Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Algebra 280 (2004) 120–131
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

The Poincaré series of the module of derivations of affine monomial curves

R. Fröberg [a], V. Micale [b,*]

[a] *Matematiska Institutionen, Stockholms Universitet, 10691 Stockholm, Sweden*
[b] *Dipartimento di Matematica, Università di Catania, Viale Andrea Doria 6, 95125 Catania, Italy*

Received 2 October 2003
Available online 21 July 2004
Communicated by Paul Roberts

Abstract

Let A be a graded k-algebra and M be a finitely generated A-module. The Poincaré series $P_M^A(z)$ is the formal power series $\sum_i \dim_k \operatorname{Tor}_i^A(k, M)z^i$. We study the Poincaré series of $\operatorname{Der}_k k[S]$, the module of derivations of a numerical semigroup ring $k[S]$, and we relate it to the Poincaré series of k over $k[S]$ and to the type of S. We then use this in order to determine the Poincaré series of $\operatorname{Der}_k k[S]$ or, at least, its rationality, for some classes of examples. We finally give an example of a non-rational $P_{\operatorname{Der}_k k[S]}^{k[S]}(z)$.

1. Introduction

If A is a commutative k-algebra, with k a field, the *module of derivations*, $\operatorname{Der}_k(A) \subseteq \operatorname{Hom}_k(A, A)$ is the set $\{\rho \in \operatorname{Hom}_k(A, A) \mid \rho(ab) = a\rho(b) + \rho(a)b$ for every $a, b \in A\}$. This set has a natural A-module structure by multiplication from the left by elements in A.

* Corresponding author.
E-mail addresses: ralff@math.su.se (R. Fröberg), vmicale@dmi.unict.it (V. Micale).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.06.001

Let A be a standard graded k-algebra, i.e., $A = \bigoplus_{i=0}^{\infty} A_i$ with $A_0 = k$ and A generated by A_1, and let M be a finitely generated A-module. The *Poincaré series* $P_M^A(z)$ is the formal power series $\sum_i \dim_k \mathrm{Tor}_i^A(k, M) z^i$.

A graded k-algebra, $A = \bigoplus_{i \geqslant 0} A_i$, is called a *Koszul algebra* if the minimal graded A-resolution of k is linear, i.e., if $(\mathrm{Tor}_i^A(k,k))_j = 0$ if $i \neq j$. If A is a Koszul algebra, then $P_k^A(z) = 1/H_A(-z)$, where $H_A(z)$ is the Hilbert series $\sum_{i \geqslant 0} \dim_k A_i z^i$ of A (cf. [16]).

1.1. Description of the content

We now make a closer description of the paper. In Section 2 we give some definitions and fundamentals about numerical semigroups S and we introduce $k[S]$, the numerical semigroup ring. Furthermore, we give a minimal set of generators of $\mathrm{Der}_k^{k[S]}$ as a left $k[S]$-module (cf. Proposition 2.1) which represents the starting point of our paper.

In Section 3 we find the minimal free resolution of $\mathrm{Der}_k^{k[S]}$ for the class of two-generated numerical semigroups S.

In Section 4 we relate the Poincaré series of $\mathrm{Der}_k k[S]$ to the Poincaré series of k over $k[S]$ and to the type of S (cf. Theorem 4.4).

In Section 5, we use Theorem 4.4 in order to determine (or just proving the rationality of) the Poincaré series of $\mathrm{Der}_k k[S]$ for some classes of example. We finally give an example of a non-rational $P_{\mathrm{Der}_k k[S]}^{k[S]}(z)$.

2. Preliminaries

Our object of study in this paper is the Poincaré series of the module of derivations on affine monomial curves, that is, on numerical semigroup rings. Therefore, we start with some definitions and fundamentals of numerical semigroups. For a general reference to properties of numerical semigroups and semigroup rings, see [4].

Let $\mathbb{N}$ be the set of natural numbers (including zero). A subset $S \subseteq \mathbb{N}$ that contains zero and is closed under addition is called a *numerical semigroup*. Every nonzero numerical semigroup is isomorphic to a numerical semigroup with finite complement to $\mathbb{N}$. Given such a numerical semigroup we always find a unique minimal finite set of generators $\{g_1, g_2, \ldots, g_m\}$ with $g_1 < \cdots < g_m$. The numbers g_1 and m are called *the multiplicity* and the *embedding dimension* of S, respectively. We will use the notation $S = \langle g_1, g_2, \ldots, g_m \rangle = \{n_1 g_1 + \cdots + n_m g_m \mid n_1, \ldots, n_m \in \mathbb{N}\}$. Now *the semigroup ring* associated to S over a field k is $k[S] = k[t^{g_1}, \ldots, t^{g_m}] \subseteq k[t]$.

One of the most important invariants of a numerical semigroup S is the *Frobenius number* $g(S)$, that is the $\max\{n \in \mathbb{Z} \mid n \notin S\}$.

A numerical semigroup is called *symmetric* if for each $n \in \mathbb{Z}$, we have $n \in S$ or $g(S) - n \in S$.

