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Representable Lexicographic Products

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Abstract. A linear ordering is said to be representable if it can be order-embedded into the reals. Representable linear orderings have been characterized as those which are separable in the order topology and have at most countably many jumps. We use this characterization to study the representability of a lexicographic product of linear orderings. First we count the jumps in a lexicographic product in terms of the number of jumps in its factors. Then we relate the separability of a lexicographic product to properties of its factors, and derive a classification of representable lexicographic products.

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1. Introduction

This paper deals with representations of linear orderings (also called chains) in ways which are useful in mathematical economics and, in particular, in utility theory. A model of *utility theory* consists of: (i) a set of elements L, usually interpreted as decision alternatives or courses of action; (ii) an individual's binary relation $\preceq$ of preference/indifference for elements of L (where $x \preceq y$ means "x is not preferred to y"); (iii) an internally-consistent set of assumptions about L and the behavior of $\preceq$ on L, together with conclusions that can be deduced from the assumptions (see Fishburn (1968) for a discussion of this point of view).

In utility theory the relation $\preceq$ of *weak preference* is usually assumed to be a total preorder, i.e., a reflexive, transitive and total binary relation. The *indifference* relation $\sim$, defined by $x \sim y$ if $x \preceq y$ and $y \preceq x$, is an equivalence relation. If indifferent elements of L are identified, $\preceq$ yields a linear order on the quotient space $L/{\sim}$. In this paper we work on the quotient space, thus assuming that the relation $\preceq$ is a linear order. A chain will be indifferently denoted by $(L, \preceq)$, $(L, \prec)$ or L. (Here $\prec$ is the relation of *strict preference*, defined by $x \prec y$ if $x \preceq y$ and $x \neq y$.)

A fundamental concept in utility theory is the representability of chains (see, e.g., Bridges and Mehta (1995) for an extensive treatment of this topic): a chain L is *representable* if there exists a map $u\colon L \to \mathbb{R}$, called a *utility function*, which is an order-embedding. Evidently such a function on L allows us to measure preferences quantitatively. Representability can be characterized as follows (see Fleischer, 1961a):

THEOREM 1.1. *A chain is representable if and only if it is separable in the order topology and has at most countably many jumps.*

(Recall that a *jump* in a chain $(L, \prec)$ is a pair $(a, b) \in L^2$ such that $a \prec b$ and the open interval (a, b) is empty.) In this paper we study the representability of lexicographic products of chains. In view of Theorem 1.1, we carry out this analysis by determining the relationships between: (a) jumps in a lexicographic product and jumps in its factors; (b) separability of a lexicographic product and separability of its factors.

The paper is organized as follows. In Section 2 we introduce some basic terminology. Section 3 deals with jumps in lexicographic products. In particular, we establish a formula which counts the jumps in a lexicographic product in terms of the number of jumps in its factors. Then we derive a characterization of lexicographic products which have at most countably many jumps. In Section 4 we characterize separability of a lexicographic product in terms of some properties of its factors. (Using the same approach, we also deal with lexicographic products which satisfy the countable chain condition.) The section ends with a classification of representable lexicographic products. In Section 5 we mention possible applications of lexicographic products to mathematical economics and outline future directions of research.

2. Preliminaries

By $\mathbb{R}$ and $\mathbb{Q}$ we mean the chains $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$, respectively; the chain $(\mathbb{N}, <)$ can be denoted either by $\mathbb{N}$ or by the ordinal number ω. As usual, an ordinal α is identified with the set of all ordinals below it, i.e., $\alpha = \{\beta : \beta < \alpha\}$. For operations on ordinals and cardinals see Kunen (1980).

An *order-homomorphism* (henceforth, *homomorphism*) is a map $f: L \to M$ between two chains such that for all $x, y \in L$, $x \preceq y$ implies $f(x) \preceq f(y)$. In particular, an *embedding* (respectively, *isomorphism*) is an injective (respectively, bijective) homomorphism; the notation $L \hookrightarrow M$ stands for embeddability of the chain L into the chain M, whereas $L \cong M$ denotes that L and M are isomorphic chains.

The *density* $\mathrm{d}(L)$ of a chain $(L, \prec)$ is the least infinite cardinal κ such that there is a set $D \subseteq L$ of cardinality κ which is dense in L (i.e., D intersects every nonempty open interval in L); in particular, L is *separable* if $\mathrm{d}(L) = \aleph_0$. The *cellularity* $\mathrm{c}(L)$ of L is the least infinite cardinal κ such that every family of pairwise disjoint nonempty open intervals of L has cardinality $\leqslant \kappa$; in particular, L has the *c.c.c.* (*countable chain condition*) if $\mathrm{c}(L) = \aleph_0$. Note that the density and the cellularity of a chain $(L, \prec)$ are equal, respectively, to the density and the cellularity of the topological space $(L, \tau_{\prec})$, where $\tau_{\prec}$ is the order topology induced by $\prec$. Observe also that $\mathrm{c}(X) \leqslant \mathrm{d}(X)$ for any topological space X. In particular, $\mathrm{c}(L) \leqslant \mathrm{d}(L)$ for any chain L, and so a chain which does not satisfy the c.c.c. is not representable.

NOTATION 2.1. Let $(L_i, \prec)_{i\in I}$ be a nonempty family of chains, where the index set I is a well-ordered set $(I, <)$. The *lexicographic product* of the family $(L_i, \prec)_{i\in I}$ is the chain $(\prod_{i\in I} L_i, \prec_{\mathrm{lex}})$, where the order relation is defined as follows: for any $x = (x_i)_{i\in I}$, $y = (y_i)_{i\in I} \in \prod_{i\in I} L_i$, set $x \prec_{\mathrm{lex}} y$ if there exists an index $\Delta = \Delta_x^y \in I$ such that $x_\Delta \prec y_\Delta$ and $x_i = y_i$ for all $i \in I$ satisfying $i < \Delta$. We denote this chain by $\prod_{i\in I}^{\mathrm{lex}} L_i$. In particular, if I is a nonzero ordinal α, we denote the corresponding lexicographic product by $\prod_{\xi<\alpha}^{\mathrm{lex}} L_\xi$. Similarly, $L \times_{\mathrm{lex}} M$ denotes the lexicographic product of the two chains L and M, whereas $L_{\mathrm{lex}}^{\alpha}$ denotes $\prod_{\xi<\alpha}^{\mathrm{lex}} L_\xi$, with $L_\xi = L$ for each $\xi < \alpha$.

