The minimal resolutions of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ with ACM support

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Abstract

Let Z be a finite set of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ and suppose further that X, the support of Z, is arithmetically Cohen–Macaulay (ACM). We present an algorithm, which depends only upon a combinatorial description of X, for the bigraded Betti numbers of I_Z, the defining ideal of Z. We then relate the total Betti numbers of I_Z to the shifts in the graded resolution, thus answering a special case of a question of Römer.
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0. Introduction

Given a set of fat points Z in $\mathbb{P}^n$, it has been the goal of many authors to describe the homological invariants encoded in the graded minimal free resolution of I_Z, the defining ideal of Z. A non-exhaustive list of references includes [1, 5–8,16,21]. Many interesting questions about these numerical characters remain open; Harbourne's survey [15] on these problems in $\mathbb{P}^2$ provides a good entry point to this material.

Recently, many authors have extended this circle of problems to include fat points in multiprojective spaces. The Hilbert function [11–13] and the Castelnuovo–Mumford regularity [14,20] are two such topics that have been investigated. Besides their intrinsic interest, motivation to study such points arises from a paper of Catalisano, Geramita, and Gimigliano [3] which exhibited a connection between specific values of the Hilbert function of a set of fat points in a multiprojective space and the dimensions of certain secant varieties of the Segre varieties. We contribute to this ongoing research program by providing an algorithm to compute the bigraded minimal free resolution of the ideal of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ whose support is arithmetically Cohen–Macaulay.

The $\mathbb{N}^2$-graded polynomial ring $S = k[x_0, x_1, y_0, y_1]$ with $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$ is the coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$. If $P = R \times Q \in \mathbb{P}^1 \times \mathbb{P}^1$ is a *point* in $\mathbb{P}^1 \times \mathbb{P}^1$, then the defining ideal of P is $I_P = (L_R, L_Q)$

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with deg $L_R = (1, 0)$ and deg $L_Q = (0, 1)$. If $X = \{P_1, \ldots, P_s\}$ is a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$, and $m_1, \ldots, m_s$ are positive integers, then the ideal $I_Z = I_{P_1}^{m_1} \cap \cdots \cap I_{P_s}^{m_s}$ is an $\mathbb{N}^2$-homogeneous ideal that defines a scheme of *fat points* $Z = \{(P_1; m_1), \ldots, (P_s; m_s)\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The set of points X is called the *support* of Z, while the integer m_i is called the *multiplicity* of P_i. When all the m_is equal two, we call Z a set of *double of points*. A set of (reduced or non-reduced) points Z is said to be *arithmetically Cohen–Macaulay* (ACM) if its associated coordinate ring S/I_Z is Cohen–Macaulay. While it is always true that Z is ACM if $Z \subseteq \mathbb{P}^n$, if $Z \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ with $r \geq 2$, then Z may or may not be ACM (e.g., see [22]).

We shall focus on sets of double points Z in $\mathbb{P}^1 \times \mathbb{P}^1$ whose support X is ACM. Such schemes were studied by the first author [11] who used combinatorial information about X to determine both the minimal generators of I_Z and its associated Hilbert function. As shown in [11,13], these schemes are rarely ACM. However, because the support X is ACM, we can associate to Z a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of the integer $s = |X|$ which is related to the relative positions of the points of X, i.e., the number of points which share the same first coordinate, and so on. We extend the results of [11] by constructing an algorithm to obtain the bigraded minimal resolution of I_Z from λ.

Our algorithm (see Algorithm 5.1) is based upon the following steps:

- Using λ we construct a scheme Y of reduced and double points, which we call the *completion of Z*, such that $Z \subseteq Y$ and Y is ACM (see Theorem 2.2). Applying a theorem of [13], we compute the bigraded minimal free resolution of I_Y from λ.
- Using [11] we use λ to construct bihomogeneous forms $\{F_1, \ldots, F_p\}$ such that $I_Z = I_Y + (F_1, \ldots, F_p)$ and where deg F_i is a function of λ (see Theorem 3.6).
- For $j = 0, \ldots, p$, we set $I_0 = I_Y$ and $I_j = (I_{j-1}, F_j)$. For each $j = 1, \ldots, p$, we show (see Lemma 4.3) that $(I_{j-1} : F_j)$ is the defining ideal of a complete intersection of points whose type (and hence minimal resolution) can be computed from λ.
- For each $j = 1, \ldots, p$, we have a short exact sequence

$$0 \rightarrow S/(I_{j-1} : F_j)(-\deg F_j) \xrightarrow{\times F_j} S/I_{j-1} \longrightarrow S/I_j \longrightarrow 0.$$

We prove (cf. Theorem 4.4) that the mapping cone construction gives the bigraded minimal free resolution of S/I_j for each j.
- Because the minimal resolution of $I_Y = I_0$ depends only upon λ, we can reiteratively use the mapping cone construction and the fact that $(I_{j-1} : F_j)$ is a complete intersection to compute the minimal resolution $I_Z = I_p$.

Römer [19] recently asked if the total graded Betti numbers of an ideal I are bounded by the shifts that appear within the minimal graded free resolution of I. As an application of Algorithm 5.1, we show (see Theorem 6.1) that the ideals I_Z satisfy this bound, thus extending work of both Römer [19] and Miró-Roig [17].

Some final observations are in order. First, our approach to computing the bigraded minimal free resolution is similar to the approach taken by Catalisano [4]. Catalisano showed that the Hilbert function and resolution of fat points on a nonsingular conic in $\mathbb{P}^2$ can be computed via an algorithm that depends only upon the multiplicities of the points, and without reference to the coordinates of the points. Second, by viewing I_Z as a graded ideal of $S = k[x_0, x_1, y_0, y_1]$, then the ideal I_Z defines a set of "fat lines" in $\mathbb{P}^3$, and our algorithm describes their graded minimal free resolutions. We are not of aware of any other such result about the resolutions of "fat lines". Finally, the ideals I_Z give a new family of examples of codimension two non-perfect ideals whose resolution can be described (see [18] for another such class arising from lattice ideals).

1. Preliminaries

In this paper k is an algebraically closed field of characteristic zero and $\mathbb{N} := \{0, 1, 2, \ldots\}$.

1.1. Points and fat points in $\mathbb{P}^1 \times \mathbb{P}^1$

We continue to use the notation and definitions from the introduction. Suppose that $P = [a_0 : a_1] \times [b_0 : b_1]$ is a point of $\mathbb{P}^1 \times \mathbb{P}^1$. The bihomogeneous ideal associated to P is the ideal $I_P = (a_1 x_0 - a_0 x_1, b_1 y_0 - b_0 y_1)$. The ideal I_P is a prime ideal of height two that is generated by an element of degree $(1, 0)$ and an element of degree $(0, 1)$. If $P = R \times Q$, then we shall usually write $I_P = (L_R, L_Q)$ where L_R is the form of degree $(1, 0)$ and L_Q is the form of

degree $(0, 1)$. Because $\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathcal{Q}$, the quadric surface in $\mathbb{P}^3$, it is useful to note that L_R defines a line in one ruling of $\mathcal{Q}$, L_Q defines a line in the other ruling, and P is the point of intersection of these two lines.

Let X be any set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ denote the projection morphism defined by $P = R \times Q \mapsto R$. Similarly, let π_2 denote the other projection morphism. The set $\pi_1(X) = \{R_1, \dots, R_r\}$ is the set of $r \leq s$ distinct first coordinates that appear in X, while $\pi_2(X) = \{Q_1, \dots, Q_t\}$ is the set of $t \leq s$ distinct second coordinates. The set X is therefore a subset of $\{R_i \times Q_j \mid R_i \in \pi_1(X) \text{ and } Q_j \in \pi_2(X)\}$. When $P \in X$, we write $P = P_{i,j}$ to mean that $P = R_i \times Q_j$.

For $i = 1, \dots, r$, let L_{R_i} denote the degree $(1, 0)$ form that vanishes at all the points of X which have first coordinate R_i. Similarly, for $j = 1, \dots, t$, let L_{Q_j} denote the degree $(0, 1)$ form that vanishes at all the points whose second coordinate is Q_j. The defining ideal of I_X is then the ideal

$$I_X = \bigcap_{P_{i,j} \in X} I_{P_{i,j}} = \bigcap_{P_{i,j} \in X} (L_{R_i}, L_{Q_j}).$$

As noted above, X is a subset of $\{R_i \times Q_j \mid R_i \in \pi_1(X) \text{ and } Q_j \in \pi_2(X)\}$. When we have equality, then X is called a *complete intersection of type* (r, t), denoted $X = CI(r, t)$, where $r = |\pi_1(X)|$ and $t = |\pi_2(X)|$. The name follows from the fact that

$$I_X = \bigcap_{P_{i,j} \in X} I_{P_{i,j}} = (L_{R_1} \cdots L_{R_r}, L_{Q_1} \cdots L_{Q_t}) = (F, G)$$

where $\deg F = (r, 0)$ and $\deg G = (0, t)$, and furthermore, F and G form a regular sequence on S. When $X = CI(r, t)$, then the bigraded resolution of I_X is

$$0 \longrightarrow S(-r, -t) \longrightarrow S(-r, 0) \oplus S(0, -t) \longrightarrow I_X \longrightarrow 0 \qquad (1.1)$$

which follows from the Koszul resolution, but also taking into account that I_X is bigraded.

If X is a finite set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$, and $m_{i_1, j_1}, \dots, m_{i_s, j_s}$ are s positive integers, then Z denotes the subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the saturated bihomogeneous ideal

$$I_Z = \bigcap_{P_{i,j} \in X} I_{P_{i,j}}^{m_{i,j}} = \bigcap_{P_{i,j} \in X} (L_{R_i}, L_{Q_j})^{m_{i,j}}.$$

We call Z a *fat point scheme* (or sometimes, a *set of fat points*) of $\mathbb{P}^1 \times \mathbb{P}^1$. When all the $m_{i,j}$ equal one, then $Z = X$, and X is called a *reduced set of points*.