The *type of* S is $|T(S)|$ where $T(S) = \{n \in \mathbb{Z} \setminus S \mid n + s \in S$ for every $s \in S \setminus \{0\}\}$ (of course $g(S) \in T(S)$ for every S). The type of S equals the CM-type of $k[S]$. An equivalent condition for a numerical semigroup to be symmetric is that $T(S) = \{g(S)\}$, hence a numerical semigroup is symmetric if and only if its type is one. Thus $k[S]$ is Gorenstein if and only if S is symmetric.

In general, there is no formula for the Frobenius number in terms of the generators. However, if the semigroup is 2-generated, say $S = \langle g_1, g_2\rangle$, then the Frobenius number is $g(S) = g_1g_2 - g_1 - g_2$ and S is symmetric.

Throughout the rest of the paper we always assume that $S \neq \mathbb{N}$ and that the characteristic of the field k is zero.

Since $\mathrm{Der}_k(A) \simeq \mathrm{Hom}_A(\Omega_{A/k}, A)$, the following proposition is easy to prove, but we refer to [9], where the result is greatly generalized.

Proposition 2.1. *Let S be a numerical semigroup with $S \neq \mathbb{N}$ and $T(S) = \{a_1, a_2, \dots, a_h\}$. Then the module of derivations of the numerical semigroup ring $k[S]$, $\mathrm{Der}_k\, k[S]$, is the left $k[S]$-module minimally generated by*

$$\{t\partial\} \cup \left\{t^{a_i+1}\partial,\ i = 1, \dots, h\right\},$$

where $\partial = \partial/\partial t$. In particular, the number of minimal generators is $|T(S)| + 1$.

3. The 2-generated case

If S is generated by two elements, then $k[S]$ is a hypersurface. Thus it is well known that the resolution in Proposition 3.1 is periodic. We will determine the resolution explicitly.

Proposition 3.1. *Let $S = \langle a, b\rangle$ be a 2-generated numerical semigroup and let $A = k[S]$. Then the left A-module $M = \mathrm{Der}_k A$ has the following minimal free resolution*:

$$\cdots \xrightarrow{\phi_4} A^2 \xrightarrow{\phi_3} A^2 \xrightarrow{\phi_2} A^2 \xrightarrow{\phi_1} A^2 \xrightarrow{\phi_0} M,$$

where

$$\phi_0 = \begin{pmatrix} t\partial & t^{ab-a-b+1}\partial \end{pmatrix}, \qquad \phi_{2r-1} = \begin{pmatrix} t^{(a-1)b} & t^{a(b-1)} \\ -t^a & -t^b \end{pmatrix},$$

$$\phi_{2r} = \begin{pmatrix} t^b & t^{a(b-1)} \\ -t^a & -t^{(a-1)b} \end{pmatrix},$$

with $r \geqslant 1$. In particular, the Poincaré series $P_M^A(z) = 2/(1-z)$.

Proof. We know by Proposition 2.1 that M is a left A-module minimally generated by $\{t\partial, t^{ab-a-b+1}\partial\}$.

Let $\phi_0 : A^2 \to M$ with $\phi_0(\mathbf{e}_1) = t\partial$ and $\phi_0(\mathbf{e}_2) = t^{ab-a-b+1}\partial$ and $\{\mathbf{e}_1, \mathbf{e}_2\}$ the canonical base for A^2. In order to determine the kernel of ϕ_0, we look for elements $f(t)$ and $g(t)$ in A such that $f(t)\mathbf{e}_1 + g(t)\mathbf{e}_2 = 0$. Since A is graded, $\ker\phi_0$ is also graded. Furthermore, $\ker\phi_0$ is 1-dimensional in each degree, hence a generator of $\ker\phi_0$ must be of the form

$(t^{s+ab-a-b}, -t^s)$ with $s \in S$. The two smallest values of s in order to obtain two independent generators are $s = a$ and $s = b$ which give $((t^{(a-1)b}, -t^a), (t^{a(b-1)}, -t^b)) \subseteq \ker \phi_0$. Furthermore, these two elements generate the kernel since a and b generate S. We also get

$$\phi_1 = \begin{pmatrix} t^{(a-1)b} & t^{a(b-1)} \\ -t^a & -t^b \end{pmatrix}.$$

Now a generator for $\ker \phi_1$ must be of the form $(t^{s+b-a}, -t^s)$. Let us consider $\max\{l \mid lb \notin a\mathbb{N}\}$. Since $\gcd(a, b) = 1$, this number is $a - 1$. Now the two smallest values of s in order to obtain two independent generators are $s = a$ and $s = (a - 1)b$. Therefore $((t^b, -t^a), (t^{a(b-1)}, -t^{(a-1)b})) \subseteq \ker \phi_1$. These elements generate $\ker \phi_1$. Indeed, if $(t^{s+b-a}, -t^s) \in \ker \phi_1$, then $s = a + s'$ for some $s' \in S$, or $s = lb$ with $l \geqslant a - 1$. We also get

$$\phi_2 = \begin{pmatrix} t^b & t^{a(b-1)} \\ -t^a & -t^{(a-1)b} \end{pmatrix}.$$

Let us consider $\ker \phi_2$. We look for generators of the form $(t^{s+ab-a-b}, -t^s)$. From $s = a$ and $s = b$, we get, as above, $\ker \phi_2 = ((t^{(a-1)b}, -t^a), (t^{a(b-1)}, -t^b))$ and

$$\phi_3 = \begin{pmatrix} t^{(a-1)b} & t^{a(b-1)} \\ -t^a & -t^b \end{pmatrix}.$$

Using induction, we have

$$\phi_{2r} = \begin{pmatrix} t^b & t^{ab-a} \\ -t^a & -t^{(a-1)b} \end{pmatrix} \quad \text{and} \quad \phi_{2r-1} = \begin{pmatrix} t^{(a-1)b} & t^{ab-a} \\ -t^a & -t^b \end{pmatrix}$$

with $r \geqslant 1$.

