Extended bicolorings of Steiner triple systems of order $2^h - 1$

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Abstract

A bicoloring of a Steiner triple system $STS(n)$ on n vertices is a coloring of vertices in such a way that every block receives precisely two colors. The maximum (resp. minimum) number of colors in a bicoloring of an $STS(n)$ is denoted by $\bar{\chi}$ (resp. χ). All bicolorable $STS(2^h-1)$s have upper chromatic number $\bar{\chi} \leq h$; also, if $\bar{\chi} = h < 10$, then lower and upper chromatic numbers coincide, namely, $\chi = \bar{\chi} = h$. In 2003, M. Gionfriddo conjectured that this equality holds whenever $\bar{\chi} = h \geq 2$.

In this paper we discuss some extensions of bicolorings of $STS(v)$ to bicoloring of $STS(2v+1)$ obtained by using the 'doubling plus one construction'. We prove several necessary conditions for bicolorings of $STS(2v+1)$ provided that no new color is used. In addition, for any natural number h we determine a triple system $STS(2^{h+1}-1)$ which admits no extended bicolorings.

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1 Introduction

A Steiner triple system $STS(v)$ is a pair $(X, \mathcal{B})$, where X is a finite set of vertices, $|X| = v$, and $\mathcal{B}$ is a family of subsets of X, called *blocks*, such that each block contains three vertices, and any two distinct vertices of X appear together in precisely one block of $\mathcal{B}$. It is well known since the 1850's that an $STS(v)$ exists if and only if $v \equiv 1$ or $3 \pmod 6$.

A *k-coloring* of $(X, \mathcal{B})$ is a surjective mapping ϕ from X onto a finite set C, with $|C| = k$, whose elements are called *colors*. It is customary to assume that $C = \{1, 2, \ldots, k\}$. For each $i \in C$, the set $\phi^{-1}(i) = \{x : \phi(x) = i\}$ is a *color class*. A k-coloring of $(X, \mathcal{B})$ is called a *k-bicoloring* if the vertices of each $b \in \mathcal{B}$ are colored with exactly two colors. Given a k-bicoloring $\mathcal{C}$, if the cardinalities of color classes are $n_1 \leq n_2 \leq \cdots \leq n_k$ (always assuming $n_1 \geq 1$), then for brevity we write $\mathcal{C} = \mathcal{C}(n_1, n_2, \ldots, n_k)$.

The systems $STS(2v+1)$ obtained by *doubling plus one construction* are fundamental in the bicoloring theory of Steiner triple systems. A system $STS(2v+1) = (X, \mathcal{B})$ is obtained from an $STS(v) = (X', \mathcal{B}')$ by a doubling plus one construction in the following way. For vertices, we write $X' = \{x_1, x_2, \ldots, x_v\}$, and introduce a new set of vertices X'' such that $|X''| = v+1$ and $X' \cap X'' = \emptyset$. We set $X = X' \cup X''$, and denote the vertices in X'' by $y_1, y_2, \ldots, y_{v+1}$. Next we consider a factorization $\mathcal{F} = (F_1, F_2, \cdots, F_v)$ of the complete graph whose vertex set is X''. (Note that $|X''|$ is even whenever an $STS(v)$ exists.) For blocks, we set $\mathcal{B}$ to include all the blocks of $\mathcal{B}'$, moreover all the blocks of type $\{x_i, y_l, y_m\}$ where $x_i \in X'$ and (y_l, y_m) is an edge in the factor F_i. The system $STS(2v+1)$ is then defined as the pair $(X, \mathcal{B})$.

Historically, the concept of bicoloring is originated from the theory of mixed hypergraph coloring, which was introduced in [12, 13]. The maximum (resp. minimum) k for which there exists a k-bicoloring of an STS is called the *upper* (resp. *lower*) *chromatic number*, and it is denoted by $\bar{\chi}$ (resp. by χ). The first results on bicolorings of $STSs$ were published in [8]; further early works on this subject are [10, 9, 3, 6]. For a survey we refer to [11].

The following theorem summarizes key results from [8] which are important for the proofs in next sections.