We assume that all factors in a lexicographic product are non-trivial chains (i.e., they have a least two elements). This assumption causes no loss of generality for our purposes. In fact, if one of the factors is the empty chain, then the whole lexicographic product is empty. On the other hand, if any of the factors is the chain with exactly one element, then we can omit that factor and obtain a lexicographic product which is isomorphic to the original one.

The following lexicographic products play an important role in studying representability.

EXAMPLE 2.2. $\mathbb{Q}_{\mathrm{lex}}^{\omega}$ *is representable, but* $\mathbb{R} \times_{\mathrm{lex}} 2$ *is not*. In fact $\mathbb{Q}_{\mathrm{lex}}^{\omega}$ is separable (eventually constant sequences form a countable dense subset) and has no jumps. On the other hand, $\mathbb{R} \times_{\mathrm{lex}} 2$ is separable but has uncountably many jumps.

EXAMPLE 2.3. *If L is an uncountable chain and M is a chain with at least three elements, then $L \times_{\mathrm{lex}} M$ fails to have the c.c.c.* (*hence it is non-representable*). Let $a, b \in M$ be such that $a \prec b$ and $(a, b) \neq \emptyset$. Then $\mathcal{G} := \{\{x\} \times (a, b) : x \in L\}$ is a set of pairwise disjoint nonempty open intervals in $L \times_{\mathrm{lex}} M$ such that $|\mathcal{G}| = |L|$. Thus, e.g., $\mathrm{c}(\mathbb{R}) = \mathrm{c}(2) = \aleph_0 < 2^{\aleph_0} = \mathrm{c}(\mathbb{R} \times_{\mathrm{lex}} 3)$.

We end this section with two monotonicity results.

LEMMA 2.4. *Let $(L_\xi)_{\xi<\alpha}$ and $(M_\xi)_{\xi<\alpha}$ be two families of chains, where α is a nonzero ordinal.*

(i) *If $L_\xi \hookrightarrow M_\xi$ for all $\xi < \alpha$, then $\prod_{\xi<\alpha}^{\mathrm{lex}} L_\xi \hookrightarrow \prod_{\xi<\alpha}^{\mathrm{lex}} M_\xi$.*
(ii) *For any $I \subseteq \alpha$, $\prod_{i\in I}^{\mathrm{lex}} L_i \hookrightarrow \prod_{\xi<\alpha}^{\mathrm{lex}} L_\xi$.*

Proof. The proof of part (i) is easy and is left to the reader. For (ii), let $I \subseteq \alpha$. For each $\xi \in \alpha \setminus I$, select an element $\widehat{l_\xi} \in L_\xi$. Further, for any $x = (x_i)_{i\in I} \in \prod_{i\in I}^{\mathrm{lex}} L_i$, let $l^x = (l_\xi^x)_{\xi<\alpha}$ be the element of $\prod_{\xi<\alpha}^{\mathrm{lex}} L_\xi$ defined by $l_\xi^x := x_\xi$ if $\xi \in I$, and

$l^x_\xi := \widehat{l_\xi}$ if $\xi \in \alpha \setminus I$. The correspondence $x \overset{\varphi}{\mapsto} l^x$ gives a well-defined embedding $\varphi\colon \prod^{\mathrm{lex}}_{i\in I} L_i \hookrightarrow \prod^{\mathrm{lex}}_{\xi<\alpha} L_\xi$. □

3. Jumps in Lexicographic Products

In this section we first obtain a formula to count jumps in a lexicographic product. Then we characterize lexicographic products with at most countably many jumps.

NOTATION 3.1. For any chain L, we denote by $\mathrm{Jump}(L)$ the set of all jumps in L, and we set $\mathrm{j}(L) := |\mathrm{Jump}(L)|$. Further, for $L = \prod^{\mathrm{lex}}_{\xi<\alpha} L_\xi$, we denote by Δ_L the least ordinal β with the property that for all ordinals ξ satisfying $\beta < \xi < \alpha$, the chain L_ξ has both a minimum and a maximum.

Remark 3.2. For any lexicographic product $L = \prod^{\mathrm{lex}}_{\xi<\alpha} L_\xi$, we have $0 \leqslant \Delta_L \leqslant \alpha$. In particular, $\Delta_L = 0$ if and only if each chain L_ξ in the factorization of L, with the possible exception of L_0, has both a minimum and a maximum. On the other hand, $\Delta_L = \alpha$ if and only if α is a limit ordinal and $\alpha \setminus \Omega(L)$ is unbounded in α.

Now we describe the jumps in a lexicographic product.

LEMMA 3.3. *Let $L = \prod^{\mathrm{lex}}_{\xi<\alpha} L_\xi$ be a lexicographic product. Further, let $a = (a_\xi)_{\xi<\alpha}$ and $b = (b_\xi)_{\xi<\alpha}$ be two elements of L such that $a \prec_{\mathrm{lex}} b$. Then (a, b) is a jump in L if and only if the following two conditions hold:*

(i) *(a_Δ, b_Δ) is a jump in L_Δ, where $\Delta = \Delta^b_a$;*
(ii) *for all $\xi > \Delta^b_a$, $a_\xi = \max L_\xi$ and $b_\xi = \min L_\xi$.*

In particular, $(a, b) \in \mathrm{Jump}(L)$ implies $\Delta^b_a \geqslant \Delta_L$.

