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Equilibria Trajectories as Stationary Solutions of Infinite-Dimensional Projected Dynamical Systems

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Abstract—We show that the problem of solving a certain class of time dependent variational inequalities is equivalent to the one of finding the stationary solutions of a related class of constrained dynamical systems. We exploit this equivalence to analyse the time dependent complementary problem. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In recent years, several time dependent equilibrium problems related with traffic or economic networks have been studied in the framework of variational inequalities [1–3]. In particular, the problems we are referring to can be cast in the following scheme.

PROBLEM 1. Given a closed and convex set $Z \subset L^2([0,T],R^n)$ and $F : Z \mapsto L^2([0,T],R^n)$, find $x^* \in Z$ such that

$$\int_{[0,T]} \langle F\left[x^*(\tau)\right], z(\tau) - x^*(\tau)\rangle \, d\tau \geq 0, \qquad \forall\, z \in Z. \tag{1.1}$$

A solution $x^*(\tau)$ of (1.1) represents a trajectory of equilibrium points in the sense that, for almost every $\tau \in [0,T]$, $x^*(\tau)$ satisfies a certain equilibrium condition whose meaning depends on the underlying application.

We shall prove that problem (1.1) is equivalent to the following.

PROBLEM 2. Find

$$x^* \in Z : \Pi_Z\left(x^*(\tau), -F\left[x^*(\tau)\right]\right) = 0. \tag{1.2}$$

Here we have defined $\Pi_Z(x(\tau), -F[x(\tau)]) = \lim_{\delta->0}(P_Z(x(\tau) - \delta F[x(\tau)]) - x(\tau))/\delta$, while P_Z is the usual projection operator on a closed convex set.

The innovative idea of connecting *static* variational inequalities with constrained dynamical systems has been introduced by Nagurney and Zhang who also coined the term *projected dynamical systems*. However, when dealing with our class of time dependent variational inequalities, we

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cannot exploit the techniques used in [4,5], because these are meaningful only in finite-dimensional spaces. On the other hand, very recently Gwinner [6] has generalized the notion of a projected dynamical system to an abstract Hilbert space, by using the notion of *quasi-interior* and *quasiboundary* of a closed convex subset of a Hilbert space. Applying Gwinner's results to our $L^2([0,T], R^n)$ space, we are led to the following.

PROBLEM 3. Find those absolutely continuous functions $x : t \in [0,T] \mapsto (x_1(t,\tau), \dots, x_n(t,\tau)) \in Z$ such that for almost every $t \in [0,T]$, there holds

$$\frac{dx(t,\tau)}{dt} = \Pi_Z(x(t,\tau), -F[x(t,\tau)]), \tag{1.3}$$

for almost every $\tau \in [0,T]$. Stationary solutions of (1.3) are defined by the condition

$$\Pi_Z(x(\tau), -F[x(\tau)]) = 0. \tag{1.4}$$

The paper is as follows. In Section 2, we use Gwinner's results to write an explicit formula for the directional derivative of the projection operator in $L^2([0,T], R^n)$ and show how to calculate equilibria trajectory using this formula. We also prove the equivalence between (1.1) and (1.2). Finally, in Section 3, we apply our result to the (*time dependent*) complementary problem, exploiting the particular structure of the *quasi-interior* of Z in this case.

2. THE DIRECTIONAL DERIVATIVE OF THE PROJECTION. AN EQUIVALENCE RESULT

Let

$$\mathcal{L} := L^2([0,T], R^n) \quad \text{and} \quad \langle\langle a(t), b(t)\rangle\rangle = \int_{[0,T]} \langle a(\tau), b(\tau)\rangle \, dt, \qquad \forall\, a, b \in \mathcal{L}.$$

When we want to stress time dependence, we shall write $a(\tau)$ for elements of $\mathcal{L}$. Let us recall some definitions of convex analysis. If $Z \subset \mathcal{L}$ is convex and closed, the *tangent cone* to Z at point $x(\tau)$ is defined as $S_Z(x(\tau)) := Cl\{\bigcup_{\lambda>0} \lambda(Z - x(\tau))\}v$, while the *normal cone* to Z at $x(\tau)$ is defined as

$$N_Z(x(\tau)) := \{\xi(\tau) \in \mathcal{L} : \langle\langle \xi(\tau), z(\tau) - x(\tau)\rangle\rangle \le 0,\ \forall\, z(\tau) \in Z\}.$$