From time to time, we will wish to represent our fat point schemes pictorially. Because $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the quadric surface $\mathcal{Q} \subseteq \mathbb{P}^3$, we can draw fat point schemes on $\mathcal{Q}$ as subschemes whose support is contained in the intersection of lines of the two rulings of $\mathcal{Q}$. For example, if $P_{i,j} = R_i \times Q_j \in \mathbb{P}^1 \times \mathbb{P}^1$, then the fat point scheme $Z = \{(P_{1,1}; 4), (P_{1,2}; 2), (P_{2,2}; 3)\}$ can be visualized as
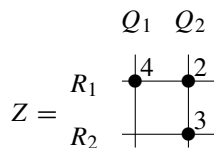
where a dot represents a point in the support and the number its multiplicity.

1.2. ACM points and fat points

As noted in the introduction, a set of (fat) points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ with $r \geq 2$ may or may not be arithmetically Cohen–Macaulay (ACM). Currently, only ACM sets of (fat) points in $\mathbb{P}^1 \times \mathbb{P}^1$ have been classified. ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ were first classified via their Hilbert function in [9]. An alternative classification was provided by the second author [22], which we recall here.

We associate to a set of points X in $\mathbb{P}^1 \times \mathbb{P}^1$ two tuples α_X and β_X as follows. Let $\pi_1(X) = \{R_1, \dots, R_r\}$ be the r distinct first coordinates in X. Then, for each $R_i \in \pi_1(X)$, let $\alpha_i := |\pi_1^{-1}(R_i)|$, i.e., the number of points in X which

have R_i as its first coordinate. After relabeling the α_i so that $\alpha_i \geq \alpha_{i+1}$ for $i = 1, \ldots, r-1$, we set $\alpha_X = (\alpha_1, \ldots, \alpha_r)$. Analogously, for each $Q_i \in \pi_2(X) = \{Q_1, \ldots, Q_t\}$, we let $\beta_i := |\pi_2^{-1}(Q_i)|$. After relabeling so that $\beta_i \geq \beta_{i+1}$ for $i = 1, \ldots, t-1$, we set $\beta_X = (\beta_1, \ldots, \beta_t)$.

Recall that a tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ is a *partition* of an integer s if $\sum \lambda_j = s$. So, by construction, α_X and β_X are partitions of $s = |X|$. The *conjugate* of a partition λ, denoted λ^*, is the tuple $\lambda^* = (\lambda_1^*, \ldots, \lambda_{\lambda_1}^*)$ where $\lambda_i^* = \#\{\lambda_j \in \lambda \mid \lambda_j \geq i\}$. With this notation, we can state Theorem 4.8 of [22]:

Theorem 1.1. *A set of reduced points X in $\mathbb{P}^1 \times \mathbb{P}^1$ is ACM if and only if $\alpha_X^* = \beta_X$.*

Example 1.2. Let $P_1 = [1 : 0]$ and $P_2 = [0 : 1]$ in $\mathbb{P}^1$, and consider $X = \{P_1 \times P_1, P_2 \times P_2\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$. In this example $\alpha_X = (1, 1)$ and $\beta_X = (1, 1)$, but $\alpha_X^* = (2) \neq \beta_X$, so X is not ACM. The set X is the simplest example of a non-ACM set of points.

Example 1.3. Consider the following set of points in $\mathbb{P}^1 \times \mathbb{P}^1$:
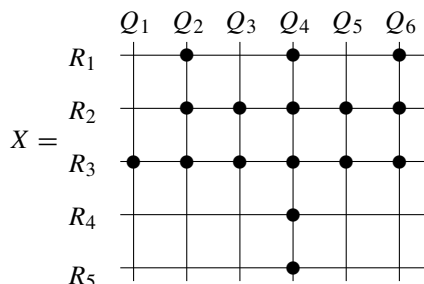
For this set of points, $\pi_1(X) = \{R_1, R_2, R_3, R_4, R_5\}$. Then

$$|\pi_1^{-1}(R_1)| = 3, \qquad |\pi_1^{-1}(R_2)| = 5, \qquad |\pi_1^{-1}(R_3)| = 6, \qquad |\pi_1^{-1}(R_4)| = 1, \quad \text{and} \quad |\pi_1^{-1}(R_5)| = 1.$$

So, $\alpha_X = (6, 5, 3, 1, 1)$. Now counting the number of points whose second coordinate is Q_i for $i = 1, \ldots, 6$, we have $\beta_X = (5, 3, 3, 2, 2, 1)$. So X is ACM because $\alpha_X^* = \beta_X$.

Remark 1.4. Suppose that X is ACM with $\alpha_X = (\alpha_1, \ldots, \alpha_r)$ and $\beta_X = (\beta_1, \ldots, \beta_t)$. Because $\alpha_X^* = \beta_X$, we can assume after relabeling that $\alpha_i = |\pi_1^{-1}(R_i)|$ for each $i = 1, \ldots, r$, and $\beta_j = |\pi_2^{-1}(Q_j)|$ for each $j = 1, \ldots, t$. So, when X is ACM, the points of X can be represented by a Ferrers diagram for the partition α_X.

The two authors [13] found a similar combinatorial description for classifying ACM fat points in $\mathbb{P}^1 \times \mathbb{P}^1$. We recall this procedure. Let X denote the support of a fat point scheme Z, and suppose that $|X| = s$. For each $R_i \in \pi_1(X)$, set

$$Z_{1,R_i} := \{(P_{i,j_1}; m_{i,j_1}), (P_{i,j_2}; m_{i,j_2}), \ldots, (P_{i,j_{\alpha_i}}; m_{i,j_{\alpha_i}})\}$$

where $P_{i,j_k} = R_i \times Q_{j_k}$ for some $Q_{j_k} \in \pi_2(X)$. Thus $\pi_1(\mathrm{Supp}(Z_{1,R_i})) = \{R_i\}$, and $I_Z = \bigcap_{i=1}^r I_{Z_{1,R_i}}$. For each $R_i \in \pi_1(X)$ define $l_i := \max\{m_{i,j_1}, \ldots, m_{i,j_{\alpha_i}}\}$. Then, for $k = 0, \ldots l_i - 1$, we set

$$a_{i,k} := \sum_{j=1}^{\alpha_i} (m_{i,j} - k)_+ \quad \text{where } (n)_+ := \max\{n, 0\}.$$

We then put all the numbers $a_{i,k}$ into a tuple; that is, let

$$\alpha_Z := (a_{1,0}, \ldots, a_{1,l_1-1}, a_{2,0}, \ldots, a_{2,l_2-1}, \ldots, a_{r,0}, \ldots, a_{r,l_r-1}).$$

Similarly, for each $Q_j \in \pi_2(X)$, define

$$Z_{2,Q_j} := \{(P_{i_1,j}; m_{i_1,j}), (P_{i_2,j}; m_{i_2,j}), \ldots, (P_{i_{\beta_j},j}; m_{i_{\beta_j},j})\}$$

where $P_{i_k,j} = R_{i_k} \times Q_j$ are those points of $\mathrm{Supp}(Z)$ whose projection onto its second coordinate is Q_j. Thus $\pi_2(\mathrm{Supp}(Z_{2,Q_j})) = \{Q_j\}$. For $Q_j \in \pi_2(X)$ define $l_j' = \max\{m_{i_1,j}, \ldots, m_{i_{\beta_j},j}\}$. Then, for each integer $0 \leq k \leq$

$l'_j - 1$, we define

$$b_{j,k} := \sum_{i=1}^{\beta_j} (m_{i,j} - k)_+ \quad \text{where } (n)_+ := \max\{n, 0\}.$$

As in the case of α_Z, we place all the values $b_{j,k}$ into a tuple:

$$\beta_Z := (b_{1,0}, \ldots, b_{1,l'_1-1}, b_{2,0}, \ldots, b_{2,l'_2-1}, \ldots, b_{t,0}, \ldots, b_{t,l'_t-1}).$$

If we reorder the entries of α_Z and β_Z in non-increasing ordering, i.e., $\alpha_i \geq \alpha_{i+1}$ and $\beta_i \geq \beta_{i+1}$ for all i, then α_Z and β_Z are partitions of deg Z. The following result of the authors [13, Theorem 4.8] then extends Theorem 1.1. Note that when $Z = X$, then $\alpha_Z = \alpha_X$ and $\beta_Z = \beta_X$, so Theorem 1.1 is a special case of the following theorem.

Theorem 1.5. *A set of fat points $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is ACM if and only if $\alpha_Z^* = \beta_Z$.*

When Z is ACM, we can in fact describe the entire resolution of I_Z using only the tuple $\alpha_Z = (\alpha_1, \ldots, \alpha_m)$. Define the following two sets from α_Z:

$$\mathcal{SZ}_0 := \{(m, 0), (0, \alpha_1)\} \cup \{(i - 1, \alpha_i) \mid \alpha_i - \alpha_{i-1} < 0\}$$
$$\mathcal{SZ}_1 := \{(m, \alpha_m)\} \cup \{(i - 1, \alpha_{i-1}) \mid \alpha_i - \alpha_{i-1} < 0\}.$$

We take $\alpha_{-1} = 0$. With this notation, we have

Theorem 1.6. *Suppose that Z is an ACM set of fat points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_Z = (\alpha_1, \ldots, \alpha_m)$. Then the bigraded minimal free resolution of I_Z is given by*

$$0 \longrightarrow \bigoplus_{(i,j) \in \mathcal{SZ}_1} S(-i, -j) \longrightarrow \bigoplus_{(i,j) \in \mathcal{SZ}_0} S(-i, -j) \longrightarrow I_Z \longrightarrow 0$$

where $\mathcal{SZ}_0$ and $\mathcal{SZ}_1$ are constructed from α_Z as above.