Proof. The proof is easy and is left to the reader. □

Remark 3.4. The previous result yields that a jump (a, b) in a lexicographic product $L = \prod^{\mathrm{lex}}_{\xi<\alpha} L_\xi$ is determined by the following parameters: (i) the least coordinate $\Delta^b_a < \alpha$ at which a and b differ; (ii) the two endpoints $a_{\Delta^b_a}$ and $b_{\Delta^b_a}$ of the jump in $L_{\Delta^b_a}$; and (iii) the sequence $(a_\xi)_{\xi<\Delta^b_a} = (b_\xi)_{\xi<\Delta^b_a}$ in $\prod_{\xi<\Delta^b_a} L_\xi$.

A chain is said to be *dense-in-itself* if it has no jumps.

PROPOSITION 3.5. *A lexicographic product $L = \prod^{\mathrm{lex}}_{\xi<\alpha} L_\xi$ is dense-in-itself if and only if L_ξ is dense-in-itself for all $\xi \geqslant \Delta_L$. In particular, if $\Delta_L = \alpha$ then L is dense-in-itself.*

Proof. We prove the contrapositive in both directions.

($\Rightarrow$) Assume that there exists $(a_\beta, b_\beta) \in \mathrm{Jump}(L_\beta)$ for some $\beta \geqslant \Delta_L$. For each $\xi < \beta$, select $l_\xi \in L_\xi$. Define two elements $a = (a_\xi)_{\xi<\alpha}$ and $b = (b_\xi)_{\xi<\alpha}$ in L as follows: $a_\xi = b_\xi := l_\xi$ if $\xi < \beta$; $a_\xi := \max L_\xi$ and $b_\xi := \min L_\xi$ if $\xi > \beta$. Then $a \prec_{\mathrm{lex}} b$ and $\Delta_a^b = \beta$, so $(a, b) \in \mathrm{Jump}(L)$ by Lemma 3.3.

($\Leftarrow$) Let $a = (a_\xi)_{\xi<\alpha}$ and $b = (b_\xi)_{\xi<\alpha}$ be such that (a, b) is a jump in L. Then Lemma 3.3 yields that $(a_{\Delta_a^b}, b_{\Delta_a^b})$ is a jump in $L_{\Delta_a^b}$ and $\Delta_a^b \geqslant \Delta_L$. □

The number of jumps in a lexicographic product can be obtained as follows.

THEOREM 3.6. *For any lexicographic product $L = \prod^{\mathrm{lex}}_{\xi<\alpha} L_\xi$, the number of jumps in L is given by*

$$\mathrm{j}(L) = \sum_{\Delta_L \leqslant \beta < \alpha} \Bigl(\mathrm{j}(L_\beta) \cdot \prod_{\xi<\beta} |L_\xi|\Bigr)$$

where $\sum$ denotes cardinal addition and $\prod$ cardinal multiplication.

Proof. The formula holds whenever L has no jumps, since in this case Proposition 3.5 yields $\mathrm{j}(L_\beta) = 0$ for all $\beta \geqslant \Delta_L$. (Note that if $\Delta_L = \alpha$, then $\mathrm{j}(L) = 0$ by Proposition 3.5, whereas the right-hand side is an empty sum.) Thus assume $\mathrm{j}(L) > 0$. Define a map

$$f\colon \mathrm{Jump}(L) \to \bigcup_{\Delta_L \leqslant \beta < \alpha} \Bigl(\prod_{\xi<\beta} L_\xi \times \mathrm{Jump}(L_\beta)\Bigr)$$

$$(a, b) \mapsto \bigl((a_\xi)_{\xi<\Delta_a^b}, (a_{\Delta_a^b}, b_{\Delta_a^b})\bigr)$$

where $a = (a_\xi)_{\xi<\alpha}$ and $b = (b_\xi)_{\xi<\alpha}$ are the endpoints of a jump in L. Note that for any $(a, b) \in \mathrm{Jump}(L)$, Lemma 3.3 yields $(a_{\Delta_a^b}, b_{\Delta_a^b}) \in \mathrm{Jump}(L_{\Delta_a^b})$ and $\Delta_a^b \geqslant \Delta_L$. Therefore f is a well-defined function. Next we show that f is a bijection; this will prove the stated equality.

To check that f is injective, let (a, b) and (c, d) be two different jumps in L. Note that their left endpoints $a = (a_\xi)_{\xi<\alpha}$ and $c = (c_\xi)_{\xi<\alpha}$ cannot be equal, since otherwise either $d \in (a, b)$ or $b \in (c, d)$, which is impossible. Without loss of generality assume that $a \prec_{\mathrm{lex}} c$. If $(a_\xi)_{\xi<\Delta_a^b} \neq (c_\xi)_{\xi<\Delta_c^d}$, then $f(a, b) \neq f(c, d)$. On the other hand, if $(a_\xi)_{\xi<\Delta_a^b} = (c_\xi)_{\xi<\Delta_c^d}$, then $\Delta_a^b = \Delta_c^d$. Since $a \prec_{\mathrm{lex}} b \preceq_{\mathrm{lex}} c \prec_{\mathrm{lex}} d$ by hypothesis, we obtain $a_{\Delta_a^b} \prec b_{\Delta_a^b} \preceq c_{\Delta_c^d} \prec d_{\Delta_c^d}$; in particular, $a_{\Delta_a^b} \neq c_{\Delta_c^d}$. It follows that $f(a, b) \neq f(c, d)$ also in this case.