Theorem 1.1 ([8]) *If S is a bicolorable $STS(v)$ with $v \leq 2^h - 1$, then $\bar{\chi}(S) \leq h$. Moreover, for any k-bicoloring $\mathcal{C} = \mathcal{C}(n_1, n_2, \ldots, n_k)$ of S, the following inequalities hold: $n_1 \geq 2^0$, $n_2 \geq 2^1$, $n_3 \geq 2^2$, …, $n_k \geq 2^{k-1}$. In particular, if $\bar{\chi}(S) = h$, then:*

1. *$v = 2^h - 1$;*

2. *in any h-bicoloring of S the cardinalities of the color classes are*

$$2^0,\ 2^1,\ 2^2,\ \ldots,\ 2^{h-1};$$

3. *S is obtained from the $STS(3)$ by repeated applications of a 'doubling plus one construction'.* □

Let $S' = (X', \mathcal{B}') = STS(v)$ be a h-bicolorable system with a bicoloring $\mathcal{C}' = \{n'_1, n'_2, \cdots, n'_h\}$, and let $S = (X, \mathcal{B}) = STS(2v+1)$ be a system obtained by doubling plus one construction from S'. The system S certainly is $(h+1)$-bicolorable with a bicoloring $\mathcal{C}'' = \{n''_1, n''_2, \cdots, n''_h, n''_{h+1}\}$ where $n''_{h+1} = v+1$ and $n''_i = n'_i$ for all $1 \leq i \leq h$. This bicoloring is obtained by coloring the subsystem S' with $\mathcal{C}'$ and the vertices in $X'' = X - X'$ with a new color denoted by $h+1$, where $|X''| = v+1$. It is absolutely non-obvious, however, what happens if one intends to color S with the same number, only h, of colors with which S' has been colored.

It was proved in [1] that if an $STS(2^h - 1)$ is obtained by a sequence of doubling plus one constructions starting from $STS(3)$, then $\bar{\chi} = \chi = h$ with all $h < 10$; i.e, their upper chromatic number and lower chromatic number are equals.

The 'sequence of doubling plus one constructions' condition above is substantial. For example, as proved in [7, Theorem 9], among the 80 pairwise non-isomorphic Steiner triple systems of order 15 there are only 23 which are bicolorable, and each of them is obtained from $STS(3)$ by iterating the doubling plus one construction. (For the history of enumerating all $STS(15)$ systems, see [2, p. 15].) Moreover, there exists a bicolorable $STS(19)$ with $\chi = 3$ and $\bar{\chi} = 4$ [7, Theorem 12].

In [5], M. Gionfriddo raised a challenging related conjecture, which is the main motivation of our current work.

Conjecture 1.1 ([5]) *If an $S = STS(2^h - 1)$ is obtained by a sequence of doubling plus one constructions from $STS(3)$, then $\bar{\chi}(S) = \chi(S) = h$.*

Should there exist counterexamples to this conjecture, an $(h-1)$-bicoloring of a smallest counterexample would induce an $(h-1)$-bicoloring of an STS$(2^{h-1} - 1)$ as well. This leads to the notion of 'extended bicoloring'. Suppose that the system $S = (X, \mathcal{B})$ of order $2v+1$ is obtained from its subsystem $S' = (X', \mathcal{B}')$ by doubling plus one construction, and that $\mathcal{C}'$ is a k-bicoloring of S' (for any k). We say that a k-bicoloring $\mathcal{C} = \{n_1, n_2, \cdots, n_k\}$

of S with the same number k of colors is an *extended bicoloring* or *extended k-bicoloring* of S' if $\mathcal{C}$ coincides with $\mathcal{C}'$ on the vertices of S'. It is important to emphasize that no new color is allowed on $S \setminus S'$. In [4] the authors found the first $STS(v)$s with extended bicolorings for $v \equiv 3$ or 7 (mod 12) and $v > 3$.

Assuming that $\mathcal{C}$ is an extended bicoloring of $\mathcal{C}' = \{n'_1, \ldots, n'_k\}$, we will denote the ith color class of $\mathcal{C}'$ by C'_i. Hence, $|C'_i| = n'_i$ for $i = 1, \ldots, k$, and $n'_1 \leq \cdots \leq n'_k$ is assumed. With a slight abuse of notation, the color classes $C_1, \ldots C_k$ of $\mathcal{C}$ are ordered such that $C'_i \subseteq C_i$ holds for each $i = 1, \ldots, k$. Then, let n_i denote $|C_i|$. (But here it would certainly make a loss of generality to assume that the values n_i are in increasing order.) For $1 \leq i \leq k$ we shall write

$$c_i = n_i - n'_i$$

to denote the number of vertices in $X'' = X \setminus X'$ which belong to C_i.