Our goal is to describe the resolution of the following special class of fat points.

Convention 1.7. For the remainder of this paper, Z will denote a set of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ with the property that $\text{Supp}(Z) = X$ is an ACM scheme and the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ will denote the partition α_X.

Example 1.8. Let X be as in Example 1.3. The scheme Z defined by $I_Z = \bigcap_{P_{i,j} \in X} I_{P_{i,j}}^2$ is an example of a set of points that satisfies Convention 1.7. For this set of points, $\lambda = \alpha_X = (6, 5, 3, 1, 1)$. In light of Remark 1.4 we can visualize this set as
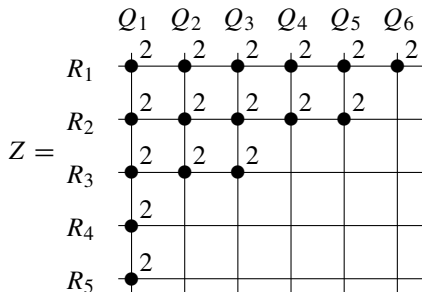
For this set of fat points, we have

$$\alpha_Z = (12, 10, 6, 6, 5, 3, 2, 2, 1, 1) \quad \text{and} \quad \beta_Z = (10, 6, 6, 5, 4, 4, 3, 3, 2, 2, 2, 1).$$

It then follows that Z is not ACM because $\alpha_Z^* = (10, 8, 6, 5, 5, 4, 2, 2, 2, 2, 1, 1) \neq \beta_Z$.

2. The completion of *Z*

Let Z be a set of double points that satisfies Convention 1.7, and let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be the partition that describes the ACM support X. In this section we build a scheme Y, which we call the completion of Z, that contains Z. The scheme Y will be an ACM set of fat points that will form the base step in our recursive formula to compute the bigraded resolution of I_Z. The notion of a completion was originally introduced by the first author in [11] to describe the minimal generators and Hilbert function of I_Z.

Geometrically, the completion of Z is formed by adding a number of simple (reduced) points to Z so that the support of the new scheme becomes a complete intersection. If X is the support of Z, and if $\pi_1(X) = \{R_1, \ldots, R_r\}$ and $\pi_2(X) = \{Q_1, \ldots, Q_t\}$, then

$$X \subseteq W = \{R_i \times Q_j \mid R_i \in \pi_1(X) \text{ and } Q_j \in \pi_2(X)\}.$$

Note that W is a complete intersection of reduced points.

Definition 2.1. Suppose that Z is set of double points that satisfies Convention 1.7. With the notation as above, the *completion* of Z is the scheme

$$Y := Z \cup (W \setminus X).$$

Note that the support of the completion is the complete intersection $CI(r, t)$. (Because of Convention 1.7, we have $t = \lambda_1$.) As first proved in [11], the completion of Z is ACM. In fact, the bigraded minimal free resolution of I_Y is a function of λ.

Theorem 2.2. *Let Y be the completion of the scheme Z. If $\lambda = (\lambda_1, \ldots, \lambda_r)$ is the tuple describing $X = \mathrm{Supp}(Z)$, then*

(i) $\alpha_Y = (\lambda_1 + \lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \lambda_r, \lambda_1, \lambda_2, \ldots, \lambda_r)$.

(ii) *Y is ACM.*

(iii) *the bigraded minimal free resolution of I_Y has the form*

$$0 \to \bigoplus_{(i,j) \in \mathcal{SY}_1} S(-i, -j) \to \bigoplus_{(i,j) \in \mathcal{SY}_0} S(-i, -j) \to I_Y \to 0$$

where

$$\mathcal{SY}_0 = \{(2r, 0), (r, \lambda_1), (0, 2\lambda_1)\} \cup \{(i-1, \lambda_1 + \lambda_i)\,(i+r-1, \lambda_i) \mid \lambda_i - \lambda_{i-1} < 0\}$$
$$\mathcal{SY}_1 = \{(2r, \lambda_r), (r, \lambda_1 + \lambda_r)\} \cup \{(i-1, \lambda_1 + \lambda_{i-1}), (i+r-1, \lambda_{i-1}) \mid \lambda_i - \lambda_{i-1} < 0\}.$$

Proof. Statement (i) follows directly from the construction of Y. For statement (ii), it suffices to note that if $\lambda^* = (\lambda_1^*, \ldots, \lambda_{\lambda_1}^*)$, then $\beta_Y = (\lambda_1^* + \lambda_1^*, \ldots, \lambda_1^* + \lambda_{\lambda_1}^*, \lambda_1^*, \ldots, \lambda_{\lambda_1}^*)$. Moreover, one can check that $\alpha_Y^* = \beta_Y$, so that by Theorem 1.5 it follows that Y is ACM. The bigraded resolution of (iii) follows from Theorem 1.6. $\square$

Example 2.3. Let Z be the scheme of Example 1.8. The completion of Z is the scheme
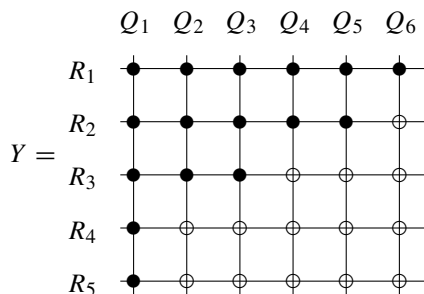
where • means a double point and ○ means a simple point (we have suppressed the multiplicities). Because $\lambda = (6, 5, 3, 1, 1)$, it follows that $\alpha_Y = (12, 11, 9, 7, 7, 6, 5, 3, 1, 1)$. Then the shifts in the bigraded minimal free resolution of I_Y are given by

$$\mathcal{SY}_0 = \{(10, 0), (8, 1), (7, 3), (6, 5), (5, 6), (3, 7), (2, 9), (1, 11), (0, 12)\}$$
$$\mathcal{SY}_1 = \{(10, 1), (8, 3), (7, 5), (6, 6), (5, 7), (3, 9), (2, 11), (1, 12)\}.$$

3. The generators of I_Z and I_Y

Using the tuple λ, we construct a matrix whose entries are either two or one. We then extract information from this matrix to describe the minimal generators of I_Z and I_Y. This technique originated with the first author [11] to describe the minimal generators and the Hilbert function of I_Z; this method can also describe the generators of I_Y.

Because $I_Y \subseteq I_Z$, we will identify a family of bigraded forms $\{F_1, \ldots, F_p\}$ such that $F_i \notin I_Y + (F_1, \ldots, F_{i-1})$ for $i = 1, \ldots, p$ and $I_Z = I_Y + (F_1, \ldots, F_p)$.

Definition 3.1. If $\lambda = (\lambda_1, \ldots, \lambda_r)$ is the partition associated to Z, then the *degree matrix* of Z is the $r \times \lambda_1$ matrix $\mathcal{M}_\lambda$ where

$$(\mathcal{M}_\lambda)_{i,j} = \begin{cases} 2 & j \leq \lambda_i \\ 1 & \text{otherwise.} \end{cases}$$

Remark 3.2. If the points in the support of Z have been relabeled according to Remark 1.4, then $(\mathcal{M}_\lambda)_{a,b}$ is the multiplicity of the point $P_{a,b}$ in Y, the completion of Z.

We now recall some definitions given in [11] using the degree matrix of Z.

Definition 3.3. The *base corners* of Z is the set:

$$\mathcal{C}_0 := \{(i, j) \mid (\mathcal{M}_\lambda)_{i,j} = 1 \text{ but } (\mathcal{M}_\lambda)_{i-1,j} = (\mathcal{M}_\lambda)_{i,j-1} = 2\}.$$

Given the base corners of Z, we then set

$$\mathcal{C}_1 := \{(i, l) \mid (i, j), (k, l) \in \mathcal{C}_0 \text{ and } i > k\}.$$

The *corners* of Z is then the set $\mathcal{C} := \mathcal{C}_0 \cup \mathcal{C}_1$. We shall assume that the elements of $\mathcal{C}$ have been ordered from largest to smallest with respect to the lex order.

Remark 3.4. The set of base corners $\mathcal{C}_0$ can be computed directly from the partition λ associated to Z. Precisely, $\mathcal{C}_0 := \{(i, \lambda_i + 1) \mid \lambda_i - \lambda_{i-1} < 0\}$.

Definition 3.5. For each $(i, j) \in \mathcal{C}$, set

$$u_{i,j} := m_{1,j} + m_{2,j} + \cdots + m_{i-1,j} \quad \text{and} \quad v_{i,j} := m_{i,1} + m_{i,2} + \cdots + m_{i,j-1}$$

were $m_{a,b} = (\mathcal{M}_\lambda)_{a,b}$. That is, $u_{i,j}$, respectively $v_{i,j}$, is the sum of the entries in $\mathcal{M}_\lambda$ in the column above, respectively in the row to the left, of the position (i, j). If $(i, j) = (i_\ell, j_\ell)$ is the ℓth largest element of $\mathcal{C}$ with respect to the lexicographical order, the form

$$F_\ell = L_{R_1}^{m_{1,j}} \cdots L_{R_{i-1}}^{m_{i-1,j}} L_{Q_1}^{m_{i,1}} \cdots L_{Q_{j-1}}^{m_{i,j-1}}$$

were $m_{a,b} = (\mathcal{M}_\lambda)_{a,b}$ is called the *form relative to the corner* (i, j).