To show that f is onto, let $(x, (y, z))$ be any element of $\bigcup_{\Delta_L \leqslant \beta<\alpha}(\prod_{\xi<\beta} L_\xi \times \mathrm{Jump}(L_\beta))$. Thus there exists an ordinal β satisfying the following properties: $\Delta_L \leqslant \beta < \alpha$, $x = (x_\xi)_{\xi<\beta}$ belongs to $\prod_{\xi<\beta} L_\xi$, and (y, z) is a jump in L_β. Define two elements $a = (a_\xi)_{\xi<\alpha}$ and $b = (b_\xi)_{\xi<\alpha}$ in L as follows. Set $a_\xi = b_\xi := x_\xi$ for all $\xi < \beta$; $a_\beta := y$ and $b_\beta := z$; $a_\xi := \max L_\xi$ and $b_\xi := \min L_\xi$ for all $\xi > \beta$. (Observe that $\beta \geqslant \Delta_L$ implies that a and b are well-defined elements of L.) Then

$\Delta_a^b = \beta \geqslant \Delta_L$, so $(a, b) \in \mathrm{Jump}(L)$. Since $f(a, b) = (x, (y, z))$, it follows that f is surjective. $\square$

Remark 3.7. Observe that the order $\prec$ on a chain L induces naturally a total order $\lhd$ on the set $\mathrm{Jump}(L)$; namely, for any two jumps $(a, b), (c, d)$ in L, let $(a, b) \lhd (c, d)$ if and only if $a \prec c$. If $L = \prod_{\xi<\alpha}^{\mathrm{lex}} L_\xi$, then we can endow the set $\bigcup_{\Delta_L \leqslant \beta < \alpha}(\prod_{\xi<\beta} L_\xi \times \mathrm{Jump}(L_\beta))$ with a linear order $\sqsubset$ defined as follows:

$$((a_\xi)_{\xi<\beta}, (a_\beta, b_\beta)) \sqsubset ((c_\xi)_{\xi<\gamma}, (c_\gamma, d_\gamma)) \Leftrightarrow ((a_\xi)_{\xi\leqslant\delta}, \gamma) \prec_{\mathrm{lex}} ((c_\xi)_{\xi\leqslant\delta}, \beta)$$

where $\delta := \min\{\beta, \gamma\}$ and $\Delta_L \leqslant \delta$. Next we show that the map f defined in the proof of Theorem 3.6 is an isomorphism between $(\mathrm{Jump}(L), \lhd)$ and $(\bigcup_{\Delta_L \leqslant \beta < \alpha}(\prod_{\xi<\beta} L_\xi \times \mathrm{Jump}(L_\beta)), \sqsubset)$.

Let $(a, b), (c, d) \in \mathrm{Jump}(L)$ be such that $(a, b) \lhd (c, d)$, i.e., $a \prec c$. Denote $\delta := \min\{\Delta_a^b, \Delta_c^d\}$. By Theorem 3.6, it suffices to show that $f(a, b) \sqsubset f(c, d)$, i.e., either $(a_\xi)_{\xi\leqslant\delta} \prec_{\mathrm{lex}} (c_\xi)_{\xi\leqslant\delta}$, or $(a_\xi)_{\xi\leqslant\delta} = (c_\xi)_{\xi\leqslant\delta}$ and $\Delta_c^d < \Delta_a^b$. Since $a \prec c$, we have $(a_\xi)_{\xi\leqslant\delta} \preceq_{\mathrm{lex}} (c_\xi)_{\xi\leqslant\delta}$. If the inequality is strict, then we are done. On the other hand, assume that $(a_\xi)_{\xi\leqslant\delta} = (c_\xi)_{\xi\leqslant\delta}$. It follows that $\Delta_a^b \neq \Delta_c^d$, because otherwise $c \in (a, b)$, contradicting $(a, b) \in \mathrm{Jump}(L)$. Thus one and only one of the following cases is possible: either (i) $a_\xi = \max L_\xi$ for all $\xi > \delta$, or (ii) $c_\xi = \max L_\xi$ for all $\xi > \delta$. Since $a \prec c$ and $(a_\xi)_{\xi\leqslant\delta} = (c_\xi)_{\xi\leqslant\delta}$, it follows that (ii) holds. Therefore $\Delta_c^d = \delta < \Delta_a^b$, and so $f(a, b) \sqsubset f(c, d)$.

Lexicographic products with at most countably many jumps can be characterized as follows.

COROLLARY 3.8. *The following statements are equivalent for a lexicographic product* $L = \prod_{\xi<\alpha}^{\mathrm{lex}} L_\xi$*:*

(i) $\mathrm{Jump}(L)$ *is countable;*

(ii) *for all ordinals* $\beta, \gamma < \alpha$*, the following conditions hold:*
- (ii.1) $\beta \geqslant \Delta_L$ *implies* $\mathrm{j}(L_\beta) \leqslant \aleph_0$*;*
- (ii.2) $\beta \geqslant \max\{\Delta_L, \omega\}$ *implies* $\mathrm{j}(L_\beta) = 0$*;*
- (ii.3) *if* $|L_\beta| > \aleph_0$*, then* $\gamma \geqslant \max\{\Delta_L, \beta + 1\}$ *implies* $\mathrm{j}(L_\gamma) = 0$.

Proof. First we assume that $\mathrm{Jump}(L)$ is countable, and prove that conditions (ii.1), (ii.2) and (ii.3) hold for all ordinals β and γ less than α. For (ii.1), let β be such that $\alpha > \beta \geqslant \Delta_L$. Then Theorem 3.6 implies $\mathrm{j}(L) \geqslant \mathrm{j}(L_\beta)$, whence $\mathrm{j}(L_\beta) \leqslant \aleph_0$. For (ii.2), assume that β is such that $\beta \geqslant \max\{\Delta_L, \omega\}$. Then, $\mathrm{j}(L) \geqslant \mathrm{j}(L_\beta) \cdot \prod_{\xi<\beta} |L_\xi|$ by Theorem 3.6, whence countability of $\mathrm{j}(L)$ implies that $\mathrm{j}(L_\beta) = 0$. For (ii.3), assume that $|L_\beta| > \aleph_0$. If $\gamma \geqslant \max\{\Delta_L, \beta + 1\}$, then

$$\mathrm{j}(L) \geqslant \mathrm{j}(L_\gamma) \cdot \prod_{\xi<\gamma} |L_\xi| \geqslant \mathrm{j}(L_\gamma) \cdot |L_\beta|$$

using Theorem 3.6. Since $\mathrm{j}(L) \leqslant \aleph_0$, it follows that $\mathrm{j}(L_\gamma) = 0$.