The *tangent cone* and the *normal cone* are polar to each other. The set of *unit inward normals* to Z at $x(\tau)$ is defined as

$$n_Z(x(\tau) = \{v(\tau) : ||v||_{\mathcal{L}} = 1,\ \langle\langle v(\tau), x(\tau) - y(\tau)\rangle\rangle \le 0,\ \forall\, y \in Z\}.$$

Let us recall that $\forall\, x(\tau) \in Z$, $\forall\, h(\tau) \in \mathcal{L}$, and $\delta > 0$, the following asymptotic relation [7] holds:

$$P_Z(x(\tau) + \delta h(\tau)) = x(\tau) + \delta P_{S_Z(x(\tau))} h(\tau) + o(\delta), \qquad \delta \to 0, \tag{2.5}$$

which allows us to write, for the directional derivative of the projection

$$\Pi_Z(x(\tau), h(\tau)) := \lim_{\delta -> 0} \frac{P_Z(x(\tau) + \delta h(\tau)) - x(\tau)}{\delta} = P_{S_Z(x(\tau))} h(\tau). \tag{2.6}$$

Following Borwein and Lewis [8], let us introduce the *quasi-relative interior* of Z, qri$Z$ as the set of those $x(\tau) \in Z$ for which $S_Z(x(\tau))$ is a subspace. In the particular case when $S_Z(x(\tau)) = \mathcal{L}$, we shall denote the same set as the *quasi-interior* of Z, qi$Z$. The set $Z\backslash$qi$Z$ will be denoted as the *quasiboundary* of Z, qbdry$Z$. In view of applications, e.g., to the complementary problem, we shall consider the important case of $Z = \{x(\tau) \in \mathcal{L} : x_j(\tau) \ge 0,\ \forall\, j = 1 \dots n, \text{ a.e. } \tau \in [0,T]\}$.

One can show that the topological interior of Z is empty while qiZ $= \{x(\tau) \in \mathcal{L} : x_j(\tau) > 0, \forall j = 1 \dots n, \text{ a.e. } \tau \in [0,T]\}$.

Now, let us give a geometric interpretation of $\Pi_Z(x(\tau), h(\tau))$.

THEOREM 1.1.

(a) *If $x(\tau) \in qiZ$, then $\forall h(\tau) \in \mathcal{L}$: $\Pi_Z[x(\tau), h(\tau)] = h(\tau)$.*

(b) *If $x(\tau) \in qbdryZ$, then $\forall v(\tau) \in \mathcal{L} \setminus S_Z(x(\tau))$, $\exists n^*(x(\tau)) \in n_Z(x(\tau))$ and $\beta > 0$:*

$$\Pi_Z[x(\tau), v(\tau)] = v(\tau) + \beta[x(\tau)]n^*[x(\tau)],$$

where

$$\beta[x(\tau)] = -\langle\langle v(\tau), n^*[x(\tau)]\rangle\rangle > 0.$$

PROOF.

(a) It follows directly from the definition of *quasi-interior* and (2.6), because in this case, we project $h(\tau)$ onto the whole space.

(b) Let $x(\tau) \in$ qbdry$Z$, and let $u(\tau) := P_{S_Z} v(\tau)$; then from the characterization of the projection and (2.6), we get

$$\langle\langle v(\tau) - u(\tau), w(\tau) - u(\tau)\rangle\rangle \le 0, \qquad \forall w \in S_Z(x).$$

Because $S_Z(x)$ is a cone, we can prove that

$$\langle\langle v(\tau) - u(\tau), u(\tau)\rangle\rangle = 0, \tag{2.7}$$

from which we can derive

$$\langle\langle v(\tau) - u(\tau), w(\tau)\rangle\rangle \le 0, \qquad \forall w(\tau) \in S_Z(x(\tau)),$$

but the last relation means, by the definition of polar cone, that

$$\begin{aligned} v(\tau) - u(\tau) \in (S_Z(x(\tau)))^* &= \{\xi(\tau) \in \mathcal{L} : \langle\langle \xi(\tau), w(\tau)\rangle\rangle \le 0, \ \forall w(\tau) \in S_Z(x(\tau))\} \\ &= N_Z(x(\tau)), \end{aligned} \tag{2.8}$$

where we have also exploited the fact that the polar of the tangent cone is the normal cone.