By construction this complex is a free resolution of M. Furthermore, since the entries in the matrices are in the graded maximal ideal of A, the resolution is minimal. □

4. The main theorem

In this section we will prove a theorem which allows us to determine the Poincaré series of $\mathrm{Der}_k\, k[S]$ over $k[S]$, whenever we know the Poincaré series of k over $k[S]$ and the type of S.

Lemma 4.1. *Let $S = \langle g_1, g_2, \ldots, g_m \rangle$ be a numerical semigroup with $T(S) = \{a_1, a_2, \ldots, a_h\}$ and let $A = k[S]$ and $\bar{A} = k[S]/((t^{g_1})k[S]) = k[\bar{t}^{g_2}, \ldots, \bar{t}^{g_m}]$. Then* $\mathrm{Soc}\, \bar{A} = \{\bar{t}^{a_1+g_1}, \ldots, \bar{t}^{a_h+g_1}\}$.

Proof. Let $\bar{t}^s \in \mathrm{Soc}\, \bar{A}$ with $\bar{t}^s \neq \bar{0}$ (hence $s \notin g_1 + S$). By the definition of socle, $\bar{t}^s \bar{t}^{g_i} = \bar{0}$ for every $i = 2, \ldots, m$ and this implies that $s + g_i \in g_1 + S$ for every $i = 1, \ldots, m$. Since

$s \in S$, $s - g_1 \notin S$ and $(s - g_1) + g_i \in S$ for every $i = 1, \ldots, m$, then $s - g_1 \in T(S)$. Hence $s - g_1 = a_j$ for some $j = 1, \ldots, h$ and $\operatorname{Soc} \bar{A} \subseteq \{\bar{t}^{a_1+g_1}, \ldots, \bar{t}^{a_h+g_1}\}$.

Let us consider now $\bar{t}^{a_i+g_1}$ with $i \in \{1, \ldots, h\}$. Since $a_i \in T(S)$, we get $a_i + g_j \in S$ for every $j = 1, \ldots, m$ and $a_i + g_j + g_1 \in g_1 + S$. So $\bar{t}^{a_i+g_1}\bar{t}^{g_j} = \bar{0}$ for every $j = 1, \ldots, m$, that is $\bar{t}^{a_i+g_1} \in \operatorname{Soc} \bar{A}$. $\square$

In [15], Levin introduces the idea of a large homomorphism of graded (or local) rings as a dual notion to small homomorphism of graded rings introduced in [1]. Namely if A and B are graded rings and $f : A \to B$ is a graded homomorphism which is surjective, then f is large if $f_* : \operatorname{Tor}^A(k,k) \to \operatorname{Tor}^B(k,k)$ is surjective.

If S, A, and $\bar{A}$ are as in Lemma 4.1, then, as a particular case of [15, Theorem 2.1], we get that the homomorphism $A \to \bar{A} = A/((t^{g_1})A)$ is large.

Hence using [15, Theorem 1.1], we have the following lemma.

Lemma 4.2. *Let S, A, and $\bar{A}$ be as in Lemma 4.1 and let M be a finitely generated left A-module such that $(t^{g_1})M = 0$. Then $P_M^A(z) = P_M^{\bar{A}}(z) P_{\bar{A}}^A(z) = (1+z) P_M^{\bar{A}}(z)$.*

Lemma 4.3. *Let S be a numerical semigroup with $T(S) = \{a_1, a_2, \ldots, a_h\}$. Then $\operatorname{Der}_k k[S]$ is isomorphic as a left $k[S]$-module to the ideal $(t^{g_1}, t^{a_1+g_1}, \ldots, t^{a_h+g_1})$ in $k[S]$.*

Proof. By Proposition 2.1, $\operatorname{Der}_k k[S]$ is minimally generated as left $k[S]$-module by $(t\partial, t^{a_1+1}\partial, \ldots, t^{a_h+1}\partial)$. Since, in our context, ∂ is just a symbol, we can delete it, and since $t^i \in S$ if $i \gg 0$, we have

$$\begin{aligned}\left(t, t^{a_1+1}, \ldots, t^{a_h+1}\right) &\simeq t^{L-1}\left(t, t^{a_1+1}, \ldots, t^{a_h+1}\right) = \left(t^L, t^{a_1+L}, \ldots, t^{a_h+L}\right) \\ &= t^{L-g_1}\left(t^{g_1}, t^{a_1+g_1}, \ldots, t^{a_h+g_1}\right) \simeq \left(t^{g_1}, t^{a_1+g_1}, \ldots, t^{a_h+g_1}\right). \qquad \square\end{aligned}$$