Next, we assume that (ii) holds, and prove (i). If $\Delta_L \geqslant \omega$, then condition (ii.2) implies that $\mathrm{j}(L_\beta) = 0$ for all $\beta \geqslant \Delta_L$, whence $\mathrm{j}(L) = 0$ by Theorem 3.6. On the other hand, if $\Delta_L < \omega$, then

$$\mathrm{j}(L) = \sum_{\Delta_L \leqslant n < \min\{\alpha,\omega\}} \left(\mathrm{j}(L_n) \cdot \prod_{i<n} |L_i| \right)$$

using Theorem 3.6 and (ii.2). Thus to prove (i) it suffices to show that each addend on the right-hand side is a countable cardinal. Consider the set $A := \{n \in \omega : |L_n| > \aleph_0\}$. If A is empty, then the result is immediate. Next assume that $A \neq \emptyset$ and denote $m := \min A$. If $m < \Delta_L$, then condition (ii.3) implies $\mathrm{j}(L_n) = 0$ for all n such that $\Delta_L \leqslant n < \min\{\alpha, \omega\}$, and so $\mathrm{j}(L) = 0$. On the other hand, if $m \geqslant \Delta_L$, then (ii.1) yields $\mathrm{j}(L_n) \leqslant \aleph_0$ for all n such that $\Delta_L \leqslant n \leqslant m$, whereas (ii.3) implies $\mathrm{j}(L_n) = 0$ for all n such that $m < n < \min\{\alpha, \omega\}$. Thus $\mathrm{Jump}(L)$ is countable. □

We conclude this section with an application of Theorem 3.6 to finite combinatorics.

Remark 3.9. For each $n < \omega$, consider the finite chain $N_n := \{0, 1, \ldots, n\}$ with the natural order; further, denote $P_n := N_0 \times_{\mathrm{lex}} \cdots \times_{\mathrm{lex}} N_n$. Note that $\mathrm{j}(N_n) = n$ and $\mathrm{j}(P_n) = (n+1)! - 1$. Since $\Delta_{P_n} = 0$, Theorem 3.6 yields (a bijective proof of) the well-known combinatorial equation

$$(n+1)! - 1 = \sum_{k=1}^{n} k \cdot k!$$

for each $n \geqslant 1$.

4. Representability of Lexicographic Products

In this section we study density and cellularity of lexicographic products. In particular, first we characterize separable and c.c.c. lexicographic products, and then derive a classification of representable lexicographic products.

Density and cellularity of topological spaces are not monotone with respect to subspaces. For example, if $\beta\mathbb{N}$ denotes the Stone–Čech compactification of the discrete topological space $\mathbb{N}$, then $\mathrm{c}(\beta\mathbb{N}) = \mathrm{d}(\beta\mathbb{N}) = \aleph_0 < 2^{\aleph_0} = \mathrm{c}(\beta\mathbb{N} \setminus \mathbb{N})$. On the other hand, both of these cardinal invariants are monotone when chains are considered.

PROPOSITION 4.1. *For any two chains L and M, if M embeds into L, then* $\mathrm{d}(M) \leqslant \mathrm{d}(L)$ *and* $\mathrm{c}(M) \leqslant \mathrm{c}(L)$.

The proof of 4.1 requires a lemma about double-jumps in chains, which we prove first.

DEFINITION 4.2. For any chain $(L, \prec)$, a *double-jump* in L is a triple $(x, c, y) \in L^3$ such that $x \prec c \prec y$ and the open intervals (x, c) and (c, y) are empty. Given a double jump (x, c, y), x and y are the *endpoints* of the double-jump, and c is the *center*. We denote by $\mathrm{Jump}_2(L)$ the set of all double-jumps in L; also, we set $\mathrm{j}_2(L) := |\mathrm{Jump}_2(L)|$.

Assume that M is a subchain of $(L, \prec)$ and x, y are two points in M such that $x \prec y$. The open interval (x, y) can be considered both in L and in M. We use the following notation: $(x, y)_L := \{l \in L : x \prec l \prec y\}$ and $(x, y)_M := \{m \in M : x \prec m \prec y\}$. Similarly, we define $[x, y]_L$ and $[x, y]_M$. Note that $(x, y)_M = (x, y)_L \cap M$ and $[x, y]_M = [x, y]_L \cap M$.

LEMMA 4.3. *For any two chains L and M, if M embeds in L, then* $\mathrm{j}_2(M) \leqslant \mathrm{c}(L)$.

Proof. Without loss of generality, let $M \subseteq L$. We argue by contradiction. Assume that $\mathrm{j}_2(M) = \lambda > \kappa = \mathrm{c}(L)$. (Recall that $\mathrm{c}(L) \geqslant \aleph_0$ by definition, even if the chain is finite.) We claim that it is possible to select a subset $\mathcal{F}$ of $\mathrm{Jump}_2(M)$, which has cardinality λ and has the property that if (x, c, y) and (v, d, w) are two different double jumps in $\mathcal{F}$, then the open intervals $(x, y)_L$ and $(v, w)_L$ are disjoint.