Theorem 3.6. *Let Z be a fat point scheme that satisfies Convention 1.7, and furthermore, assume that the points in the support have been relabeled using Remark 1.4. If $(i, j) = (i_\ell, j_\ell)$ is the ℓth largest element of $\mathcal{C}$ with respect to the lex order, then let*

$$F_\ell = L_{R_1}^{m_{1,j}} \cdots L_{R_{i-1}}^{m_{i-1,j}} L_{Q_1}^{m_{i,1}} \cdots L_{Q_{j-1}}^{m_{i,j-1}}$$

be the form relative to the corner (i, j). Set $I_0 := I_Y$, and $I_\ell := (I_{\ell-1}, F_\ell)$ for $\ell = 0, \ldots, |\mathcal{C}|$. Then

(i) deg $F_\ell = (u_{i,j}, v_{i,j})$.
(ii) $F_\ell \notin I_{\ell-1}$.
(iii) $I_Z = I_Y + (F_1, \ldots, F_p)$ where $p = |\mathcal{C}|$.
(iv) I_ℓ is generated by the generators of I_Y, and all the forms relative to corners (a, b) with (a,b) bigger than or equal to (i_ℓ, j_ℓ).

Proof. Statement (i) is immediate from the definition of F_ℓ. For statement (ii), note that after relabeling, $P_{i_\ell, j_\ell} = R_{i_\ell} \times Q_{j_\ell}$ is a reduced point of Y. Furthermore, every element of $I_{\ell-1}$ vanishes at the point P_{i_ℓ, j_ℓ}, i.e., $I_{\ell-1} \subseteq I_{P_{i_\ell, j_\ell}} = (L_{R_{i_\ell}}, L_{Q_{j_\ell}})$, but the form $F_\ell \notin I_{P_{i_\ell, j_\ell}}$. Statements (iii) and (iv) are Theorem 3.15 of [11]. $\square$

A slight variation of the above technique enables us to describe the generators of I_Y.

Definition 3.7. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be the partition associated to Z, and suppose $\mathcal{M}_\lambda$ is the *degree matrix* of Z. The *degree matrix* of Y is the $(r + 1) \times (\lambda_1 + 1)$ matrix

$$\mathcal{M}_Y = \begin{bmatrix} \mathcal{M}_\lambda & \mathbf{1} \\ \mathbf{1} & 1 \end{bmatrix}$$

where $\mathbf{1}$ denotes the appropriately sized matrix consisting only of ones.

Definition 3.8. Let $\mathcal{C}_0$ be the base corners of Z constructed from $\lambda = (\lambda_1, \ldots, \lambda_r)$. The *outside corners* of Z is the set

$$\mathcal{OC} = \{(r + 1, 1), (1, \lambda_1 + 1), (r + 1, \lambda_1 + 1)\} \cup \{(r + 1, j), (i, \lambda_1 + 1) \mid (i, j) \in \mathcal{C}_0\}.$$

Theorem 3.9. *Let Z be a fat point scheme that satisfies Convention 1.7, and furthermore, assume that the points in the support have been relabeled using Remark 1.4. If $(i, j) = (i_\ell, j_\ell) \in \mathcal{OC}$, then set*

$$G_\ell = L_{R_1}^{m_{1,j}} \cdots L_{R_{i-1}}^{m_{i-1,j}} L_{Q_1}^{m_{i,1}} \cdots L_{Q_{j-1}}^{m_{i,j-1}} \quad \text{where } m_{a,b} = (\mathcal{M}_Y)_{a,b}.$$

Then $\{G_1, \ldots, G_q\}$ where $q = |\mathcal{OC}|$ is a minimal set of generators of I_Y.

Proof. For each $\ell = 1, \ldots, q$, one can show that G_ℓ passes through all the points of Y to the correct multiplicity. By comparing the degrees of each G_ℓ with the degrees of the minimal generators of I_Y from the bigraded minimal free resolution in Theorem 2.2, we then see that the G_ℓ's form a minimal set of generators of I_Y. $\square$

We end this section with an example illustrating these ideas.

Example 3.10. Let $\lambda = (6, 5, 3, 1, 1)$ be the λ associated to the fat point scheme Z of Example 1.8. Then the degree matrices of Z and Y are given by

$$\mathcal{M}_\lambda = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & \underline{1} \\ 2 & 2 & 2 & \underline{1} & 1 & \underline{1} \\ 2 & \underline{1} & 1 & \underline{1} & 1 & \underline{1} \\ 2 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad \mathcal{M}_Y = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & \underline{1} \\ 2 & 2 & 2 & 2 & 2 & 1 & \underline{1} \\ 2 & 2 & 2 & 1 & 1 & 1 & \underline{1} \\ 2 & 1 & 1 & 1 & 1 & 1 & \underline{1} \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ \underline{1} & \underline{1} & 1 & \underline{1} & 1 & \underline{1} & \underline{1} \end{bmatrix}.$$

Then $\mathcal{C}_0 = \{(4, 2), (3, 4), (2, 6)\}$, ordered lexicographically. The corners of Z is the set

$$\mathcal{C} := \mathcal{C}_0 \cup \{(4, 4), (4, 6), (3, 6)\} = \{(4, 6), (4, 4), (4, 2), (3, 6), (3, 4), (2, 6)\}.$$

The positions of the underlined 1's in $\mathcal{M}_\lambda$ correspond to the elements of $\mathcal{C}$.

The outside corners, which correspond to the positions of the underlined 1's in the matrix $\mathcal{M}_Y$, is the set $\mathcal{OC} = \{(6, 1), (6, 2), (6, 4), (6, 6), (6, 7), (1, 7), (2, 7), (3, 7), (4, 7)\}$. As an example of Theorem 3.9, consider $(6, 6) \in \mathcal{OC}$. Associated to this tuple is the form

$$G = L_{R_1}^2 L_{R_2}^1 L_{R_3}^1 L_{R_4}^1 L_{R_5}^1 L_{Q_1}^1 L_{Q_2}^1 L_{Q_3}^1 L_{Q_4}^1 L_{Q_5}^1.$$

We see from the picture of Example 2.3 that G passes through all the points (with correct multiplicity) of Y. Also, deg $G = (6, 5)$ is one of the degrees of the minimal generators.

Observation 3.11. The following fact will be used implicitly in the next section. For each $(i, j) \in \mathcal{C}$ there exists non-negative integers c and d such that $(i + c + 1, j)$, $(i, j + d + 1)$ and $(i + c + 1, j + d + 1)$ are either elements of $\mathcal{C}$ or $\mathcal{OC}$. Although we leave the proof of this fact to the reader, we can illustrate this observation using the above example. Note that $(4, 2)$ is a corner of Z. There exists two integers $c = 1$ and $d = 1$ such that $(4 + 1 + 1, 2)$, $(4, 2 + 1 + 1)$ and $(4 + 1 + 1, 2 + 1 + 1)$ are also corners or outside corners.

4. The resolution of I_Z

Let $F_1, \ldots, F_p$ be the p forms of Theorem 3.6 where F_ℓ is the form relative to the corner $(i_\ell, j_\ell) \in \mathcal{C}$. As in Theorem 3.6, we set $I_0 = I_Y$ and $I_\ell = (I_{\ell-1}, F_\ell)$ for $\ell = 1, \ldots, p$. Then, for each $1 \leq \ell \leq p$, we have a short exact sequence

$$0 \to S/(I_{\ell-1} : F_\ell)(-u_{i_\ell, j_\ell}, -v_{i_\ell, j_\ell}) \xrightarrow{\times F_\ell} S/I_{\ell-1} \to S/I_\ell = S/(I_{\ell-1}, F_\ell) \to 0 \tag{4.1}$$

where $\deg F_\ell = (u_{i_\ell, j_\ell}, v_{i_\ell, j_\ell})$. Using the short exact sequence and the mapping cone construction, we will reiteratively describe the bigraded minimal free resolution of I_Z.

To use the mapping cone construction in conjunction with (4.1), we will prove that $(I_{\ell-1} : F_\ell)$ is a complete intersection for each $\ell = 1, \ldots, p$ whose type can be determined through the following family of matrices. Let $\mathcal{C} = \{(i_1, j_1), \ldots, (i_p, j_p)\}$ be the corners of Z ordered from largest to smallest with respect to the lex order. Then set $\mathcal{M}_0 = \mathcal{M}_\lambda$, and for $\ell = 1, \ldots, p$, let $\mathcal{M}_\ell$ be the $r \times \lambda_1$ matrix where

$$(\mathcal{M}_\ell)_{i,j} = \begin{cases} 0 & \text{if } (i, j) \succeq (i_\ell, j_\ell) \\ (\mathcal{M}_{\ell-1})_{i,j} & \text{otherwise.} \end{cases}$$

Here $\succeq$ denotes the partial order where $(i_1, j_1) \succeq (i_2, j_2)$ if and only if $i_1 \geq i_2$ and $j_1 \geq j_2$.

Example 4.1. Before preceding to the main results of this paper, we describe in more detail what our algorithm does geometrically, and how we shall use the matrices $\mathcal{M}_\ell$. Let Z_ℓ denote the scheme of fat points defined by the ideal I_ℓ, where $Z_0 = Y$ is the completion of Z. Roughly speaking, at each step in our algorithm, we are removing a set of points from $Z_{\ell-1}$ to form the set of points Z_ℓ. In particular, at each step we are removing a complete intersection whose type can be ascertained from the matrix $\mathcal{M}_{\ell-1}$.