Theorem 4.4. *Let $S = \langle g_1, g_2, \ldots, g_m\rangle$ be a numerical semigroup with $T(S) = \{a_1, a_2, \ldots, a_h\}$ and let $A = k[S]$. Then $P_{\operatorname{Der}_k A}^A(z) = 1 + hP_k^A(z)$. In particular, $P_{\operatorname{Der}_k A}^A(z)$ is rational if and only if $P_k^A(z)$ is rational.*

Proof. By Lemma 4.3, we can replace $\operatorname{Der}_k A$ with $I = (t^{g_1}, t^{a_1+g_1}, \ldots, t^{a_h+g_1})$ in A and use the equality $P_{\operatorname{Der}_k A}^A(z) = P_I^A(z)$. We finally note that, by Lemma 4.1, $\bar{I}$, the image of I in $\bar{A} = k[S]/((t^{g_1})k[S])$, is the socle of $\bar{A}$ and that $P_{A/I}^A(z) = 1 + zP_I^A(z)$.

We note that A/I and $\bar{A}/\bar{I}$ are isomorphic as $\bar{A}$-modules, hence, using Lemma 4.2 with $M = A/I$ and that $\bar{I} \cong k \oplus \cdots \oplus k$ (h times), we have

$$\begin{aligned}P_I^A(z) &= \frac{P_{A/I}^A(z) - 1}{z} = \frac{(1+z)P_{\bar{A}/\bar{I}}^{\bar{A}}(z) - 1}{z} = \frac{(1+z)(1+zP_{\bar{I}}^{\bar{A}}(z)) - 1}{z} \\ &= \frac{(1+z)(1+hzP_k^{\bar{A}}) - 1}{z} = \frac{(1+z)(1+hzP_k^A(z)/(1+z)) - 1}{z} \\ &= \frac{hzP_k^A(z) + z}{z} = 1 + hP_k^A(z). \qquad \square\end{aligned}$$

5. Examples

In this section we will use Theorem 4.4 in order to determine the Poincaré series of $\mathrm{Der}_k\, k[S]$ for some classes of examples.

5.1. *The 3-generated case*

Let us start with considering the class of rings $A = k[S]$ where $S = \langle g_1, g_2, g_3\rangle$ is a 3-generated numerical semigroup. Then $\bar{A} = A/((t^{g_1})A)$ has embedding dimension 2, and $\bar{A}$ is either a complete intersection or a Golod ring (c.f. [18, Satz 9]). If $\bar{A}$ is a complete intersection, we have $P_k^{\bar{A}}(z) = 1/(1-z)^2$ (cf. [19, Theorem 6]), so $P^A_{\mathrm{Der}_k A}(z) = (2-z+z^2)/(1-z)^2$ since the type is one. Otherwise $\bar{A} = k[x_1, x_2]/(f_1, f_2, f_3)$ (c.f. [13, Theorem 3.7]), which gives $P_k^{\bar{A}}(z) = (1+z)^2/(1-3z^2-2z^3)$ (cf. [18]), so $P^A_{\mathrm{Der}_k A}(z) = (3+6z+3z^2)/(1-3z^2-2z^3)$, since the type equals 2.

5.2. *The case $S = \langle a, a+1, \dots, a+d\rangle$*

Let us now consider the class of ring $A = k[S]$ with $S = \langle a, a+1, \dots, a+d\rangle$ with $2d \geqslant a-1$. Let $N := S \setminus \{0\}$ (the so called *maximal ideal of* S). We denote $\{n_1 + \cdots + n_t \mid n_i \in N\}$ by tN.

Lemma 5.1. *Let $S = \langle a, a+1, \dots, a+d\rangle$ with $2d \geqslant a-1$. Then $3N = a + 2N$.*

Proof. From $2d \geqslant a-1$, we get that $3a + 2d \geqslant 4a - 1$. This implies that $\{3a, 3a+1, \dots, 4a-1\} \subseteq a + \{2a, 2a+1, \dots, 2a+2d\}$, which gives the proof. □

Let $\bar{A} = A/t^aA$. Then $\bar{A}$ has an induced t-grading from A, and $\bar{A}$ exists (and is 1-dimensional) only in degrees $S \setminus (a+S)$. We can regard $\bar{A}$ as $k[x_1, \dots, x_d]/I$, where I is the kernel of the epimorphism which sends x_i to $\bar{t}^{a+i}$, so $t\deg(x_i) = a+i$. We thus have $x_ix_j \in I$ if $(a+i)+(a+j) \in a+S$, $x_ix_j - x_kx_l \in I$ if $i+j = k+l$, and, by Lemma 5.1, $(x_1, \dots, x_d)^3 \subseteq I$. Let $H = \{d+1, d+2, \dots, a-1\}$, $F = \{x_ix_j \mid i+j \notin H\} \cup \{x_ix_j - x_{n-d}x_d \mid i+j = n \in H\}$, and $\bar{B} = k[x_1, \dots, x_d]/(F)$.

Lemma 5.2. *Let $\bar{A} = A/(t^aA) = k[x_1, \dots, x_d]/I$. Then $I = (F) + (x_1, \dots, x_d)^3$. Thus $\bar{A}$ is a standard graded algebra* ($\deg x_i = 1$ *for each* i) *with Hilbert series* $1 + dz + (a-(d+1))z^2$.