To prove the claim, define a binary relation $\sim$ on $\mathrm{Jump}_2(M)$ as follows: for any two double-jumps (x, c, y), (v, d, w) in M, let $(x, c, y) \sim (v, d, w)$ if the interval $[c, d]_M$ in M (or the interval $[d, c]_M$, if $d \prec c$) is finite. Then $\sim$ is an equivalence relation whose equivalence classes are countable or finite. Since $\mathrm{j}_2(M) = \lambda > \aleph_0$, there are λ equivalence classes. The set $\mathcal{F} \subseteq \mathrm{Jump}_2(M)$, obtained by selecting one element from each equivalence class, has the required properties.

Now let $\mathcal{G}$ be the set of open intervals in L determined by the endpoints of all the double-jumps in $\mathcal{F}$. Then $\mathcal{G}$ is a set of pairwise disjoint nonempty open intervals in L, whose cardinality is λ. This contradicts $\mathrm{c}(L) < \lambda$. □

Proof of Proposition 4.1. Without loss of generality, assume that $M \subseteq L$. The result is immediate for cellularity, because if $\{(a_i, b_i)_M : i \in I\}$ is a set of pairwise disjoint nonempty open intervals in M, then $\{(a_i, b_i)_L : i \in I\}$ is a set of pairwise disjoint nonempty open intervals in L.

For density, assume that $\mathrm{d}(L) = \kappa$ and let D be a dense subset of L which has cardinality κ. Let $P := \{(a, b) \in D^2 : M \cap (a, b) \neq \emptyset\}$. For each $(a, b) \in P$, select $m_a^b \in M$ such that $a \prec m_a^b \prec b$, and set $E := \{m_a^b : (a, b) \in P\}$. Also, let F be the subset of M composed of all the centers of the double-jumps in M. Let $G := E \cup F$. We prove that $|G| \leqslant \kappa$. Note that $|E| \leqslant \kappa$. By Lemma 4.3, we have $\mathrm{j}_2(M) \leqslant \mathrm{c}(L) \leqslant \mathrm{d}(L)$, whence $|F| \leqslant \kappa$. Thus $|G| \leqslant \kappa$. Next, it is easy to show that the set G is a dense subset of M. This proves that $\mathrm{d}(M) \leqslant \mathrm{d}(L)$. □

Now we analyze density and cellularity of lexicographic products.

LEMMA 4.4. *For any chain L, we have:*

(i) $\mathrm{d}(L \times_{\mathrm{lex}} 2) = \max\{\mathrm{d}(L), \mathrm{j}(L)\}$;
(ii) $\mathrm{c}(L \times_{\mathrm{lex}} 2) = \max\{\mathrm{c}(L), \mathrm{j}(L)\}$.

In particular, L is representable if and only if $L \times_{\mathrm{lex}} 2$ is separable.

Proof. We prove (i). First observe that $\mathrm{d}(L\times_{\mathrm{lex}}2) \geqslant \mathrm{d}(L)$, using Proposition 4.1. Furthermore $\mathrm{c}(L\times_{\mathrm{lex}}2) \geqslant \mathrm{j}(L)$, because a jump (a, b) in L yields a nonempty open interval $((a, 0), (b, 1))$ in $L \times_{\mathrm{lex}} 2$. It follows that $\mathrm{d}(L \times_{\mathrm{lex}} 2) \geqslant \max\{\mathrm{d}(L), \mathrm{j}(L)\}$. On the other hand, let D be a dense subset of L with cardinality $\mathrm{d}(L)$, and C the set composed of the endpoints of the jumps in L. Set $E := C \cup D$. It is easy to check that $E \times_{\mathrm{lex}} 2$ is dense in $L \times_{\mathrm{lex}} 2$. Since $|E \times_{\mathrm{lex}} 2| = \max\{\mathrm{d}(L), \mathrm{j}(L)\}$, this proves that $\mathrm{d}(L \times_{\mathrm{lex}} 2) \leqslant \max\{\mathrm{d}(L), \mathrm{j}(L)\}$.

For (ii), observe that the inequality $\mathrm{c}(L \times_{\mathrm{lex}} 2) \geqslant \max\{\mathrm{c}(L), \mathrm{j}(L)\}$ follows from Proposition 4.1 and the remark made in the previous paragraph. To prove the reverse inequality, let $\mathcal{F} = \{((a^i, h^i), (b^i, k^i)) : i \in I\}$ be any set of pairwise disjoint nonempty open intervals in $L \times_{\mathrm{lex}} 2$. Since the intervals in $\mathcal{F}$ are nonempty, we get $a^i \prec b^i$ for all $i \in I$. Set $I_1 := \{i \in I : (a^i, b^i) \neq \emptyset\}$ and $I_2 := \{i \in I : (a^i, b^i) = \emptyset\}$. Note that $|I_1| \leqslant \mathrm{c}(L)$ and $|I_2| \leqslant \mathrm{j}(L)$. Let $\mathcal{F}_1 = \{((a^i, h^i), (b^i, k^i)) \in \mathcal{F} : i \in I_1\}$ and $\mathcal{F}_2 = \{((a^i, h^i), (b^i, k^i)) \in \mathcal{F} : i \in I_2\}$. Then $|\mathcal{F}| = \max\{|\mathcal{F}_1|, |\mathcal{F}_2|\} \leqslant \max\{\mathrm{c}(L), \mathrm{j}(L)\}$, which proves that $\mathrm{c}(L \times_{\mathrm{lex}} 2) \leqslant \max\{\mathrm{c}(L), \mathrm{j}(L)\}$. □

LEMMA 4.5. *Let $L = \prod_{n<\omega}^{\mathrm{lex}} L_n$ be a lexicographic product. If L_n is countable for all $n < \omega$, then L is representable.*

Proof. If L_n is countable, then it embeds into $\mathbb{Q}$ by Cantor's theorem (see, e.g., Rosenstein, 1982). It follows that $L \hookrightarrow \mathbb{Q}^{\omega}_{\mathrm{lex}} \hookrightarrow \mathbb{R}$, using Lemma 2.4 (i) and Example 2.2. □

LEMMA 4.6. *Let $L = \prod_{k<n}^{\mathrm{lex}} L_k$ be a lexicographic product, where L_k is countable for all $k < n - 1$, and L_{n-1} is uncountable. If L_{n-1} is separable* (*respectively, has the c.c.c.*)*, then L is separable* (*respectively, has the c.c.c.*).