We illustrate some of these ideas by using our running example (Example 1.8) of $\lambda = (6, 5, 3, 1, 1)$. The matrix $\mathcal{M}_0 = \mathcal{M}_\lambda$ of Example 3.10 describes the multiplicities of the fat points $Z_0 = Y$. By Example 3.10 the largest corner of Z is $(4, 6)$. The element

$$F_1 = L_{R_1}^2 L_{R_2} L_{R_3} L_{Q_1}^2 L_{Q_2} L_{Q_3} L_{Q_4} L_{Q_5}$$

is the form relative to the corner $(4, 6)$. The form F_1 passes through all the points of $Z_0 = Y$ with correct multiplicity, except the points $P_{a,b} = R_a \times Q_b$ with $(4, 6) \preceq (a, b) \preceq (5, 6)$. These points are $C = \{R_4 \times Q_6, R_5 \times Q_6\}$, a complete intersection of points of type $(2, 1)$ defined by $I_C = (L_{R_4} L_{R_5}, L_{Q_6})$. The type can be found by starting at the location of the first corner $(4, 6)$ in $\mathcal{M}_0$, and summing the entry in position $(4, 6)$ and all those below it (in this case, $1 + 1 = 2$), to get the first coordinate of the type, and summing the entry in position $(4, 6)$ and all those to right (in this case, only 1) to get the second coordinate.
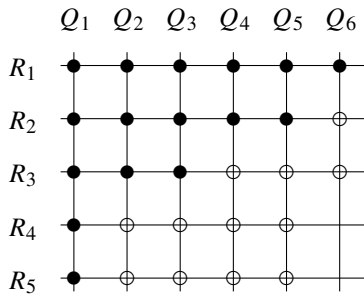
The ideal $I_1 = (I_0, F_1)$ is then the defining ideal of Z_1, where

$$Z_1 = Y \setminus CI(2, 1) = Y \setminus \{P_{4,6}, P_{5,6}\}.$$

Observe now that the matrix

$$\mathcal{M}_1 = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

describes the multiplicities of the fat point scheme Z_1:

$$
\begin{array}{c|cccccc}
 & Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \\
\hline
R_1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
R_2 & \bullet & \bullet & \bullet & \bullet & \bullet & \circ \\
R_3 & \bullet & \bullet & \bullet & \circ & \circ & \circ \\
R_4 & \bullet & \circ & \circ & \circ & \circ & \\
R_5 & \bullet & \circ & \circ & \circ & \circ & \\
\end{array}
$$

where $\bullet$ means a double point and $\circ$ means a simple point.

The next largest corner of Z is $(4, 4)$, and the form

$$F_2 = L_{R_1}^2 L_{R_2}^2 L_{R_3} L_{Q_1}^2 L_{Q_2} L_{Q_3}$$

is the form relative to the second corner $(4, 4)$. The form F_2 now passes through all the points of the scheme Z_1 with correct multiplicity, except the points $P_{a,b}$ with $(4, 4) \preceq (a, b) \preceq (5, 5)$. These points are $C = \{R_4 \times Q_4, R_4 \times Q_5, R_5 \times Q_4, R_5 \times Q_5\}$, a complete intersection of type $(2, 2)$ defined by $I_C = (L_{R_4} L_{R_5}, L_{Q_4} L_{Q_5})$. The type can be found by starting at the location of the second corner $(4, 4)$ in $\mathcal{M}_1$, and summing the entry in position $(4, 4)$ and all those below it (in this case, $1 + 1 = 2$), to get the first coordinate of the type, and summing the entry in position $(4, 4)$ and all those to right (in this case, $1 + 1 + 0 = 2$) to get the second coordinate.

The ideal $I_2 = (I_1, F_1)$ now defines the scheme

$$Z_2 = Z_1 \setminus CI(2, 2) = Z_1 \setminus \{P_{4,4}, P_{4,5}, P_{5,4}, P_{5,5}\} = Y \setminus \{P_{4,4}, P_{4,5}, P_{4,6}, P_{5,4}, P_{5,5}, P_{5,6}\},$$

and analogously, the matrix $\mathcal{M}_2$ describes the multiplicities of the fat point scheme Z_2:

$$
\begin{array}{c|cccccc}
 & Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \\
\hline
R_1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
R_2 & \bullet & \bullet & \bullet & \bullet & \bullet & \circ \\
R_3 & \bullet & \bullet & \bullet & \circ & \circ & \circ \\
R_4 & \bullet & \circ & \circ & & & \\
R_5 & \bullet & \circ & \circ & & & \\
\end{array}
$$
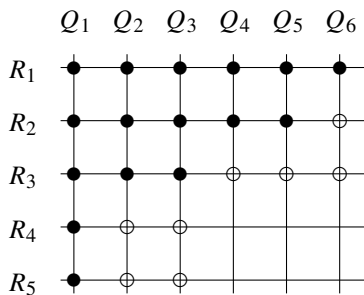
Continuing in this fashion, we remove all the simple points from Y by removing a suitably sized complete intersection at each step until we get $Z_6 = Z$. In general, the matrices $\mathcal{M}_\ell$ allow us to keep track of the size of the complete intersection we are cutting out from Z_ℓ at each step.

Remark 4.2. Let $\{(i_1, j_1), \ldots, (i_p, j_p)\}$ be the corners of Z starting from the largest corner of Z; the complete intersection C that we remove at each step from Y is formed from the points $P_{a,b}$ with $(i_\ell, j_\ell) \preceq (a, b) \preceq (i_\ell + c, j_\ell + d)$ and such that (i_ℓ, j_ℓ), $(i_\ell, j_\ell + c + 1)$ and $(i_\ell + d + 1, j_\ell)$ are either corners or outside corners of Z.

In the next lemma we show $(I_{\ell-1} : F_\ell)$ is a complete intersection of points.

Lemma 4.3. *With the notation as above, let $(i, j) = (i_\ell, j_\ell)$ be the ℓth corner of C. Then*

$$(I_{\ell-1} : F_\ell) = I_{CI(a_{i,j}, b_{i,j})}$$

where $a_{i,j} = m_{i,j} + \cdots + m_{r,j}$, $b_{i,j} = m_{i,j} + \cdots + m_{i,\lambda_1}$ and $m_{a,b} = (M_{\ell-1})_{a,b}$.

Proof. Without loss of generality, assume that the points of Z have been relabeled in accordance to Remark 1.4. From the construction of $\mathcal{M}_{\ell-1}$ there exists integers c and d such that $m_{i,j} = m_{i+1,j} = \cdots = m_{i+c,j} = 1$, but $m_{i+c+1,j} = \cdots = m_{r,j} = 0$, and similarly, $m_{i,j} = \cdots = m_{i,j+d} = 1$, but $m_{i,j+d+1} = \cdots = m_{i,\lambda_1} = 0$. Set

$$A = L_{R_i}^{m_{i,j}} \cdots L_{R_{i+c}}^{m_{i+c,j}} = L_{R_i} \cdots L_{R_{i+c}} \quad \text{and} \quad B = L_{Q_j}^{m_{i,j}} \cdots L_{Q_{j+d}}^{m_{i,j+d}} = L_{Q_j} \cdots L_{Q_{j+d}}.$$

It will now suffice to show that $(I_{\ell-1} : F_\ell) = (A, B)$.

Note that (A, B) defines a complete intersection $C = CI(a_{i,j}, b_{i,j})$. Because the points have been rearranged in accordance to Remark 1.4, $P_{a,b} = R_a \times Q_b \in C$ if and only if $(i, j) \preceq (a, b) \preceq (i + c, j + d)$. The points of C form a subset of the reduced points of Y.

By Theorems 3.6 and 3.9, $I_{\ell-1} = (G_1, \ldots, G_q, F_1, \ldots, F_{\ell-1})$. The forms G_i vanish at all the points of $C \subseteq Y$. By Theorem 3.6 we have $F_i \in I_C$ for $1 \leq i \leq \ell - 1$. However,

$$F_\ell = L_{R_1}^{m_{1,j}} \cdots L_{R_{i-1}}^{m_{i-1,j}} L_{Q_1}^{m_{i,1}} \cdots L_{Q_{j-1}}^{m_{i,j-1}}$$

from which it follows that for every $P_{a,b} \in C$, $F_\ell(P_{a,b}) \neq 0$. So, if $HF_\ell \in I_{\ell-1} \subseteq I_C$, then $H \in I_C$. That is, $(I_{\ell-1} : F_\ell) \subseteq I_C = (A, B)$.

From the construction of $\mathcal{M}_{\ell-1}$, $(i + c + 1, j)$ is either a corner or outside corner of Z. In either case, set

$$F = L_{R_1}^{n_{1,j}} \cdots L_{R_{i-1}}^{n_{i-1,j}} L_{R_i}^{n_{i,j}} \cdots L_{R_{i+c}}^{n_{i+c,j}} L_{Q_1}^{n_{i+c+1,1}} \cdots L_{Q_{j-1}}^{n_{i+c+1,j-1}}$$

where $n_{a,b}$ refers to the entries in $\mathcal{M}_Y = (n_{a,b})$, the degree matrix of Y. If $(i + c + 1, j) \in \mathcal{C}$, then $F \in I_{\ell-1}$ by Theorem 3.6; if $(i + c + 1, j) \in \mathcal{OC}$, then $F \in I_{\ell-1}$ by Theorem 3.9. Now set

$$F_\ell A = L_{R_1}^{m_{1,j}} \cdots L_{R_{i-1}}^{m_{i-1,j}} L_{R_i} \cdots L_{R_{i+c}} L_{Q_1}^{m_{i,1}} \cdots L_{Q_{j-1}}^{m_{i,j-1}}.$$

We claim that F divides $F_\ell A$, and hence $F_\ell A \in I_{\ell-1}$. To see this we compare the matrices $\mathcal{M}_Y$ and $\mathcal{M}_{\ell-1}$. By construction $(\mathcal{M}_Y)_{a,b} = (\mathcal{M}_\lambda)_{a,b} = (\mathcal{M}_{\ell-1})_{a,b}$ for all $(a, b) \preceq (i + c, j)$. So, the exponents of the L_{R_i}'s in $F_\ell A$ and F are actually the same.

On the other hand, note that $n_{a,j} \geq n_{b,j}$ if $a \geq b$ in $\mathcal{M}_Y$, i.e., the columns are non-increasing. Since $m_{i,t} = n_{i,t}$ for $t = 1, \ldots, j - 1$, we have that the exponents of the L_{Q_j}'s in F are less than or equal than those that appear in $F_\ell A$. So, F divides $F_\ell A$. So $A \in (I_{\ell-1} : F_\ell)$. A similar argument using the fact that $(i, j + d + 1) \in \mathcal{C}$ or $\mathcal{OC}$ will now show that $B \in (I_{\ell-1} : F_\ell)$. Hence $(A, B) \subseteq (I_{\ell-1} : F_\ell)$. $\square$

We now come to the main result of this section, which forms the basis of our recursive algorithm to compute the resolution of I_Z.