Concerning extended bicolorings the following important results were proved in [1, 4]; they will be useful in the next sections.

Theorem 1.2 ([1, 4]) *If $\mathcal{C}$ is an extended h-bicoloring of an $STS(2^{h+1} - 1)$ then it satisfies the following equalities:*

$$\begin{cases} \sum_{i=1}^{h} c_i^2 + \sum_{i=1}^{h} 2^i c_i = 2^{2h} \\ \sum_{i=1}^{h} c_i = 2^h. \end{cases} \tag{1}$$

□

Theorem 1.3 ([1]) *If $\mathcal{C}$ is an extended h-bicoloring of an $STS(2^{h+1} - 1)$ with $c_i > 0$ and $c_j > 0$ for some $1 \leq i < j \leq h$, then $c_i \leq 2^{i-1} + 2^{j-1}$ and $c_j \leq 2^{i-1} + 2^{j-1}$.* □

Theorem 1.4 ([1]) *If $\mathcal{C}$ is an extended h-bicoloring of an $STS(2^{h+1} - 1)$, then there exists at least one $c_i = 0$ and all the $c_j > 0$ are even.* □

Theorem 1.5 ([1]) *Let $\mathcal{C}$ be an extended h-bicoloring of an $STS(2^{h+1} - 1)$, and let $1 \leq i < j - 1 \leq h - 1$ such that $c_i > 0$, $c_j > 0$, and $c_k = 0$ for every $k \in \{1, 2, \ldots, j-1\} \setminus \{i\}$. Then, $c_{j+t} > 0$ for every $t \geq 0$.*

□

Our goal in Section 2 is to strengthen the necessary conditions listed above; our results will restrict the possible candidates of $(c_1, \ldots, c_h)$-sequences to only two basic types. In Section 3 we explicitly construct an infinite class of Steiner triple systems STS$(2^{h+1}-1)$ obtained by a sequence of doubling plus one constructions from STS(3), which do not admit any extended h-bicolorings.

2 Necessary conditions

Throughout this section we assume that $\mathcal{C}$ is an extended h-bicoloring of an $STS(2^{h+1}-1)$ which corresponds to a solution $(c_1, \ldots, c_h)$ of the system (1). We will strengthen Theorem 1.5 by excluding some further subclasses of the possible solutions $(c_1, \ldots, c_h)$. To state our first observation we introduce the notation $I^0 = \{q : c_q = 0\}$.

Lemma 2.1 *Let $\mathcal{C}$ be an extended h-bicoloring of an $STS(2^{h+1}-1)$ which corresponds to a solution $(c_1, \ldots, c_h)$ of the system (1). Then, for every i with $c_i > 0$,*

$$\sum_{q \in I^0} 2^{q-1} + 1 \leq c_i$$

must hold.

Proof.
Since the system $STS(2^{h+1}-1)$ is obtained by doubling plus one construction, there is a corresponding factorization $\mathcal{F}$ of X''. If $x' \in X'' \cap C_i$ and $x_l \in \bigcup_{q \in I^0} C_q$, then the only way to bicolor the unique triple containing the vertex pair (x', x_l) is that its third vertex — which is the other end of the edge incident to x' in $F_l \in \mathcal{F}$ — belongs to $X'' \cap C_i$. By definition, for distinct vertices of $\bigcup_{q \in I^0} C_q$ the corresponding third vertices are distinct. $\square$

Proposition 2.1 *Let $\mathcal{C}$ be an extended h-bicoloring of an $STS(2^{h+1}-1)$ which corresponds to a solution $(c_1, \ldots, c_h)$ of the system (1). Then, $(c_1, \ldots, c_h)$ must be one of the following two types.*

- (a) $(0, \ldots, 0, c_j, \ldots, c_h)$ *where $j \geq 2$ and $0 < c_k$ for every $j \leq k \leq h$;*
- (b) $(0, \ldots, 0, c_i, 0, \ldots, 0, c_j, \ldots, c_h)$ *where $2 \leq i \leq j-2$, $0 < c_i$, and $0 < c_k$ for every $j \leq k \leq h$.*

Proof. Assume that c_k and c_ℓ are positive entries and $k \neq \ell$. Lemma 2.1 and Theorem 1.3 imply that

$$\sum_{q \in I^0} 2^{q-1} + 1 \leq 2^{k-1} + 2^{\ell-1}.$$

In particular, for any s with $c_s = 0$

$$2^{s-1} < 2^{k-1} + 2^{\ell-1},$$

and $k < \ell < s$ cannot be valid. In other words, in the sequence $(c_1, \ldots, c_h)$ none of the zero entries are preceded by more than one positive entry. If all zeros precede every positive entry, we get case (a). Note that $j \geq 2$ follows from Theorem 1.4.

