) or (42) is such that $B \in \mathbb{R}^{\nu\times\nu}$ is lower triangular with $b_{ii} = 0$, $i = 1, \ldots, \nu$, $b_{ii-1} = 0$, $i = 2, \ldots, \nu$ and $v^T \in \mathbb{R}^\nu$ with $v_i = 0$, $i = \nu - 1, \nu$. In addition, for definition (42) we also have $b = (0, b_3, \ldots, b_\nu) \in \mathbb{R}^{\nu-1}$ where $b_2 = 0$.*

The GSA property implies $v_1 = \hat{\tilde{w}}^T \hat{A}^{-1}\tilde{a} = e^T_{\nu-1}\hat{\tilde{A}}\hat{A}^{-1}\tilde{a} = e^T_{\nu-1}b = b_\nu$, and as a consequence of this fact we have that if $\nu = 2$, from Proposition 3 we get $v_1 = v_2 = 0$. Thus, we need at least $\nu \geq 3$ in order to derive a scheme of type II that satisfies the first order conditions of Table 1 and (41). After some algebraic computations, we obtain that $\nu = 4$ stages are required to design first order ISA and GSA IMEX-RK schemes of type II. We do not report the double Butcher tableau for this type of schemes.

Now, we investigate second order GSA and ISA IMEX-RK schemes of types I and II that satisfy the order conditions of Table 1. We start by saying that there is no second order GSA IMEX scheme of type I satisfying such order conditions with $\nu = 3$ (see for details [33,40]). For this reason we derived a second order GSA IMEX-RK scheme of type I with $\nu = 4$ stages that satisfies the classical first and second order conditions and the additional order condition of Table 1, i.e., $v^T\mathbf{e} = 1$, $\tilde{w}^T d = 1/2$ and $v^T\tilde{c} = 1/2$.

The resulting second order GSA IMEX scheme of type I with $\nu = 4$, called IMEX-I-GSA2, is reported below

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 0 \\ 2/3 & 7/24 & 3/8 & 0 & 0 \\ 1 & 1/2 & -1/2 & 1 & 0 \\ \hline & 1/2 & -1/2 & 1 & 0 \end{array} \qquad \begin{array}{c|cccc} 1/4 & 1/4 & 0 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 0 \\ 1/2 & 1/16 & 3/16 & 1/4 & 0 \\ 1 & 1/4 & 1/4 & 1/4 & 1/4 \\ \hline & 1/4 & 1/4 & 1/4 & 1/4 \end{array}. \tag{A.2}$$

We also constructed a second order IMEX-RK scheme of type I, not GSA but only ISA, with $\tilde{w}_i = w_i$ for all i, satisfying the first and second order conditions in Table 1. We call this scheme IMEX-I-ISA2

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 0 \\ 2/3 & 2/3 & 0 & 0 & 0 \\ 1 & -1/2 & 3/2 & 0 & 0 \\ \hline & 1/4 & 1/4 & 1/4 & 1/4 \end{array} \qquad \begin{array}{c|cccc} 1/4 & 1/4 & 0 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 0 \\ 1/2 & 1/4 & 0 & 1/4 & 0 \\ 1 & 1/4 & 1/4 & 1/4 & 1/4 \\ \hline & 1/4 & 1/4 & 1/4 & 1/4 \end{array}. \tag{A.3}$$

Subsequently, we investigate second order GSA IMEX-RK schemes of type II, that satisfy the conditions (41) and the order ones in Table 1 up to order two. We note that it is not possible to have a second order GSA IMEX-RK scheme of type II with $\nu = 3, 4$. Then, for such a scheme, we consider $\nu = 5$ and we require $\tilde{c}_i = c_i$ for $i = 1, \cdot \nu - 1$ (from the GSA assumption we get $\tilde{c}_\nu = c_\nu = 1$) with the aim to simplify the order conditions. We get the following GSA IMEX RK scheme of type II and we call it IMEX-II-GSA2

$$\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 1/3 & 1/6 & 1/6 & 0 & 0 & 0 \\ 2/3 & -2/3 & 0 & 4/3 & 0 & 0 \\ 1 & -1/6 & 1/2 & 0 & 9/16 & 0 \\ \hline & -1/6 & 1/2 & 0 & 9/16 & 0 \end{array} \qquad \begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 0 & 0 \\ 1/3 & 0 & 1/12 & 1/4 & 0 & 0 \\ 2/3 & 0 & -11/12 & 4/3 & 1/4 & 0 \\ 1 & 0 & 9/31 & 12/31 & 9/124 & 1/4 \\ \hline & 0 & 9/31 & 12/31 & 9/124 & 1/4 \end{array}. \tag{A.4}$$