Theorem 4.4. *With the notation as above, suppose that* $(i, j) = (i_\ell, j_\ell)$ *is the ℓth largest element of $\mathcal{C}$, and furthermore, suppose that*

$$0 \to \mathbb{F}_2 \to \mathbb{F}_1 \to \mathbb{F}_0 \to I_{\ell-1} \to 0$$

is the bigraded minimal free resolution of $I_{\ell-1}$. Then

$$0 \to \begin{matrix} \mathbb{F}_2 \\ \oplus \\ S(-u_{i,j} - a_{i,j}, -v_{i,j} - b_{i,j}) \end{matrix} \to \begin{matrix} \mathbb{F}_1 \\ \oplus \\ S(-u_{i,j} - a_{i,j}, -v_{i,j}) \\ \oplus \\ S(-u_{i,j}, -v_{i,j} - b_{i,j}) \end{matrix} \to \begin{matrix} \mathbb{F}_0 \\ \oplus \\ S(-u_{i,j}, -v_{i,j}) \end{matrix} \to I_\ell \to 0 \qquad (4.2)$$

is a bigraded minimal free resolution of $I_\ell = (I_{\ell-1}, F_\ell)$ where

$$u_{i,j} = m_{1,j} + m_{2,j} + \cdots + m_{i-1,j} \quad \text{and} \quad v_{i,j} = m_{i,1} + m_{i,2} + \cdots + m_{i,j-1}$$

$$a_{i,j} = m_{i,j} + \cdots + m_{r,j} \quad \text{and} \quad b_{i,j} = m_{i,j} + \cdots + m_{i,\lambda_1}$$

and $m_{a,b} = (\mathcal{M}_{\ell-1})_{a,b}$.

Proof. Let $(i, j) = (i_\ell, j_\ell) \in \mathcal{C}$ denote the ℓth largest corner of Z, and assume that the points of Z have been rearranged in accordance to Remark 1.4. Let

$$F_\ell = L_{R_1}^{m_{1,j}} \cdots L_{R_{i-1}}^{m_{i-1,j}} L_{Q_1}^{m_{i,1}} \cdots L_{Q_{j-1}}^{m_{i,j-1}}$$

be the form relative to the corner (i, j) with $\deg F_\ell = (u_{i,j}, v_{i,j})$. Note that for all (a, b) with $(a, b) \preceq (i, j)$, we have $(\mathcal{M}_{\ell-1})_{a,b} = (\mathcal{M}_\lambda)_{a,b}$. So, the integers $u_{i,j}$ and $v_{i,j}$ as defined above are the same as those of Theorem 3.6.

By Lemma 4.3, we know that $(I_{\ell-1} : F_\ell) = I_{CI(a_{i,j}, b_{i,j})}$. By using (1.1), a minimal bigraded free resolution of $(I_{\ell-1} : F_\ell)$ is:

$$0 \to S(-a_{i,j}, -b_{i,j}) \to S(-a_{i,j}, 0) \oplus S(0, -b_{i,j}) \to (I_{\ell-1} : F_\ell) \to 0.$$

When we apply the mapping cone construction to the short exact sequence (4.1), we get that (4.2) is a bigraded free resolution of I_ℓ. It therefore suffices to verify that this resolution is minimal.

The map in (4.1)

$$S/(I_{\ell-1} : F_\ell)(-u_{i,j}, -v_{i,j}) \xrightarrow{\times F_\ell} S/I_{\ell-1}$$

lifts to a map from the minimal resolution of $S/(I_{\ell-1} : F_\ell)$ to that of $S/I_{\ell-1}$:

$$
\begin{array}{ccccccccccc}
0 & \to & S & \xrightarrow{\phi_1} & S^2 & \xrightarrow{\phi_0} & S & \xrightarrow{\epsilon} & S/(I_{\ell-1} : F_\ell) & \to & 0 \\
& & \downarrow \delta_2 & & \downarrow \delta_1 & & \downarrow \times F_\ell & & \downarrow \times F_\ell & & \\
0 & \to & \mathbb{F}_2 & \xrightarrow{\varphi_2} & \mathbb{F}_1 & \xrightarrow{\varphi_1} & \mathbb{F}_0 & \xrightarrow{\varphi_0} & S & \xrightarrow{\epsilon} & S/I_{\ell-1} & & \to & 0.
\end{array}
$$

We have suppressed all the shifts in the resolutions. The maps in each square commute. Again suppressing the shifts, the resolution of S/I_ℓ given by the mapping cone construction has the form

$$0 \to S \oplus \mathbb{F}_2 \xrightarrow{\Phi_2} S^2 \oplus \mathbb{F}_1 \xrightarrow{\Phi_1} S \oplus \mathbb{F}_0 \xrightarrow{\Phi_0} S \to S/I_\ell \to 0$$

where the maps are

$$\Phi_2 = \begin{bmatrix} -\phi_1 & 0 \\ \delta_2 & \varphi_2 \end{bmatrix}, \qquad \Phi_1 = \begin{bmatrix} -\phi_0 & 0 \\ \delta_1 & \varphi_1 \end{bmatrix} \quad \text{and} \quad \Phi_0 = \begin{bmatrix} F_\ell & \varphi_0 \end{bmatrix}.$$

After fixing a basis, each map ϕ_i, φ_i, and δ_i can be represented by a matrix with entries in S. It will therefore suffice to show that all the nonzero entries of the matrix corresponding to the map Φ_i for $i = 0, 1, 2$ belong to the maximal ideal (x_0, x_1, y_0, y_1) of S. The matrices corresponding to ϕ_i and φ_i already have this property because they are the maps in the minimal resolution of $S/(I_{\ell-1} : F_\ell)$ and $S/I_{\ell-1}$, respectively. So, we need to show that there exists maps δ_1 and δ_2 that make each square commute, and when these maps are represented as a matrices, all the nonzero entries belong to (x_0, x_1, y_0, y_1).

From Observation 3.11, because $(i, j) \in \mathcal{C}$, there exist integers c and d such that $(i + c + 1, j)$, $(i, j + d + 1)$, and $(i + c + 1, j + d + 1)$ are either corners or outside corners of Z; in particular, we choose c and d as in the proof of Lemma 4.3, that is, $m_{i,j} = m_{i+1,j} = \cdots = m_{i+c,j} = 1$, but $m_{i+c+1,j} = \cdots = m_{r,j} = 0$, and similarly, $m_{i,j} = \cdots = m_{i,j+d} = 1$, but $m_{i,j+d+1} = \cdots = m_{i,\lambda_1} = 0$ with $m_{a,b} = (\mathcal{M}_{\ell-1})_{a,b}$. Set

$$A = L_{R_i}^{m_{i,j}} \cdots L_{R_{i+c}}^{m_{i+c,j}} = L_{R_i} \cdots L_{R_{i+c}} \quad \text{and} \quad B = L_{Q_j}^{m_{i,j}} \cdots L_{Q_{j+d}}^{m_{i,j+d}} = L_{Q_j} \cdots L_{Q_{j+d}}.$$

Because $(I_{\ell-1} : F_\ell) = (A, B)$ is a complete intersection, the maps ϕ_0 and ϕ_1 are simply the Koszul maps. As matrices, these maps are

$$\phi_1 = \begin{bmatrix} B \\ -A \end{bmatrix} \quad \text{and} \quad \phi_0 = \begin{bmatrix} A & B \end{bmatrix}.$$

We also let

$$H_1 = L_{R_1}^{n_{1,j}} \cdots L_{R_{i+c}}^{n_{i+c,j}} L_{Q_1}^{n_{i+c+1,1}} \cdots L_{Q_{j-1}}^{n_{i+c+1,j-1}}$$

$$H_2 = L_{R_1}^{n_{1,j+d+1}} \cdots L_{R_{i-1}}^{n_{i-1,j+d+1}} L_{Q_1}^{n_{i,1}} \cdots L_{Q_{j+d}}^{n_{i,j+d}}$$

$$H_3 = L_{R_1}^{n_{1,j+d+1}} \cdots L_{R_{i+c}}^{n_{i+c,j+d+1}} L_{Q_1}^{n_{i+c+1,1}} \cdots L_{Q_{j+d}}^{n_{i+c+1,j+d}}$$

where $n_{a,b} = (\mathcal{M}_Y)_{a,b}$.

Now $(i + c + 1, j)$, $(i, j + d + 1)$, and $(i + c + 1, j + d + 1)$ are either corners or outside corners of Z. In the case that they are corners of Z, then they are larger than the corner (i, j). So by Theorems 3.6 and 3.9 we have that the forms H_1, H_2, H_3 are minimal generators of $I_{\ell-1}$.

After a suitable change of basis, we can then write φ_0 as

$$\varphi_0 = \begin{bmatrix} H_1 & H_2 & H_3 & K_1 & \cdots & K_s \end{bmatrix}$$

where $K_1, \ldots, K_s$ denote the other minimal generators of $I_{\ell-1}$.