If there is exactly one positive entry c_i which precedes one or more zeros, we have a sequence $(0, \ldots, 0, c_i, 0, \ldots, 0, c_j, \ldots, c_h)$ where $1 \leq i \leq j-2$, $0 < c_i$, and $0 < c_k$ for every $j \leq k \leq h$. To obtain case (b), it suffices to prove that $i \geq 2$. Suppose for a contradiction that $i = 1$ that is, $c_1 > 0$ holds. By Theorem 1.3, $c_1 \leq 2^{j-1} + 1$ and by Theorem 1.4 c_1 must be even. Hence, $c_1 \leq 2^{j-1}$. Let x' be a vertex from $C_1 \cap X''$, and consider a c_k with $k \geq j$. By Theorems 1.3 and 1.4 we have that $c_k \leq 2^{k-1}$. Thus,

$$2^h = |X''| = c_1 + \sum_{k=j}^{h} c_k \leq 2^{j-1} + \sum_{k=j}^{h} 2^{k-1} = 2^h.$$

Consequently, $c_1 = 2^{j-1}$ and $c_k = 2^{k-1}$ for all $j \leq k \leq h$. Plugging in, we derive that the left-hand side of (1) is equal to

$$\sum_{k=1}^{h} c_k^2 + \sum_{k=1}^{h} 2^k c_k = 2^{2j-2} + \sum_{k=j}^{h} 2^{2k-2} + 2^j + \sum_{k=j}^{h} 2^{2k-1}.$$

But this is $2^{2h} + 2^j$, rather than 2^{2h}, a contradiction. □

We proceed with two propositions related to the solutions of type $(0, \cdots,$ $0,\ c_i,\ 0, \cdots, 0,\ c_j,\ c_{j+1}, \cdots, c_h)$ with $i \geq 2$. They are of a technical nature; their eventual goal is to exclude the possibility of $c_2 > 0$.

Proposition 2.2 *Let $\mathcal{C}$ be an extended h-bicoloring of an $STS(2^{h+1}-1)$, associated with a solution $(0, \cdots, 0, c_i, 0, \cdots, 0, c_j,\ c_{j+1}, \cdots, c_h)$, for some $i \geq 2$ and $j \geq i+2$. Then there exist nonnegative integers $T \leq 2^{i-1}$, $P_k \leq 2^{i-1}$, and $P'_k \leq 2T$ (for $k = j, \ldots, h$), such that*

$$c_i = 2^{j-1} - 2^{i-1} + 2T$$

and

$$c_k = 2^{k-1} + P_k - P'_k$$

hold. Moreover, $\sum_{k=j}^{h} P_k = 2^{i-1}$ *and* $\sum_{k=j}^{h} P'_k = 2T$.

Proof.
Consider an extended h-bicoloring $\mathcal{C}$ of S satisfying the conditions, and fix a vertex $x' \in C_i \cap X''$. This vertex x' is in $2^{j-1} - 2^{i-1} - 1$ monochromatic pairs of X'', all contained in factors of $\mathcal{F}$ corresponding to vertices of X' colored with k such that $1 \le k \le j-1$ and $k \ne i$. So, $c_i \ge 2^{j-1} - 2^{i-1}$. By Theorem 1.4 it is necessary that $c_i \le 2^{j-1} + 2^{i-1}$, therefore $c_i = 2^{j-1} - 2^{i-1} + 2T$ holds for some $0 \le T \le 2^{i-1}$. Note that $2T$ is the number of monochromatic pairs containing x' inside factors belonging to the vertices from $X' \cap \bigcup_{k=j}^{h} C_k$; or equivalently, $2T$ is the number of blocks $B = \{x', x'', u\}$ in S with $x'' \in C_i \cap X''$ and $u \in X' \cap C_k$ for some $j \le k \le h$.