Proof. It suffices to prove the result in the case $L = L_0 \times_{\mathrm{lex}} L_1$. Let L_0 be a countable chain. First assume that L_1 is separable and let D_1 be a countable dense subset of L_1. (If L_1 has a minimum and/or a maximum, add $\min L_1$ and/or $\max L_1$ to D_1.) We claim that the countable set $D := L_0 \times_{\mathrm{lex}} D_1$ witnesses the separability of L. To show that D is dense in L, let $a = (a_0, a_1)$ and $b = (b_0, b_1)$ be any two elements of L such that the open interval (a, b) is nonempty; we will find $d = (d_0, d_1) \in D$ such that $a \prec_{\mathrm{lex}} d \prec_{\mathrm{lex}} b$. We consider two cases: (i) $a_0 \prec b_0$; (ii) $a_0 = b_0$ and $a_1 \prec b_1$.

Assume that (i) holds. If the open interval (a_0, b_0) is nonempty, we can select $d_0 \in (a_0, b_0)$ and $d_1 \in D_1$. The point $d = (d_0, d_1) \in D$ satisfies $a \prec_{\mathrm{lex}} d \prec_{\mathrm{lex}} b$. On the other hand, assume that (a_0, b_0) is an empty interval. Since by hypothesis (a, b) is not a jump in L, Lemma 3.3 implies that there exists $x_1 \in L_1$ such that either $a_1 \prec x_1$ or $x_1 \prec b_1$; without loss of generality, let $a_1 \prec x_1$. By definition of D_1, we can select $d_1 \in D_1$ such that $a_1 \prec d_1 \preceq x_1$. The point $d = (a_0, d_1)$ satisfies the claim. Next, assume that (ii) holds. Then $(a, b) \notin \mathrm{Jump}(L)$ implies that $(a_1, b_1) \notin \mathrm{Jump}(L_1)$. Let $d_1 \in D_1 \cap (a_1, b_1)$; then the point $d = (a_0, d_1)$ belongs to $D \cap (a, b)$.

Finally we prove that if L_1 satisfies the countable chain condition, then so does $L = L_0 \times_{\mathrm{lex}} L_1$. We prove the contrapositive. Assume that L fails to have the c.c.c. and let $\mathcal{F} = \{(a^i, b^i) : i \in I\}$ be an uncountable set of pairwise disjoint nonempty open intervals in L. Since L_0 is countable, we can extract from $\mathcal{F}$ an uncountable subset $\mathcal{G} = \{(a^j, b^j) : j \in J\}$ of pairwise disjoint nonempty open intervals such that $a_0^j = a_0^k$ and $b_0^j = b_0^k$ for all $j, k \in J$. It follows that $\{(a_1^j, b_1^j) : j \in J\}$ is an uncountable set of pairwise disjoint nonempty open intervals in L_1, i.e., the countable chain condition fails for L_1. □

We are ready to classify separable and c.c.c. lexicographic products.

THEOREM 4.7. *A lexicographic product $L = \prod_{\xi<\alpha}^{\mathrm{lex}} L_\xi$ is separable (respectively, has the c.c.c.) if and only if one of the following cases holds:*

(a.1) *$\alpha < \omega$; L_ξ is countable for all $\xi < \alpha$;*
(a.2) *$\alpha < \omega$; L_ξ is countable for all $\xi < \alpha - 1$, and $L_{\alpha-1}$ is uncountable but is separable (respectively, has the c.c.c.);*
(a.3) *$\alpha < \omega$; L_ξ is countable for all $\xi < \alpha - 2$, $L_{\alpha-2}$ is uncountable, but it is separable (respectively, has the c.c.c.) and has at most countably many jumps, and $L_{\alpha-1}$ is isomorphic to 2;*
(b) *$\alpha = \omega$; L_ξ is countable for all $\xi < \alpha$;*
(c) *$\alpha = \omega + 1$; L_ξ is countable for all $\xi < \omega$, and L_ω is isomorphic to 2.*

Proof. ($\Leftarrow$) The chain L is obviously separable in case (a.1), since it is even countable. In case (a.2), the result follows at once from Lemma 4.6. For case (b), Lemma 4.5 yields separability of L. Similarly, a joint application of Lemma 4.4 and Lemma 4.5 ensures that L is separable also in case (c). Finally, we show that case (a.3) is an instance of case (a.2). Indeed, let $L = \prod_{k<n}^{\mathrm{lex}} L_k$ be as in (a.3), and consider $L' := L_{n-2} \times_{\mathrm{lex}} L_{n-1}$. Since L_{n-2} is uncountable, separable (respectively, has the c.c.c.) and with at most countably many jumps, whereas L_{n-1} is isomorphic to 2, then Lemma 4.4 yields that L' is a chain which is uncountable and separable (respectively, has the c.c.c.). Thus, we are in case (a.2), with the last factor equal to L'.

($\Rightarrow$) Assume that $L = \prod_{\xi<\alpha}^{\mathrm{lex}} L_\xi$ is separable (respectively, has the c.c.c.). First, note that we cannot have any factor L_ξ in the product which fails to be separable (respectively, fails to have the c.c.c.). Otherwise, Lemma 2.4 (ii) and Proposition 4.1 would imply that L is not separable (respectively, fails to have the c.c.c.), too.