Let

$$C = \frac{F_\ell A}{H_1} = \frac{L_{R_1}^{m_{1,j}} \cdots L_{R_{i-1}}^{m_{i-1,j}} L_{Q_1}^{m_{i,1}} \cdots L_{Q_{j-1}}^{m_{i,j-1}} L_{R_i} \cdots L_{R_{i+c}}}{L_{R_1}^{n_{1,j}} \cdots L_{R_{i+c}}^{n_{i+c,j}} L_{Q_1}^{n_{i+c+1,1}} \cdots L_{Q_{j-1}}^{n_{i+c+1,j-1}}}.$$

Now, by the construction of $\mathcal{M}_Y$ and $\mathcal{M}_{\ell-1}$, we also have $(\mathcal{M}_Y)_{a,b} = (\mathcal{M}_{\ell-1})_{a,b}$ for all $(a, b) \preceq (i + c, j + d)$. The exponents of the L_{R_i}'s in the above expression are then the same on the top and bottom, and thus they cancel out, i.e.,

$$C = \frac{F_\ell A}{H_1} = \frac{L_{Q_1}^{m_{i,1}} \cdots L_{Q_{j-1}}^{m_{i,j-1}}}{L_{Q_1}^{n_{i+c+1,1}} \cdots L_{Q_{j-1}}^{n_{i+c+1,j-1}}}.$$

Because (i, j) is a corner and $(i + c + 1, j)$ is either a corner or outside corner of Z, by construction of the $\mathcal{M}_Y$, there exist some $j' \leq j - 1$ such that $n_{i+c+1,j'} < n_{i,j'} = m_{i,j'}$. (The columns of $\mathcal{M}_Y$ are non-increasing, so if $n_{i+c+1,j'} = n_{i,j'}$ for all $j' \leq j - 1$, then the first $j - 1$ entries of rows i through $i + c + 1$ are the same, and thus there would not be a corner (or outside corner) in position $(i + c + 1, j)$.) Because of this fact, we have $\deg C > 0$. A similar argument implies that if $D = \frac{F_\ell B}{H_2}$, then $\deg D > 0$.

Because $F_\ell H_3 = H_1 H_2$, we have the following two syzygies:

$$BH_1 - DH_3 = 0 \quad \text{and} \quad AH_2 - CH_3 = 0.$$

That is, $(B, 0, -D, 0, \ldots, 0)^{\mathrm{T}}$ and $(0, A, -C, 0, \ldots, 0)^{\mathrm{T}}$ are two elements of $\mathbb{F}_0$, written as vectors, in $\ker \varphi_0 = \operatorname{Im} \varphi_1$. Let $\underline{a} = (a_1, \ldots, a_m)^{\mathrm{T}}$, respectively, $\underline{b} = (b_1, \ldots, b_m)^{\mathrm{T}}$ denote an element of $\mathbb{F}_1$ with $\varphi_1(\underline{a}) = (B, 0, -D, 0, \ldots, 0)^{\mathrm{T}}$, respectively, $\varphi_1(\underline{b}) = (0, A, -C, 0, \ldots, 0)^{\mathrm{T}}$. With this notation, we can now prove:

Claim 4.5. *The maps δ_1 and δ_2 are given by*

$$\delta_2 = \begin{bmatrix} Ca_1 - Db_1 \\ \vdots \\ Ca_m - Db_m \end{bmatrix} \quad \text{and} \quad \delta_1 = \begin{bmatrix} C & 0 \\ 0 & D \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}.$$

Proof. We just need to show that each square containing a δ_i commutes. Now $\varphi_0 \delta_1 = \begin{bmatrix} H_1 C & H_2 D \end{bmatrix} = \begin{bmatrix} F_\ell A & F_\ell B \end{bmatrix}$. This map is the same as composing the map ϕ_0 with the map defined by multiplication by F_ℓ. For the second square,

$$\begin{aligned} \varphi_1 \delta_2 &= C \varphi_1(\underline{a}) - D \varphi_1(\underline{b}) = C(B, 0, -D, 0, \ldots, 0)^{\mathrm{T}} - D(0, A, -C, 0, \ldots, 0)^{\mathrm{T}} \\ &= (CB, -DA, 0, \ldots, 0)^{\mathrm{T}} = \delta_1 \phi_1. \end{aligned}$$

This completes the proof of the claim. $\quad\square$

Because C and D are nonconstant bihomogeneous forms, every nonzero entry of δ_1 and δ_2 belongs to $(x_0, x_1, y_0, y_1) \subseteq S$. Therefore, the resolution of I_ℓ is minimal, as desired. $\quad\square$

Remark 4.6. As observed in Example 4.1, the ideal I_ℓ corresponds to a subscheme of Y formed by removing a number of complete intersections of reduced points. The above theorem allows us to calculate the bigraded minimal free resolution for each such subscheme "between" Y and Z, that is, those schemes we called Z_ℓ in Example 4.1.

5. The algorithm

The resolution of $I_0 = I_Y$ depends only upon λ. By repeatedly applying Theorem 4.4, we obtain the minimal resolution of $I_p = I_Z$. Furthermore, the shifts that appear at each step only depend upon $\mathcal{M}_{\ell-1}$ which is constructed from λ. Thus, there is an algorithm to compute the bigraded minimal free resolution of a fat point scheme Z which satisfies Convention 1.7. For the convenience of the reader, we explicitly write out this algorithm.

Algorithm 5.1 (*Computing Bigraded Resolution*).

Input: $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ where λ describes the ACM support of Z.
Output: The shifts in the bigraded minimal free resolution of I_Z.

Step 1:   Compute the shifts in the bigraded resolution of I_Y where Y is the completion of Z.
- $\mathcal{SY}_0 := \{(2r, 0), (r, \lambda_1), (0, 2\lambda_1)\} \cup \{(i-1, \lambda_1 + \lambda_i), (i+r-1, \lambda_i) \mid \lambda_i - \lambda_{i-1} < 0\}$
- $\mathcal{SY}_1 := \{(2r, \lambda_r), (r, \lambda_1 + \lambda_r)\} \cup \{(i-1, \lambda_1 + \lambda_{i-1}), (i+r-1, \lambda_{i-1}) \mid \lambda_i - \lambda_{i-1} < 0\}$

Step 2:   Locate the corners
- $\mathcal{C}_0 := \{(\lambda_i + 1, i) \mid \lambda_i - \lambda_{i-1} < 0\} = \{(i_1, j_1), \ldots, (i_s, j_s)\}$ (lex ordered from largest to smallest)
- $\mathcal{C}_1 := \{(i_a, j_b) \mid (i_a, j_a), (i_b, j_b) \in \mathcal{C}_0 \text{ and } a > b\}$
- $\mathcal{C} := \mathcal{C}_0 \cup \mathcal{C}_1$ and order $\mathcal{C}$ in lexicographical order (largest to smallest)

Step 3:   Calculate the shifts in the resolution of I_Z.
- Let $\mathcal{M}_\lambda$ be the $r \times \lambda_1$ matrix where $(\mathcal{M}_\lambda)_{i,j} = \begin{cases} 2 & \text{if } j \leq \lambda_i \\ 1 & \text{otherwise} \end{cases}$
- Set $\mathcal{SZ}_0 := \mathcal{SY}_0$, $\mathcal{SZ}_1 := \mathcal{SY}_1$, and $\mathcal{SZ}_2 := \{\}$
- For each $(i, j) \in \mathcal{C}$ (working largest to smallest) do
$$u_{i,j} := (\mathcal{M}_\lambda)_{1,j} + \cdots + (\mathcal{M}_\lambda)_{i-1,j}$$
$$v_{i,j} := (\mathcal{M}_\lambda)_{i,1} + \cdots + (\mathcal{M}_\lambda)_{i,j-1}$$
$$a_{i,j} := (\mathcal{M}_\lambda)_{i,j} + \cdots + (\mathcal{M}_\lambda)_{r,j}$$
$$b_{i,j} := (\mathcal{M}_\lambda)_{i,j} + \cdots + (\mathcal{M}_\lambda)_{i,\lambda_1}$$
$$\mathcal{SZ}_0 := \mathcal{SZ}_0 \cup \{(u_{i,j}, v_{i,j})\}$$
$$\mathcal{SZ}_1 := \mathcal{SZ}_1 \cup \{(u_{i,j} + a_{i,j}, v_{i,j}), (u_{i,j}, v_{i,j} + b_{i,j})\}$$
$$\mathcal{SZ}_2 := \mathcal{SZ}_2 \cup \{(u_{i,j} + a_{i,j}, v_{i,j} + b_{i,j})\}$$
$$(\mathcal{M}_\lambda)_{ij} := \begin{cases} 0 & \text{if } (i', j') \succeq (i, j) \\ (\mathcal{M}_\lambda)_{ij} & \text{otherwise} \end{cases}$$

Step 4:   Return $\mathcal{SZ}_0$, $\mathcal{SZ}_1$, and $\mathcal{SZ}_2$ (the shifts at the 0th, 1st, and 2nd step of the resolution, respectively).

Remark 5.2. The above algorithm has been implemented in CoCoA [2] and *Macaulay* 2 [10], and can be downloaded from the second author's web page.[1]

Example 5.3. We use Algorithm 5.1 to compute the bigraded resolution of the fat points of Example 1.8. We have already computed $\mathcal{SY}_0$ and $\mathcal{SY}_1$ in Example 2.3. To calculate the remaining elements of $\mathcal{SZ}_0$, $\mathcal{SZ}_1$, and $\mathcal{SZ}_2$, where $\mathcal{SZ}_i$ is the set of shifts in ith free module appearing the resolution of I_Z, we need the numbers $u_{i,j}, v_{i,j}, a_{i,j}, b_{i,j}$ for each corner $(i, j) \in \mathcal{C}$. We have presented these numbers in the table below:

$(i, j) \in \mathcal{C}$	$u_{i,j}$	$v_{i,j}$	$a_{i,j}$	$b_{i,j}$
(4,6)	4	6	2	1
(4,4)	5	4	2	2
(4,2)	6	2	2	2
(3,6)	3	8	1	1
(3,4)	4	6	1	2
(2,6)	2	10	1	1

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[1] http://flash.lakeheadu.ca/~avantuyl/research/DoublePoints_Guardo_VanTuyl.html.