Since $\mathcal{C}$ is a bicoloring, each block B of S meets exactly two color classes. Moreover, $|B \cap X''|$ is either 0 or 2, as S is obtained by doubling plus one construction from S'. Now, consider a color class C_k with $j \le k \le h$. We may have three types of blocks $B \in \mathcal{B}$ incident with x' and contained entirely in $C_i \cup C_k$.

1. $B = \{x', y, z\}$ with $y \in C_k \cap X''$ and $z \in C_i \cap X'$. The number of these blocks is denoted by P_k.
2. $B = \{x', y, u\}$ with $y \in C_k \cap X''$ and $u \in C_k \cap X'$. The number of these blocks is denoted by Y_k.
3. $B = \{x', x'', u\}$ with $x'' \in C_i \cap X''$ and $u \in C_k \cap X'$. The number of these blocks is denoted by P'_k.

There exist $n'_i = 2^{i-1}$ pairs (x', z) with $z \in C_i \cap X'$, and each of them is covered by exactly one block B of type 1. Hence, $\sum_{k=j}^{h} P_k = 2^{i-1}$. We have already observed that $2T$ is the total number of blocks belonging to type 3. This implies $\sum_{k=j}^{h} P'_k = 2T$.

The number of pairs (x', u) with $u \in C_k \cap X'$ is exactly $n'_k = 2^{k-1}$, each of them is covered by a block of type 2 or 3. Hence, $Y_k + P'_k = 2^{k-1}$ holds for every $j \le k \le h$. On the other hand, each of the c_k pairs (x', y) with $y \in C_k \cap X''$ is contained in a block of type 1 or 2. Hence, $P_k + Y_k = c_k$ follows. From these equalities, we obtain $c_k = 2^{k-1} + P_k - P'_k$. □

In the proposition above, by Theorem 1.4 we have that c_i and all c_k must be even. It means that if some P_k is odd ($j \leq k \leq h$), then the corresponding P'_k must be odd as well; and vice versa.

Proposition 2.3 *If an extended h-bicoloring is defined by a solution of type $(0, \cdots, 0, c_i, 0, \cdots, 0, c_j, c_{j+1}, \cdots, c_h)$, then with the notation above, the following equation holds:*

$$4T^2 + 2^{j+1}T + \sum_{k=j}^{h}[(P_k - P'_k)^2 + 2^{k+1}(P_k - P'_k)] = 2^{2i-2} \tag{2}$$

where $i \geq 2$, $0 < T \leq 2^{i-1}$, $0 \leq P_k \leq 2^{i-1}$, and $0 \leq P'_k \leq 2T$.

Proof.
The proposition is easily derived from the first equation of system (1), by replacing c_i and c_k (for $j \leq k \leq h$) with the values obtained in the previous proposition.

It is easy to verify that T must be positive. In fact, if $T = 0$, then, since $\sum_{k=j}^{h}(P_k^2 + 2^{k+1}P_k) > 2^{2i-2}$, the equation (2) does not hold for any $i \geq 2$. □

Proposition 2.4 *If $c_i > 0$ and $c_{i+1} = 0$, then $i = 2$ is not possible.*

Proof. We have that $0 < T \leq 2$, $0 \leq P_k \leq 2$, and $0 < P'_k \leq 4$ for all $j \leq k \leq h$. The following Table 1 shows all possible values of T, P_k, and P'_k.

T	P_k	P'_k
1	$P_{k_1} = 1$, $P_{k_2} = 1$	$P'_{k_1} = 1$, $P'_{k_2} = 1$
1	$P_{k_1} = 2$, $P_{k_2} = 0$	$P'_{k_1} = 0$, $P'_{k_2} = 2$
2	$P_{k_1} = 2$	$P'_{k_1} = 4$
2	$P_{k_1} = 2$, $P_{k_2} = 0$	$P'_{k_1} = 0$, $P'_{k_2} = 4$
2	$P_{k_1} = 2$, $P_{k_2} = 0$	$P'_{k_1} = 2$, $P'_{k_2} = 2$
2	$P_{k_1} = 1$, $P_{k_2} = 1$	$P'_{k_1} = 3$, $P'_{k_2} = 1$
2	$P_{k_1} = 2$, $P_{k_2} = 0$, $P_{k_3} = 0$	$P'_{k_1} = 0$, $P'_{k_2} = 2$, $P'_{k_3} = 2$
2	$P_{k_1} = 1$, $P_{k_2} = 1$, $P_{k_3} = 0$	$P'_{k_1} = 1$, $P'_{k_2} = 1$, $P'_{k_3} = 2$