Proof. This follows since $S = \{0, a, a+1, \dots, a+d, 2a, \rightarrow\}$ and since, using Lemma 5.1, $S \setminus (a+S) = \{0, a+1, \dots, a+d, 2a+d+1, \dots, 3a-1\}$. □

We will now show that the set F is a Gröbner basis of (F) in Degrevlex. We will prove it considering two different cases. We recall that $a-1 \leqslant 2d$. We start to consider the case $a-1 < 2d$.

Lemma 5.3. *Let $a-1<2d$. Then $I=(F)$ and F is a Gröbner bases of (F) (hence of I) in Degrevlex.*

Proof. Let us denote the elements of F by $f_1,\dots,f_r$ (in no special order), the Gröbner basis of (F) by $G(F)$, the initial term of f_i by $\mathrm{in}(f_i)$ and $k[x_1,\dots,x_d]/(\mathrm{in}(f_1),\dots,\mathrm{in}(f_r))$ by $\bar{C}$.

Since $H_{\bar{A}}(z)\leqslant H_{\bar{B}}(z)=H_{k[x_1,\dots,x_d]/G(I_2)}(z)\leqslant H_{\bar{C}}(z)$ (coefficientwise), we only need is to show that $H_{\bar{A}}(z)=H_{\bar{C}}(z)$.

By Lemma 5.2, we know that $H_{\bar{A}}(z)=1+dz+(a-(d+1))z^2$. Since $\mathrm{in}(x_ix_j-x_{n-d}x_d)=x_ix_j$ (as we use Degrevlex), we get that the only monomials in $\bar{C}$ different from zero in degree less than or equal to two are $1,\bar{x}_1,\dots,\bar{x}_d,\bar{x}_1\bar{x}_d,\dots,\bar{x}_{(3a-1)-(2a+d+1)+1}\bar{x}_d=\bar{x}_{a-(d+1)}\bar{x}_d$.

The only possible nonzero monomials of degree three are $\bar{x}_i\cdot\bar{x}_j\bar{x}_d$ with $1\leqslant i\leqslant d$ and $1\leqslant j\leqslant a-(d+1)$. Since $a-1<2d$ we have $j<d$. If $i=d$, we get $\bar{x}_i\bar{x}_d=0$, if $i<d$, we get $\bar{x}_i\bar{x}_j=0$, hence all monomials of degree three are 0. Hence $H_{\bar{C}}(z)=1+dz+(a-(d+1))z^2=H_{\bar{A}}(z)$. □

Corollary 5.4. *Let $a-1<2d$. Then $P_k^{\bar{A}}(z)=1/(1-dz+(a-(d+1))z^2)$.*

Proof. By Lemma 5.3 and [7, Theorem 2.2], $\bar{A}$ is a Koszul algebra. □

Theorem 5.5. *Let $S=\langle a,a+1,\dots,a+d\rangle$ with $2d>a-1$ and $A=k[S]$. Then $P^A_{\mathrm{Der}_k A}(z)=(a-d+(a-2d-1)z+(a-d-1)z^2)/(1-dz+(a-d-1)z^2)$ if $d<a-1$ and $P^A_{\mathrm{Der}_k A}(z)=a/(1-(a-1)z)$ if $d=a-1$.*

Proof. By Lemma 4.2 and Corollary 5.4, we get $P_k^A(z)=(1+z)P_k^{\bar{A}}(z)=(1+z)/(1-dz+(a-(d+1))z^2)$.

We note that for this kind of semigroup S, we have $T(S)=\{a+d+1,\dots,2a-1\}$ (hence $h=|T(S)|=a-(d+1)$) if $d<a-1$ and $T(S)=\{1,\dots,a-1\}$ (hence $h=|T(S)|=a-1$) if $d=a-1$. Using Theorem 4.4, we get

$$P^A_{\mathrm{Der}_k A}(z)=1+\big(a-(d+1)\big)P_k^A(z)=\frac{a-d+(a-2d-1)z+(a-d-1)z^2}{1-dz+(a-d-1)z^2}$$

if $d<a-1$ and

$$P^A_{\mathrm{Der}_k A}(z)=1+(a-1)P_k^A(z)=\frac{a}{1-(a-1)z}$$

if $d=a-1$. □

Let us consider now the remaining case $a-1=2d$. In this case we cannot use the proof of Lemma 5.3 to show that the set F is a Gröbner basis of I as $\bar{A}$ and $\bar{B}$ have not the same Hilbert series (and, in particular, $I\neq(F)$). This is the case, for example, for $S = \langle 13, 14, 15, 16, 17, 18, 19\rangle$, where $H_{\bar{A}}(z) = 1 + 6z + 6z^2$ (cf. Lemma 5.2) and $H_{\bar{B}}(z) = 1 + 6z + 6z^2 + 6z^3 + \cdots = (1 + 5z)/(1 - z)$.

Lemma 5.6. *Let $a - 1 = 2d$. Then F is a Gröbner basis of (F) in Degrevlex. In particular, $\bar{B} = k[x_1, \ldots, x_d]/(F)$ is a Koszul algebra.*

Proof. We note that in this case $H = \{d + 1, \ldots, 2d\}$. As in Lemma 5.3, let us denote the elements of F by $f_1, \ldots, f_r$ and the initial term of a polynomial f by in f. We need to show that the S-polynomials $S_{i,j}$ of each pair (f_i, f_j) of the elements from F is zero modulo F.