By using Theorem 4.4 and the above information, we have

$$\mathcal{SZ}_0 = \{(6, 2), (5, 4), (4, 6), (4, 6), (3, 8), (2, 10)\} \cup \mathcal{SY}_0$$
$$\mathcal{SZ}_1 = \{(8, 2), (7, 4), (6, 6), (6, 4), (5, 6), (5, 6), (4, 8), (4, 8), (4, 7), (3, 10), (3, 9), (2, 11)\} \cup \mathcal{SY}_1$$
$$\mathcal{SZ}_2 = \{(8, 4), (7, 6), (6, 7), (5, 8), (4, 9), (3, 11)\}.$$

Remark 5.4. From Algorithm 5.1 we see that Z is ACM if and only if $\mathcal{C} = \emptyset$ if and only if $\lambda = (\lambda_1, \ldots, \lambda_1)$, that is, if the support of Z is a complete intersection and $Z = Y$.

6. An application: A question of Römer

Let I be a homogeneous ideal of $R = k[x_1, \ldots, x_n]$ and consider the minimal graded free resolution of R/I

$$0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{F}_{p-1} \rightarrow \cdots \rightarrow \mathbb{F}_1 \rightarrow R \rightarrow R/I \rightarrow 0$$

where $\mathbb{F}_i = \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{i,j}(R/I)}$. The number $p = \mathrm{projdim}(R/I)$ is the *projective dimension*, while the numbers $\beta_{i,j}(R/I)$ are the i, jth *graded Betti numbers* of R/I. Römer [19] recently initiated an investigation into the relationship between the ith *Betti number of R/I*, i.e., $\beta_i(R/I) = \sum_{j \in \mathbb{Z}} \beta_{i,j}(R/I)$, and the shifts that appear with the minimal free resolution. Among other things, Römer asked what ideals satisfy the bound

$$\beta_i(R/I) \leq \frac{1}{(i-1)!(p-i)!} \prod_{j \neq i} M_j \tag{6.1}$$

where $M_i = \max\{j \mid \beta_{i,j}(R/I_Z) \neq 0\}$ denotes the maximum shift that appears in $\mathbb{F}_i$. In this section, we show the ideals I_Z studied in this paper satisfy (6.1). Precisely,

Theorem 6.1. *Let Z be a set of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ with ACM support. Then all the ith Betti numbers of S/I_Z satisfy the upper bound* (6.1).

Although we have viewed S/I_Z as a bigraded ring up to this point, the ring S/I_Z also can be given a graded structure by defining the ith graded piece to be $(S/I_Z)_i = \bigoplus_{a+b=i} (S/I_Z)_{a,b}$. As noted, S/I_Z is rarely Cohen–Macaulay, so this family provides further evidence that (6.1) holds for all codimension 2 ideals (Römer showed (6.1) is true for all codimension 2 Cohen–Macaulay ideals).

We continue to use the notation we developed in previous sections. In particular, we continue to assume Z satisfies Convention 1.7. We first show how to obtain precise formulas for $\beta_i(R/I_Z)$ for $i = 1, 2$ and 3, and lower bounds for M_1, M_2 and M_3 using λ. With this information, the verification of the bound (6.1) is a straightforward exercise.

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be any partition, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$. We set

$$d(\lambda) = \#\{i \mid \lambda_i - \lambda_{i-1} < 0\}.$$

Also, let $i^\star = \min\{i \mid \lambda_i - \lambda_{i-1} < 0\}$. This means $\lambda_1 = \lambda_2 = \cdots = \lambda_{i^\star - 1} > \lambda_{i^\star}$.

Lemma 6.2. *Let Z be a set of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ with ACM support with associated tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$. Let $d = d(\lambda)$. Then*

(i) $\beta_1(S/I_Z) = 2d + 3 + \binom{d+1}{2}$.

(ii) $\beta_2(S/I_Z) = 2d + 2 + 2\binom{d+1}{2}$.

(iii) $\beta_3(S/I_Z) = \binom{d+1}{2}$.

Proof. Let Y be the completion of Z. By Theorem 2.2, R/I_Y is ACM, and $\beta_1(R/I_Y) = 3 + 2d$ and $\beta_2(R/I_Y) = 2 + 2d$. By Theorem 3.6 there exist p forms $F_1, \ldots, F_p$ such that $I_Z = I_Y + (F_1, \ldots, F_p)$. Here, p is the number of corners

which is $p = \binom{d+1}{2}$. So $\beta_1(R/I_Z) = 2d + 3 + \binom{d+1}{2}$. By Theorem 4.4, each generator F_i contributes two first syzygies and one second syzygy. Hence $\beta_2(R/I_Z) = 2d + 2 + 2\binom{d+1}{2}$ and $\beta_3(R/I_Z) = \binom{d+1}{2}$. $\square$

Lemma 6.3. *Let Z be a set of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ with ACM support with associated tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$, and $d(\lambda) > 0$. Then*

(i) $2\lambda_1 \leq M_1$.
(ii) $2\lambda_1 + 1 \leq M_2$.
(iii) $\lambda_1 + \lambda_{i^\star} + 3 \leq M_3$.

Proof. Let Y be the completion of Z. By Theorem 2.2 there is a generator of I_Y of bidegree $(0, 2\lambda_1)$ and a first syzygy of I_Y of bidegree $(i^\star - 1, \lambda_1 + \lambda_{i^\star - 1})$. By Algorithm 5.1 we thus have that the bigraded shift $(0, -2\lambda_1)$ appears in $\mathbb{F}_1$ and $(-i^\star + 1, -\lambda_1 - \lambda_{i^\star - 1})$ appears as a shift in $\mathbb{F}_2$. So, if we only consider the graded resolution of S/I_Z, we have that there must be a shift of $-2\lambda_1$ in $\mathbb{F}_1$ and a shift of $-i^\star + 1 - \lambda_1 - \lambda_{i^\star - 1} \leq -1 - \lambda_1 - \lambda_1$ in $\mathbb{F}_2$. So $M_1 \geq 2\lambda_1$ and $M_2 \geq 2\lambda_1 + 1$.

Note that $(i^\star, \lambda_{i^\star} + 1)$ is a base corner of Z, and is in fact the smallest corner of Z with respect to the lexicographical ordering. Consider the matrix $(\mathcal{M}_p)$ as defined before Lemma 4.3. It must have the following form:

$$\begin{bmatrix} 2 & 2 & \cdots & 2 & 2 & \cdots & 2 \\ & \vdots & & & & & \\ 2 & 2 & \cdots & 2 & 1 & \cdots & 1 \\ & \vdots & & & & & \end{bmatrix}.$$

That is, the first row contains λ_1 twos, and row $i^\star$ contains $\lambda_i^\star$ twos and $\lambda_1 - \lambda_{i^\star}$ ones. By Theorem 4.4 there is a second syzygy of I_Z whose bidegree is (u, v) where u is the sum of the entries in column $\lambda_{i^\star} + 1$ and v is the sum of the entries in row $i^\star$ of the above matrix. Hence $u \geq 2 + 1$ and $v = 2\lambda_i^\star + (\lambda_1 - \lambda_{i^\star}) = \lambda_1 + \lambda_{i^\star}$. So, in the graded resolution of R/I_Z, there is a shift of $-u - v \leq -3 - \lambda_1 - \lambda_i^\star$, from which we deduce $M_3 \geq \lambda_1 + \lambda_i^\star + 3$. $\square$

With the above lemmas, we now prove Theorem 6.1.

Proof (*Of Theorem* 6.1). Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be the tuple associated to the support Z, and set $d = d(\lambda)$. If $d = 0$, then $\lambda = (\lambda_1, \ldots, \lambda_1)$, and in this case S/I_Z is Cohen–Macaulay of codimension 2, and thus satisfies the bound (6.1) by [19, Corollary 4.2].

So, we can assume that $d \geq 1$. In this case S/I_Z is not ACM because $\beta_3(S/I_Z) = \binom{d+1}{2} > 0$. Before proceeding, we note that $\lambda_1 - 1 \geq d$ and $\lambda_{i^\star} \geq d$. We need to verify (6.1) for $i = 1, 2$ and 3 where $p = 3$ in this case. We consider each case separately.

Case: $i = 1$. In this case, we have

$$\beta_1(S/I_Z) = 2d + 3 + \binom{d+1}{2} = \frac{1}{2}(d+2)(d+3).$$

But $(d + 2) \leq (2d + 3)$ and $(d + 3) \leq (2d + 3)$ for all $d \geq 1$, so

$$\beta_1(S/I_Z) \leq \frac{1}{2}(2d+3)(2d+3) \leq \frac{1}{2}(2(d+1)+1)((d+1)+d+2)$$

$$\leq \frac{1}{2}(2\lambda_1 + 1)(\lambda_1 + \lambda_{i^\star} + 2) \leq \frac{1}{(1-1)!(3-1)!} M_2 M_3.$$

Case: $i = 2$. For this case

$$\beta_2(S/I_Z) = 2d + 2 + 2\binom{d+1}{2} = 2d + 2 + (d+1)d = (d+1)(d+2)$$

$$\leq 2(d+1)(d+3) = (2(d+1))(2(d+2)) = (2(d+1))((d+1)+d+3)$$

$$\leq (2\lambda_1)(\lambda_1 + \lambda_{i^\star} + 3) \leq \frac{1}{(2-1)!(3-2)!} M_1 M_3.$$

Case: $i = 3$. In our final case we have

$$\beta_3(S/I_Z) = \binom{d+1}{2} \le \binom{\lambda_1 + 1}{2} \le \lambda_1(\lambda_1 + 1)$$

$$\le \lambda_1(2\lambda_1 + 1) = \frac{1}{2}2\lambda_1(2\lambda_1 + 1) \le \frac{1}{(3-1)!(3-3)!}M_1M_2.$$

So, the bound (6.1) is satisfied for all i. $\square$

Acknowledgement

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