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Triangles with a given special figure

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Abstract

A figure F is said to be special for a triangle T if F can be determined from T by a ruler and compass construction that does not depend on the order of vertices of T. In this paper, we deal with the problem of the existence of triangles with a given special figure that is a point, or a line, or a circle, or a triangle.

Keywords: : *triangles, ruler and compass constructions, similarities*

Trokuti s danim posebnim likom

Sažetak

Kažemo da je lik F poseban za trokut T ukoliko F može biti dobiven iz T korištenjem ravnala i šestara neovisno o redoslijedu vrhova od T. U ovom radu razmatra se postojanje trokuta s danim posebnim likom koji je točka, pravac, kružnica ili trokut.

Ključne riječi: *trokuti, konstrukcija ravnalom i šestarom, sličnosti*

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1 The problem

A triangle T is determined by its vertices, no matter what their order is, i.e., T is determined by a non-ordered triple of non-collinear points, $\{A_1, A_2, A_3\}$. Consider a ruler and compass construction χ that associates a figure F to any triangle $T = \{A_1, A_2, A_3\}$; precisely, χ is a construction that, starting from the vertices of T, determines F no matter what the order of the vertices is, i.e.,:

$$\chi(A_i, A_j, A_k) = F$$

for any permutation (i, j, k) of indices $1, 2, 3$.
We call such a construction χ a *special construction* and we say that $F = \chi(T)$ is a *special figure* for T. Any special construction is invariant with respect to similarities, that is, given any similarity σ of the plane, for any triangle $T = A_1, A_2, A_3$, it is:

$$\chi(\sigma(T)) = \sigma(\chi(T)),$$

with $\sigma(T)$ being the correspondent triangle of T with respect to σ.
We will consider the following types of special figures: a point, a line, a circle, and a triangle. For example, the centroid and the orthocenter of a triangle T are special points for T; the Lemoine axis [1] and the Brocard axis [1] of a triangle T are special lines for T; the circumcircle and the nine-point circle [2] of a triangle T are special circles for T; the medial triangle [2] and the orthic triangle [2] of a triangle T are special triangles for T.
Let χ be any special construction that associates a point (or a line, or a circle, or a triangle) to any triangle. We consider the following problem: given a point P (or a line l, or a circle c, or a triangle T), do there exist triangles T' such that $\chi(T') = P$ (or $\chi(T') = l$, or $\chi(T') = c$, or $c(T') = T$, respectively? And if they do, how many are there? Here we give a general procedure allowing us to solve the problem for special points, special lines, and special circles, and we will also show that for each of those cases there exist continuum many triangles T', not similar to each other, that solve the problem. In the case of special triangles, the procedure cannot be used and the property does not hold.

2 A representative subset of the set of triangles

Let T be the set of all triangles in the plane. In order to solve the problem posed in the previous section, we define a set S of representatives for each class of similar triangles, i.e., a subset S of T such that any triangle in T is similar to one and only one triangle in S. Fix a line a and two distinct points, A_1 and A_2, on a; let d be one of the half-planes bounded by a. Consider any triangle $A'_1A'_2A'_3$ and suppose that $A'_1A'_2 \geq A'_1A'_3 \geq A'_2A'_3$. It is known that there exist only two similarities, one direct and one opposite, that transform A'_1 and A'_2 to A_1 and A_2, respectively [3, p. 73]. Let σ be the one that transforms A'_3 to the point A_3 in the half-plane d. The similarity σ determines the triangle $A_1A_2A_3$, completely contained in the half-plane d, such that $A_1A_2 \geq A_1A_3 \geq A_2A_3$. Denote by S the set of all triangles $A_1A_2A_3$ that are obtained by varying the triangles $A'_1A'_2A'_3$ in T. Let γ be the circle with center A_1 and radius A_1A_2. Let b be the perpendicular bisector of the segment A_1A_2. Let b intersect γ at the point B of d and the segment A_1A_2 at M; thus M is the midpoint of A_1A_2. We define the region R as the subset of d bounded by the arc A_2B of γ and the segments BM and MA_2. Since $A_1A_2 \geq A_1A_3 \geq A_2A_3$, we see that the point A_3 lies in the region R. The triangle $A_1A_2A_3$ (see Figure 1) is scalene if A_3 is in the interior of R; it is isosceles if A_3 lies on the segment BM or on the arc A_2B. The triangle A_1A_2B is equilateral. Observe that since no two distinct triangles of S are similar, then every triangle in T is similar to exactly one triangle in S. Moreover, S contains continuum many triangles.