Similarly, as was done for the type I, we also consider type II schemes ISA with $\tilde{w}^T = w^T$. This scheme, called IMEX-II-ISA2, is given by

$$\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 0 & 0 \\ 2/3 & 0 & -2/3 & 4/3 & 0 & 0 \\ 1 & 0 & 16/15 & -5/6 & 23/30 & 0 \\ \hline & 0 & 3/5 & 0 & 3/20 & 1/4 \end{array} \qquad \begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 0 & 0 \\ 1/3 & 0 & 1/12 & 1/4 & 0 & 0 \\ 2/3 & 0 & 5/12 & 0 & 1/4 & 0 \\ 1 & 0 & 3/5 & 0 & 3/20 & 1/4 \\ \hline & 0 & 3/5 & 0 & 3/20 & 1/4 \end{array}. \tag{A.5}$$

Finally we give the Butcher tableau of two third order IMEX-RK scheme of type II that satisfies the classical order conditions and conditions up to order three in Table 1. The first scheme is GSA and has $\tilde{c}_i = c_i$ for $i = 1, \ldots, \nu - 1$ and $\tilde{c}_\nu = c_\nu = 1$. We denote it as IMEX-II-GSA3 scheme

$$\text{Explicit part}: \begin{array}{c|ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 43/100 & 43/100 & 0 & 0 & 0 & 0 & 0 & 0 \\ 336/929 & 0 & 336/929 & 0 & 0 & 0 & 0 & 0 \\ -29/42 & 0 & -29/42 & 0 & 0 & 0 & 0 & 0 \\ 581/527 & 0 & -1213/770 & 2491/956 & 267/3701 & 0 & 0 & 0 \\ 2/3 & 0 & -197/1238 & 499/743 & 0 & 581/3768 & 0 & 0 \\ 1 & 0 & 263/620 & 134/16589 & 1040/22119 & 0 & 4777/9174 & 0 \\ \hline & 0 & 263/620 & 134/16589 & 1040/22119 & 0 & 4777/9174 & 0 \end{array} \tag{A.6}$$

$$\text{Implicit part}: \begin{array}{c|ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 43/100 & 0 & 43/100 & 0 & 0 & 0 & 0 & 0 \\ 336/929 & 0 & -168/2459 & 43/100 & 0 & 0 & 0 & 0 \\ -29/42 & 0 & -2353/2100 & 0 & 43/100 & 0 & 0 & 0 \\ 581/527 & 0 & 889/1322 & 0 & 0 & 43/100 & 0 & 0 \\ 2/3 & 0 & 247/2416 & 0 & 408/3035 & 0 & 43/100 & 0 \\ 1 & 0 & 872/1201 & 0 & 139/4081 & -50/237 & 434/20817 & 43/100 \\ \hline & 0 & 872/1201 & 0 & 139/4081 & -50/237 & 434/20817 & 43/100 \end{array}.$$

The second scheme is only ISA with $\tilde{w}^T = w^T$ and $\tilde{c}_i = c_i$ for $i = 1, \ldots, \nu$ and it will be denoted by IMEX-II-ISA3

$$
\text{Explicit part}: \begin{array}{c|ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/5 & 1/5 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 \\
2/3 & 0 & 557/867 & 7/289 & 0 & 0 & 0 & 0 \\
3/4 & 0 & 16/289 & 803/1156 & 0 & 0 & 0 & 0 \\
1 & 0 & 13348/3993 & -9355/3993 & 0 & 0 & 0 & 0 \\
1 & 0 & 75/154 & 0 & -3/14 & 8/11 & 0 & 0 \\
\hline
 & 0 & -155/112 & 251/80 & -547/280 & 2/3 & 1/3 & 1/5
\end{array} \qquad \text{(A.7)}
$$

$$
\text{Implicit part}: \begin{array}{c|ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/5 & 0 & 1/5 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 0 & 2/15 & 1/5 & 0 & 0 & 0 & 0 \\
2/3 & 0 & 7/15 & 0 & 1/5 & 0 & 0 & 0 \\
3/4 & 0 & 1137/1004 & -731/1255 & 0 & 1/15 & 0 & 0 \\
1 & 0 & 447/565 & 0 & -636/613 & 519/496 & 1/5 & 0 \\
1 & 0 & -155/112 & 251/80 & -547/280 & 2/3 & 1/3 & 1/5 \\
\hline
 & 0 & -155/112 & 251/80 & -547/280 & 2/3 & 1/3 & 1/5
\end{array}.
$$

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