Table 1

For the values in Table 1 the equation (2) does not hold. Consequently, $i > 2$ and $c_2 = 0$ must hold. □

From the previous propositions we observe that solutions of system (1) which may allow extended bicolorings are only of the following two types:

(*a*) $(0, \ldots, 0, c_j, \ldots, c_h)$ with some $j \geq 2$;

(*b*) $(0, \ldots, 0, c_i, 0, \ldots, 0, c_j, c_{j+1}, \ldots, c_h)$ with some $i \geq 3$ and $j \geq i + 2$.

3 A class of systems with no extended bicolorings

As we mentioned in the Introduction, it was shown in [1] that no $S' = STS(2^h - 1)$ has an extended bicoloring for $h < 10$.

In the next theorem we prove that at least some triple systems have this property for each h.

Theorem 3.1 *For every integer $h \geq 2$ there exists an $STS(2^h - 1)$ S_h obtained from $STS(3)$ by a sequence of doubling plus one constructions, such that $\bar{\chi}(S_h) = \chi(S_h) = h$.*

Proof.
For every $h \geq 2$ we describe an explicit (non-recursive) algebraic construction of a system $STS(2^h - 1)$ with the required properties, which will be denoted by S_h.

Let $h \geq 2$, and consider the h-dimensional vector space F^h over the field $GF(2)$. Then, the vertex set X_h of S_h consists of the non-zero vectors of F^h, and the set of blocks is defined as

$$\mathcal{B} = \{\{\underline{x}, \underline{y}, \underline{z}\} \subseteq X_h \mid \underline{x} + \underline{y} + \underline{z} = \underline{0}\}.$$

Clearly, for every two distinct non-zero vectors $\underline{x}$ and $\underline{y}$ there exists a unique $\underline{z} \in X_h$ satisfying $\underline{x} + \underline{y} + \underline{z} = \underline{0}$ which can equivalently be written as $\underline{x} + \underline{y} = \underline{z}$. Hence, S_h is an $STS(2^h - 1)$. Inside S_h, the vertices having zero as their last coordinate together determine an $STS(2^{h-1} - 1)$ which is isomorphic to S_{h-1}. Thus, for each $h \geq 2$, S_h can be obtained by a sequence of doubling plus one constructions from $STS(3)$. By Theorem 1.1, every S_h is bicolorable and $\bar{\chi}(S_h) = h$. We will show that none of the constructed systems S_h has extended bicolorings.

Let $\mathcal{C}$ be a bicoloring of S_h with color classes $C_1, \ldots, C_\ell$. We need to prove $\ell = h$. For this purpose we are going to analyze the positions of blocks with respect to $C_1, \ldots, C_\ell$.

Claim A. *For any two color classes C_i and C_j, if there exists a block which contains one vertex from C_i and two vertices from C_j, then for every block $B \subseteq C_i \cup C_j$ the same property holds; that is, $|B \cap C_i| = 1$ and $|B \cap C_j| = 2$.*

Proof. Assume to the contrary that there are two blocks $B_1 = \{\underline{a}, \underline{b}, \underline{c}\}$ and $B_2 = \{\underline{d}, \underline{x}, \underline{y}\}$ inside $C_i \cup C_j$ such that $\underline{a}, \underline{x}, \underline{y} \in C_i$ and $\underline{b}, \underline{c}, \underline{d} \in C_j$.

If these are six different vertices, consider the block incident with $\underline{a}$ and $\underline{d}$. Its third vertex, say $\underline{e}$, has color either i or j; we may assume without loss of generality that $\underline{e} \in C_i$. On the other hand, if B_1 and B_2 are not disjoint, then they share exactly one vertex, and renaming the vertices (and changing the indices i and j if necessary) we may assume again that $\{\underline{a}, \underline{b}, \underline{c}\}$ and $\{\underline{a},\underline{d},\underline{e}\}$ are two blocks with $\underline{a}, \underline{e} \in C_i$ and $\underline{b}, \underline{c}, \underline{d} \in C_j$.