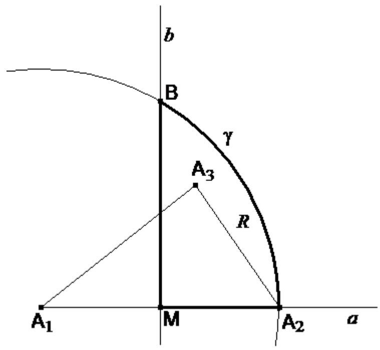

Figure 1: $A_1A_2A_3$ construction

3 Solution of the problem

Denote by $\chi_i, i = 1, 2, 3, 4$, a special construction that associates to any triangle a point if $i = 1$, a line if $i = 2$, a circle if $i = 3$, a triangle if $i = 4$. Denote by F_i a point if $i = 1$, a line if $i = 2$, a circle if $i = 3$, and a triangle if $i = 4$.

Theorem 1. *Given a special construction $\chi_i, i = 1,2,3$, for every figure F_i there exist continuum many triangles T', not similar to each other, such that $\chi_i(T') = F_i$.*

Proof. Fix a special construction $\chi_i, i = 1,2,3$, and a figure F_i. Let T be any triangle in S and let $\Omega_i(T)$ be the set of similarities that transform $\chi_i(T)$ to F_i. It is easy to see that for any i $\Omega_i(T)$ is non-empty. Let $\omega \in \Omega_(T)$ and let $T' = w(T)$. Since χ_i is invariant under similarities, we have

$$\chi_i(T') = \chi_i(\omega(T)) = \omega(\chi_i(T)) = F_i.$$

Then, since $\Omega_i(T)$ is non-empty, there exists at least one triangle T' such that $\chi_i(T') = F_i$. Moreover, if $w' \in \Omega_i(T)$ and $w'(T) = T''$, then the triangles T' and T'' are similar, because T'' corresponds to T' by means of the similarity $\omega^{-1} \cdot \omega'$.

In order to see how many non-similar triangles solve the problem, we can observe that each triangle T in S determines a family $S(T)$ of similar triangles. Precisely, $S(T) = \{T' = \omega(T) : \omega \in \Omega_i(T)\}$. These families are disjoint. In fact, suppose that T_1, $T_2 \in S$, with $T_1 \neq T_2$. If $\omega_1 \in \Omega_i(T_1)$, $\omega_2 \in \Omega_i(T_2)$, $\omega_1(T_1) = T_1'$ and $\omega_2(T_2) = T_2'$, then T_1' and T_2' are triangles of the families $S(T_1)$ and $S(T_2)$, respectively. T_1' and T_2' are non-similar because if there exists a similarity ω^* such that $\omega^*(T_1') = T_2'$, then T_2 corresponds to T_1 through the similarity $\omega_1 \cdot \omega^* \cdot \omega_2^{-1}$, which cannot happen because distinct triangles in S are not similar to each other. Since S contains continuum many triangles, there are continuum many non-similar triangles that solve the problem, one per each family $S(T)$. □

Fix now a special construction χ_4 and any figure F_4. Consider the problem: do there exist triangles T' such that $\chi_4(T') = F_4$? The general method that we have just illustrated cannot be used to solve such problem, because given two triples of non-aligned points, (H_1, H_2, H_3) and (H_1', H_2', H_3'), generally there exists no similarity ω such that $\omega'(Hi) = H'i$ for $i = 1,2,3$ (unless the triangles $H_1H_2H_3$ and $H_1'H_2'H_3'$ are similar). Nevertheless, it is known that, given a triangle T, there exists one and only one triangle whose medial triangle is T, the anticomplementary triangle of T [1]. Then we can state that for $i = 4$ the theorem does not hold. Moreover, it is still an open problem to see for which special constructions χ_4 it happens that, for any figure F_4, there exist triangles T' such that $\chi_4(T') = F_4$.

References

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