Since this happens whenever f_i, f_j are both monomials or gcd(in f_i, in f_j) $= 1$, without loss of generality, we can restrict to consider only two cases, that is $f_a = x_i x_l$, $f_b = x_i x_j - x_{n-d} x_d$ (with $i + l \notin H$ and $i + j = n \in H$), and $f_a = x_i x_j - x_{n_1-d} x_d$, $f_b = x_i x_l - x_{n_2-d} x_d$ (with $i + l = n_2 \in H$, $i + j = n_1 \in H$).

In the first case, $S_{a,b} = x_j(x_i x_l) - x_l(x_i x_j - x_{n-d} x_d) = x_l x_{n-d} x_d$. Since $i + l \notin H$ and $i, l \leqslant d$, then $i + l < d + 1$. By $j \leqslant d$, we get $i + l + j < 2d + 1$, that is $l + (n - d) < d + 1$. This implies $l + (n - d) \notin H$ and $x_l x_{n-d} \in F$.

Let us now consider the second case. Here we have

$$S_{a,b} = x_l(x_i x_j - x_{n_1-d} x_d) - x_j(x_i x_l - x_{n_2-d} x_d) = -x_l x_{n_1-d} x_d + x_j x_{n_2-d} x_d.$$

By $i + j = n_1$ and $i + l = n_2$, we get $l + n_1 = n_2 - i + i + j = n_2 + j$. Hence $\bar{x}_l \bar{x}_{n_1-d} \bar{x}_d = \bar{x}_j \bar{x}_{n_2-d} \bar{x}_d$ modulo F as $\bar{B}$ is graded and one-dimensional in each degree.

The second part of the lemma follows by [7, Theorem 2.2]. □

Lemma 5.7. *Let $a - 1 = 2d$ and $\bar{B}$ be as above. Then the Hilbert series $H_{\bar{B}}(z) = (1 + (d - 1)z)/(1 - z)$.*

Proof. We first note that $|H| = |\{d + 1, \ldots, 2d\}| = d$.

Since, by Lemma 5.6, F if a Gröbner basis in Degrevlex, then $H_{\bar{B}}(z) = H_{\bar{C}}(z)$. Moreover, $\mathrm{in}(x_i x_j - x_{n-d} x_d) = x_i x_j$, hence the only elements in $\bar{C}$ of degree two different from zero are of the kind $\bar{x}_{n-d} \bar{x}_d$ with $n \in H$.

Using induction it is easy to see that the only elements in $\bar{C}$ of degree i different from zero are of the kind $\bar{x}_{n-d} \bar{x}_d^{i-1}$ with $n \in H$.

This gives $H_{\bar{B}}(z) = H_{\bar{C}}(z) = 1 + dz + dz^2 + dz^3 + \cdots = 1 + (d - 1)z/(1 - z)$. □

Theorem 5.8. *Let $S = \langle a, a + 1, \ldots, a + d\rangle$ with $2d = a - 1$ and $A = k[S]$. Then $P^A_{\mathrm{Der}_k A}(z) = 1 + d/(1 - dz)$.*

Proof. Since $\bar{A} = k[x_1, \ldots, x_d]/((F) + \mathfrak{m}^3)$, where $\mathfrak{m} = (x_1, \ldots, x_d)$, and $\bar{B}$ is a Koszul algebra (cf. Lemma 5.6), we get, as a particular case of [16, Theorem 1.5],

$$P_k^{\bar{A}}(-z) = \frac{z}{(z - 1)H_{\bar{B}}(z) + H_{\bar{A}}(z)}$$

$$= \frac{z}{(z-1)(1+(d-1)z)/(1-z)+1+dz+dz^2} = \frac{1}{1+dz}.$$

Finally, as above we get $h = |T(S)| = a - (d+1) = d$ and by Theorem 4.4 and Lemma 4.2, we have

$$P^A_{\mathrm{Der}_k A}(z) = 1 + dP^A_k(z) = 1 + d\big((1+z)/(1-dz)\big) = \frac{1+d}{1-dz}. \qquad \square$$

Remark 5.9. Let $S_1 = \langle a, a+1, \ldots, a+d\rangle$ and $S_l = \langle a, a+l, \ldots, a+ld\rangle$ with $2d \geqslant a-1$, $l \geqslant 1$ and $\gcd(a,l) = 1$.

Since $k[S_1]/((t^a)k[S_1]) = \bar{A} = k[S_l]/((t^a)k[S_l])$, then Theorems 5.5 and 5.8 hold for $S = S_l$.

5.3. *The case of multiplicity less than or equal 7*

In [14] there is a classification of all possible algebras $\bar{A}$, for $A = k[S]$ of multiplicity at most 7. We will show that $P^A_{\mathrm{Der}_k A}(z)$ is rational in all these cases.