By definition of S_h, we have $\underline{a} = \underline{b} + \underline{c} = \underline{d} + \underline{e}$. Consider first the element $\underline{z} = \underline{b} + \underline{e}$. Since $\mathcal{C}$ is a bicoloring, and $\{\underline{z}, \underline{b}, \underline{e}\}$ is a block in S_h, we obtain $\underline{z} \in C_i \cup C_j$. On the other hand, $\underline{z} = \underline{c} + \underline{d}$ also holds and $\underline{z} \notin C_j$ follows. Thus, $\underline{z} \in C_i$. (Note for this analysis that $\underline{z}$ is non-zero because $\underline{b} \neq \underline{e}$, and also it is distinct from each of $\underline{a}, \underline{c}, \underline{d}$ since $\underline{z} = \underline{c} + \underline{d}$, and $\underline{z} = \underline{a}$ would imply $\underline{b} + \underline{c} = \underline{b} + \underline{e}$, contradicting $\underline{c} \neq \underline{e}$.)

Next, let us consider $\underline{z}' = \underline{a} + \underline{b} + \underline{e}$. Since both $\underline{a}$ and $\underline{z} = \underline{b} + \underline{e}$ belong to C_i, we see that $\underline{z}' \notin C_i$. But $\underline{z}' = \underline{d}+\underline{e}+\underline{b}+\underline{e} = \underline{b}+\underline{d}$ and $\underline{z}' = \underline{b}+\underline{c}+\underline{b}+\underline{e} = \underline{c}+\underline{e}$ are valid as well. The former one implies $\underline{z}' \notin C_j$, while the latter implies $\underline{z}' \in C_i \cup C_j$. This contradiction proves the claim. (□)

By Claim A and by the basic property of Steiner triple systems, we can define an orientation $\vec{K}_\ell$ of the complete graph on the vertex set $\{1, \ldots, \ell\}$ such that an edge ij is oriented from i to j if and only if there exists a block B in S_h with $|B \cap C_i| = 1$ and $|B \cap C_j| = 2$.

Claim B. *The orientation $\vec{K}_\ell$ is transitive.*

Proof. For three color classes of $\mathcal{C}$, namely for C_i, C_j, and C_k, assume that we have the orientations $\vec{ij}$ and $\vec{jk}$ in $\vec{K}_\ell$. Take three vertices, one from each color class, $\underline{x} \in C_i$, $\underline{y} \in C_j$ and $\underline{z} \in C_k$. Since $\mathcal{C}$ is a bicoloring, $\underline{a} = \underline{x} + \underline{z}$ belongs to $C_i \cup C_k$. On the other hand, $\underline{a} = (\underline{x} + \underline{y}) + (\underline{z} + \underline{y})$, and our condition $\vec{ij}, \vec{jk} \in E(\vec{K}_\ell)$ implies that $\underline{x} + \underline{y} \in C_j$ and $\underline{z} + \underline{y} \in C_k$. Hence, $\underline{a} \in C_k$ and ik is oriented from i to k. This proves the transitivity of the orientation $\vec{K}_\ell$. (□)

Consequently, the defined orientation is a linear ordering. So, for every bicoloring $\mathcal{C}$ of S_h, there is an order $C_{i_1}, \ldots, C_{i_\ell}$ of the color classes satisfying the following condition. If $j < k$, every block intersecting C_{i_j} and C_{i_k} contains one vertex from the former class and two vertices from the latter one.

Claim C. *For every j, the color class C_{i_j} contains exactly 2^{j-1} vertices.*

Proof. Observe that every block meets C_{i_1} in at most one vertex, hence $|C_{i_1}| = 1$. Then, we may proceed by induction on j. In the color class C_{i_j}, each pair $\underline{x}, \underline{y}$ of vertices is covered by a block whose third vertex is from $\bigcup_{p=1}^{j-1} C_{i_p}$. Each block of this type covers exactly two vertex pairs between C_{i_j} and $\bigcup_{p=1}^{j-1} C_{i_p}$. Hence, by the induction hypothesis, we obtain

$$\binom{|C_{i_j}|}{2} = \frac{1}{2}|C_{i_j}| \sum_{p=1}^{j-1} |C_{i_p}| = \frac{1}{2}|C_{i_j}|(2^{j-1} - 1)$$

that proves $|C_{i_j}| = 2^{j-1}$. (□)