Next observe that the number α of factors cannot exceed $\omega + 1$. Indeed, if $\alpha \geqslant \omega + 2$, then Lemma 2.4 yields that $2_{\mathrm{lex}}^{\omega+2}$ embeds into L. Since $2_{\mathrm{lex}}^{\omega+2}$ fails to satisfy the c.c.c. (by Example 2.3), so does L by Proposition 4.1, which is a contradiction. Thus, only three cases are possible: either $\alpha < \omega$, or $\alpha = \omega$, or $\alpha = \omega + 1$.

If $\alpha = n < \omega$, let L_k be a factor which is uncountable and separable (respectively, has the c.c.c.). (If there is no such factor, we are in case (a.1).) We claim that $k \geqslant n - 2$. If not, consider the chain $M := L_k \times_{\text{lex}} 2 \times_{\text{lex}} 2$. Since $k < n - 2$, Lemma 2.4 yields $M \hookrightarrow L$, and so M has the c.c.c. by Proposition 4.1; this contradicts the result in Example 2.3. Thus, either $k = n-1$, or $k = n-2$. In the first case, (a.2) holds. In the second case, we have $|L_{n-1}| = 2$, since otherwise $L_{n-2} \times_{\text{lex}} 3$ is a chain which fails to have the c.c.c. (by Example 2.3), but embeds into L (by Lemma 2.4), and so Proposition 4.1 gives again a contradiction. Therefore (a.3) holds.

Now assume that $\alpha = \omega$. In this case, no factor can be uncountable, since otherwise an argument similar to the previous one would show that L fails to have the c.c.c. This proves that the only possibility is (b).

Finally, if $\alpha = \omega + 1$, then $|L_\omega| = 2$, because otherwise $2^\omega_{\text{lex}} \times_{\text{lex}} 3$ is a chain which embeds in L, yet it fails to have the c.c.c. Since no factor can be uncountable, (c) holds. □

COROLLARY 4.8. *A lexicographic product* $L = \prod^{\text{lex}}_{\xi<\alpha} L_\xi$ *is representable if and only if one of the following cases holds:*

(R.1) $\alpha \leqslant \omega$; L_ξ *is countable for all* $\xi < \alpha$;
(R.2) $\alpha < \omega$; L_ξ *is countable for all* $\xi < \alpha - 1$, *and* $L_{\alpha-1}$ *is uncountable but representable.*

Proof. We use Corollary 3.8 to rule out some of the cases in Theorem 4.7. It is immediate to check that conditions (ii.1), (ii.2) and (ii.3) of Corollary 3.8 are satisfied in cases (a.1) and (b) of Theorem 4.7. Therefore a lexicographic product L as in (R.1) is representable.

For case (a.2) of Theorem 4.7, conditions (ii.2) and (ii.3) hold vacuously, but (ii.1) is satisfied if and only if $\mathrm{j}(L_{\alpha-1}) \leqslant \aleph_0$. This proves that lexicographic products of type (R.2) are representable.

Finally we show that if L is as in cases (a.3) and (c) of Theorem 4.7, then the last factor (isomorphic to 2) causes L to have uncountably many jumps; thus L is not representable. Indeed, in case (a.3) condition (ii.3) fails (because $\mathrm{j}(L_{\alpha-1}) = 1$), whereas in case (c) condition (ii.2) fails (because $\mathrm{j}(L_\omega) = 1$). □

5. Final Remarks

Lexicographic products appear quite often in the economic literature. The most well-known example of non-representable lexicographic product is the lexicographic plane $\mathbb{R}^2_{\text{lex}}$: in a famous paper, Debreau (1954) used this chain to disprove the inveterate belief of economists that every preference relation is representable by a utility function. More recently, several authors have been focusing their attention on non-representable preference relations. Research on this topic has been done in at least two directions: (a) exploring preference representations via embeddings into particular lexicographic products; and (b) determining the structure of chains which fail to be representable.

Concerning (a), some authors have considered representations of preference relations which use embeddings into a chain Z different from $\mathbb{R}$. For example, Wakker (1988) examines the case $Z = \mathbb{R} \times_{\mathrm{lex}} 2$, whereas Knoblauch (2000) deals with the case $Z = \mathbb{R}^n_{\mathrm{lex}}$, $n \in \omega$. More generally, we define the *representability number of L in Z* as the least ordinal α such that L embeds into Z^α_{lex}; we denote this number by $\mathrm{repr}_Z(L)$. Kuhlmann (1995) has shown that $\mathrm{repr}_{\mathbb{R}}(\mathbb{R}^\alpha_{\mathrm{lex}}) = \alpha$ for any ordinal α (cf. Corollary 2.4, p. 2660). Further, the following results are proved in Giarlotta (200?): (i) if κ is a regular cardinal which does not embed into M, then $\mathrm{repr}_M(\kappa) = \kappa$; (ii) if L is either an Aronszajn line or a Souslin line, then $\mathrm{repr}_{\mathbb{R}}(L) = \omega_1$.

In the direction of research (b) mentioned above, Beardon *et al.* (2002) have recently proved the following result: a chain is non-representable if and only if it is long (i.e., it embeds ω_1 or its reverse ordering $\omega_1{}^*$), or it embeds a non-representable subchain of the lexicographic plane, or it embeds an Aronszajn line. A more direct classification of non-representable chains (and, more generally, of all chains) can be obtained by using the notion of representability number in $\mathbb{R}$. Since long chains do not embed into $\mathbb{R}^\alpha_{\mathrm{lex}}$ for any countable ordinal α (see Fleischer, 1961b), the class of chains can be partitioned as follows: (i) long chains; (ii) non-long chains which have an uncountable representability number in $\mathbb{R}$; (iii) chains with a countable representability number in $\mathbb{R}$. We are currently working on a description of class (ii), which is surprisingly rich in variety. In fact one can construct a nested hierarchy of chains which embed neither ω_1 nor $\omega_1{}^*$ nor an Aronszajn line, and yet have representability number in $\mathbb{R}$ equal to ω_1.

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