Since $P^R_k(z)$ is rational for all rings R of embedding dimension at most 3 (c.f. [20, Corollary 4.4]), we get that $P^A_k(z)$, and thus $P^A_{\mathrm{Der}_k A}(z)$, is rational for all $A = k[S]$, where S has at most 4 generators. This takes care of all $\bar{A}$ for semigroups of multiplicity at most 5, except one for which $\bar{A} = k[x_1, \ldots, x_4]/(x_i x_j, 1 \leqslant i, j \leqslant 4)$, which is a Koszul algebra (cf. [10]).

If A has multiplicity 6, there are 5 different $\bar{A}$ of embedding dimension 4 or 5. All these have relations of degree 2 which constitute a Gröbner basis, so they are Koszul algebras (c.f. [7, Theorem 2.2]).

Finally, if A has multiplicity 7, there are 25 (out of 55) $\bar{A}$ of embedding dimension larger than 3. They are all of the form $k[x_1, \ldots, x_n]/I$, with $I = I_2 + J$, where I_2 (the part of the ideal in degree 2) is a Gröbner basis in all cases and $J = 0$ (so the ring is Koszul), or $J = (x_1, \ldots, x_n)^3$, or (in one case) $J = (x_1, \ldots, x_n)^4$. In the first case $\bar{A}$ is a Koszul algebra, so $P^{\bar{A}}_k(z)$, and thus $P^A_{\mathrm{Der}_k A}(z)$, is rational. For the last two cases we can use [16, Theorem 1.5] to conclude that $P^{\bar{A}}_k(z)$, and thus $P^A_{\mathrm{Der}_k A}(z)$, is rational.

5.4. *The case of maximal embedding dimension, maximal length, or almost maximal length*

A one-dimensional ring is of maximal embedding dimension if its embedding dimension equals its multiplicity (the same definition holds for the numerical semigroups and S is of maximal embedding dimension if and only if $k[S]$ is of maximal embedding dimension). Let $S = \langle g_1, \ldots, g_m\rangle$. If $k[S]$ is of maximal embedding dimension, then $\bar{A} = k[x_1, \ldots, x_{m-1}]/(x_1, \ldots, x_{m-1})^2$, which is a Koszul algebra, so $P^{\bar{A}}_k(z) = 1/(1-(m-1)z)$, and thus $P^A_{\mathrm{Der}_k A}(z) = m/(1-(m-1)z)$ since the type is $m-1$.

For a one-dimensional ring R we have the inequality $l(\bar{R}/R) \leqslant l(R/C)t(R)$, where $\bar{R}$ is the integral closure of R in its field of fractions, C is the conductor, and $t(R)$ the CM-type (c.f. [4]). The ring R is called of maximal length if there is equality, and it is called of almost maximal length if $l(\bar{R}/R) = l(R/C)t(R) - 1$.

If $k[S]$ is of maximal length it is either Gorenstein or of maximal embedding dimension (c.f. [5] or [6]) thus, rings of maximal embedding dimension and with type at least two have rational $P^A_{\mathrm{Der}_k A}(z)$.

If $k[S]$ is of almost maximal length, then $S = \langle 4, 5, 11\rangle$, $S = \langle 4, 7, 13\rangle$, $S = \langle 3, 3d+2, 3d+4\rangle$ for some $d \geqslant 1$, or $S = \langle p, dp+1, dp+2, \ldots, dp+p-1\rangle$ for some $p \geqslant 3$ and $d \geqslant 1$ (c.f. [5, Theorems 3 and 4]). For the first three examples $\bar{A}$ is a Golod ring, so $P^A_{\mathrm{Der}_k A}(z) = (3+6z+3z^2)/(1-3z^2-2z^3)$ (see Section 5.1). The last class is of maximal embedding dimension.

5.5. *The case of monomial semigroups*

Let R be a one-dimensional Noetherian domain with $k \subset R \subseteq k[\![t]\!]$ and v be the natural valuation for nonzero elements of $k(\!(t)\!)$. Then $v(R)$ is a numerical semigroup. If $S = \langle g_1, \ldots, g_m\rangle$, then by $k[\![t^S]\!]$ we mean $k[\![t^{g_1}, \ldots, t^{g_m}]\!]$. An equivalent definition of semigroup ring for this kind of rings R is that $R = k[\![x^S]\!]$ for some $x \in (t) \setminus (t^2)$. In general, if S is fixed and we consider all rings R as above with $v(R) = S$, it is not true that all these rings are semigroup rings.

A numerical semigroup S is called monomial if each ring R with $v(R) = S$ is a semigroup ring.

If S is a monomial semigroup, then S is one from the following list:

(i) S is such that the only elements smaller than the Frobenius number are multiples of g_1,
(ii) $l \notin S$ only for one $l > g_1$,
(iii) $g_1 \geqslant 3$ and the only elements greater than g_1 that are not in S are g_1+1 and $2g_1+1$ (cf. [17, Theorem 3.12]).

The first class is of maximal embedding dimension.

Let us consider the second class. In this case $S = \{0, g_1, \ldots, g_1+\alpha-1, g_1+\alpha+1, \rightarrow\}$. If $\alpha = 1$, then S is of maximal embedding dimension. If $\alpha = 2$, then $\bar{A} = k[x_1, \ldots, x_{m-1}]/I = k[x_1, \ldots, x_{m-1}]/(I_2 + \tilde{\mathfrak{m}}^3)$, where I_2 is the part of I in degree two and $k[x_1, \ldots, x_{m-1}]/I_2$ is a Koszul algebra. Using the same argument as in the proof of Theorem 5.5, we get $P^A_{\mathrm{Der} A}(z)$. Finally, for $\alpha > 2$, $I = I_2$ and even in this case, using the same argument as in the proof of Theorem 5.8, we get $P^A_{\mathrm{Der} A}(z)$.