By Claim C, the $2^h - 1$ vertices of S_h are partitioned into exactly h color classes in any bicoloring. Therefore, $\chi(S_h) = \bar{\chi}(S_h) = h$, and consequently S_h admits no extended bicolorings. □

The system S_h has a large number of bicolorings; we can take the coordinates $i_1, i_2, \ldots, i_h$ in any order, and define the jth color class as the set of those vectors in which the coordinate i_j is 1, and all later coordinates are zero. In particular, fixing one coordinate to be 1 provides h different positions for a set of cardinality 2^{h-1} that can be taken as the largest color class. Hence, S_h has at least $h!$ different bicolorings.

As we have seen, S_h is obtained by doubling plus one construction, contains S_{h-1} as a subsystem and admits no extended bicolorings. Another kind of construction is described in the following result.

Theorem 3.2 *Suppose that S' is an $STS(2^h - 1)$ such that $\chi(S') = \bar{\chi}(S') = h$, moreover there is a (unique) set of 2^{h-1} vertices which is the largest color class in every bicoloring of S'. Then S' can be extended to an $STS(2^{h+1} - 1)$ system S via doubling plus one construction such that S does not admit any extended bicoloring.*

Proof.
Let $S' = (X', \mathcal{B}')$ be an $STS(2^h - 1)$ satisfying the assumptions of the theorem. Let $m = 2^{h-1}$ and denote by Y the set $\{y_1, \ldots, y_m\}$ of 2^{h-1}

vertices, which has to be the largest color class independently of the actual choice of the bicoloring. We extend S' to an $STS(2^{h+1}-1)$ denoted as $S=(X,\mathcal{B})$, and write X in the form $X=X'\cup Y'\cup Y''$, where the sets $Y'=\{y'_1,\dots,y'_m\}$ and $Y''=\{y''_1,\dots,y''_m\}$ are viewed as two copies of Y.

We keep all blocks of S' to form a subsystem of S. The further blocks of S are defined as follows. If $xy_iy_j\in\mathcal{B}'$ for an $x\in X'\setminus Y$ and some $y_i,y_j\in Y$, then its copies $xy'_iy'_j$ and $xy''_iy''_j$ are blocks in $\mathcal{B}$. To partition the remaining vertex pairs into triples, we note that the complete bipartite graph with vertex classes Y' and Y'' admits a 1-factorization into perfect matchings $F_1,\dots,F_m$. Then the following blocks are created: $y_i\cup f$, where $f\in F_i$, for $i=1,2,\dots,m$.

Consider any bicoloring $\mathcal{C}$ of S. On the set X' this induces a bicoloring of S', therefore the entire Y is monochromatic. Analogously, since the subsystems induced by $(X'\setminus Y)\cup Y'$ and $(X'\setminus Y)\cup Y''$ are isomorphic to S', each of Y' and Y'' is monochromatic. Moreover, the colors on $Y\cup Y'\cup Y''$ cannot occur in $X'\setminus Y$. Since $Y\cup Y'\cup Y''$ cannot be monochromatic, we obtain that $\mathcal{C}$ uses more than h colors (in fact precisely $h+1$ ones). Thus, S does not have an extended bicoloring. $\square$

We note that the assumption on the unique largest color class implies that S' is obtained by doubling plus one construction from a Steiner triple system S'' of order 2^{h-1}. It is not required, however, that also S'' is created in the same way.

4 Concluding remarks

For the systems $STS(2^{h+1}-1)$ we gave new necessary conditions for the existence of extended bicolorings. In particular, we obtained that the solutions of the equation system (1), which possibly may define extended bicolorings, can only be of two types: $(0,\cdots,0,c_i,0,\cdots,0,c_j,c_{j+1},\cdots,c_h)$ with $i\geq 3$, and $(0,\cdots,0,c_i,\ c_{i+1},\ \cdots,c_h)$.

It is important to stress that extended bicolorings of triple systems $STS(2^h-1)$ have not been found yet, while on the other hand the construction of Section 3 — which provably does not admit an extended bicoloring — does not include all triple systems obtainable by a sequence of doubling plus one constructions from STS(3). Hence, Gionfriddo's conjecture [5] remains open.

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