Finally, let S be in the third class. If $g_1 = 3$, then S is two-generated. If $g_1 > 3$, then $I = I_2$ and $\bar{A}$ is Koszul.

In all these cases $P^A_{\mathrm{Der} A}(z)$ is rational.

5.6. *Further rational cases*

There are some more classes of semigroups for which we can say that $P^A_{\mathrm{Der}_k A}(z)$ is rational. If $k[S]$ is a complete intersection, then $P^A_k(z)$, and thus $P^A_{\mathrm{Der}_k A}(z)$, is rational (cf. [19]). Semigroups defining complete intersections are classified in [8].

Gorenstein rings R of codimension at most 4 have rational Poincaré series $P_k^R(z)$ (c.f. [12]). They treat the case $\mathrm{char}(k) \neq 2$, the general case is in [2]. Thus symmetric semigroups with at most 5 generators have rational series $P^A_{\mathrm{Der}_k A}(z)$.

In [2] also other classes of rings are shown to have rational series, e.g., almost complete intersections of codimension 4. Thus, if S has 5 generators and $k[S]$ (or $\bar{A}$) 5 relations, then $P^A_{\mathrm{Der}_k A}(z)$ is rational. Also if $\bar{A}$ has monomial relations, the series are rational, c.f. [3].

5.7. An example of non-rational $P^{k[S]}_{\mathrm{Der}_k k[S]}(z)$

If $S = \langle 18, 24, 25, 26, 28, 30, 33\rangle$, it is shown in [11] that $P_k^{k[S]}(z)$ is not rational. Thus $P^{k[S]}_{\mathrm{Der}_k k[S]}(z)$ is not rational.

Acknowledgment

We thank the referee for suggestions and a very careful reading.

References

[1] L.L. Avramov, Small homomorphism of local rings, J. Algebra 50 (1978) 400–453.
[2] L.L. Avramov, A. Kustin, M. Miller, Poincaré series of modules over local rings of small embedding codepth or small linking number, J. Algebra 118 (1988) 162–204.
[3] J. Backelin, Les anneaux locaux à relations monomiales ont des séries de Poincaré–Betti rationelles, C. R. Acad. Sci. Paris Sér. I Math. 295 (1982) 607–610.
[4] V. Barucci, D.E. Dobbs, M. Fontana, Properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains, Mem. Amer. Math. Soc. 125 (1997) 598.
[5] W.C. Brown, F. Curtis, Numerical semigroups of maximal and almost maximal length, Semigroup Forum 42 (1991) 219–235.
[6] W.C. Brown, J. Herzog, One dimensional local rings of maximal and almost maximal length, J. Algebra 51 (1992) 332–347.
[7] W. Bruns, J. Herzog, U. Vetter, Syzygies and walks, in: A. Simis, N.V. Trung, G. Valla (Eds.), Commutative Algebra, Trieste 1992, World Scientific, Singapore, 1994.
[8] C. Delorme, Sous-monoïdes d'intersection complète de N, Ann. Sci. Ècole Norm. Sup. 9 (1976) 145–154.
[9] E. Eriksen, Differential operators on monomial curves, J. Algebra 264 (1) (2003) 186–198.
[10] R. Fröberg, Determination of a class of Poincaré series, Math. Scand. 37 (1975) 29–39.
[11] R. Fröberg, J.E. Roos, An affine monomial curve with irrational Poincaré–Betti series, J. Pure Appl. Algebra 152 (2000) 89–92.
[12] C. Jacobsson, A. Kustin, M. Miller, The Poincaré series of a codimension four Gorenstein ring is rational, J. Pure Appl. Algebra 38 (1985) 255–275.
[13] J. Herzog, Generators and relations of abelian semigroups and semigroup rings, Manuscripta Math. 3 (1970) 175–193.
[14] E. Kunz, Über die Klassifikation numerischer Halbgruppen, in: Regensburger Math. Schriften, vol. 11, Universität Regensburg, Fachbereich Mathematik, Regensburg, 1987.
[15] G. Levin, Large homomorphisms of local rings, Math. Scand. 46 (1980) 209–215.
[16] C. Löfwall, On the subalgebra generated by one-dimensional elements in the Yoneda Ext-algebra, in: J.-E. Roos (Ed.), Algebra, Algebraic Topology and their Interactions, in: Lecture Notes in Math., vol. 1183, Springer-Verlag, Berlin, 1986, pp. 291–338.
[17] V. Micale, On monomial semigroups, Comm. Algebra 30 (10) (2002) 4687–4698.

[18] G. Scheja, Über die Bettizahlen lokaler Ringe, Math. Ann. 155 (1964) 155–172.
[19] J. Tate, Homology of Noetherian rings and local rings, Illinois J. Math. 1 (1957) 14–27.
[20] J. Weyman, On the structure of free resolutions of length 3, J. Algebra 126 (1980) 1–33.