Let us observe that each element of $N_Z(x(\tau))$ can be written as an element of $n(x(\tau))$ multiplied for a negative constant. Thus, $\exists n^*(x(\tau)) \in n(x(\tau))$, $\beta > 0$, such that $u(\tau) - v(\tau) = \beta n^*(x(\tau))$. From (2.7), one gets immediately $\langle\langle n^*[x(\tau)], u(\tau)\rangle\rangle = 0$, and as a consequence, $\beta = -\langle\langle n^*[x(\tau)], v(\tau)\rangle\rangle$. ∎

A procedure to calculate $u(\tau)$ is given by the following.

COROLLARY 1.1. *Let $x(\tau) \in Z$ and $v(\tau) \in \mathcal{L}$. Then*

$$u[x(\tau)] := \Pi_Z(x(\tau), v(\tau)) = P_{v(\tau) - N_Z(x(\tau))}(0). \tag{2.9}$$

PROOF. If $x(\tau) \in$ qi$Z$, then $N_Z(x) = \{0\}$ and the thesis is immediately reached. On the other hand, if $x(\tau) \in$ qbdry$Z$, we have shown that $v(\tau) - u(\tau) \in N_Z(x(\tau))$, i.e., $u(\tau) \in v(\tau) - N_Z(x(\tau))$, but from (2.6), one has $u(\tau) \in S_Z(x(\tau)) = (N_Z(x(\tau)))^* = \{\xi(\tau) \in \mathcal{L} : \langle\langle \xi(\tau), z(\tau)\rangle\rangle \le 0, \ \forall z(\tau) \in N_Z(x(\tau))\}$, hence, $\langle\langle u(\tau), z(\tau)\rangle\rangle \le 0$, $\forall z \in N_Z(x)$; but from (2.7), one has $\langle\langle u(\tau), u(\tau) - v(\tau)\rangle\rangle = 0$. Finally, $\langle\langle -u(\tau), v(\tau) - u(\tau) - z(\tau)\rangle\rangle \le 0$, $\forall z \in N_Z(x)$, which is equivalent to state that $u(\tau)$ is the projection of the vector 0 on the closed and convex set $v(\tau) - N_Z(x(\tau))$. ∎

Thus, in order to calculate $u(\tau)$, one should be able to describe the normal cone

$$N_Z(x(\tau)) := \left\{ \xi(\tau) \in \mathcal{L} : \int_{[0,T]} \langle \xi(\tau), z(\tau) - x(\tau) \rangle \leq 0, \ \forall\, z(\tau) \in Z \right\},$$

for those $x(\tau) \in \text{qbdry}Z$, and then calculate

$$\begin{aligned} &\operatorname{argmin} \{ \|v - \xi\|_{\mathcal{L}} : v - \xi \in v - N_Z(x) \} \\ &= \operatorname{argmin} \left\{ \int_{[0,T]} |v(\tau) - \xi(\tau)|^2 \, d\tau, \ v(\tau) - \xi(\tau) \in v(\tau) - N_Z(x(\tau)) \right\}. \end{aligned} \tag{2.10}$$

Putting $v(\tau) = -F[x(\tau)]$, we can explicitly write equation (1.2):

$$\operatorname{argmin} \left\{ \int_{[0,T]} |F[x(\tau)] + \xi(\tau)|^2 \, d\tau, \ F[x(\tau)] - \xi(\tau) \in -F[x(\tau)] - N_Z(x(\tau)) \right\} = 0. \tag{2.11}$$

In order to show the above-mentioned equivalence, suppose first that $x^*(\tau)$ is a solution of (1.1). Then, $\forall\, \delta > 0$, there holds $x^*(\tau) = P_Z(x^*(\tau) - \delta F[x^*(\tau)])$, which implies that the derivative of $P_Z(x^*(\tau)) = 0$. Suppose now that $x^*(\tau)$ solves (1.2), and that $x^*(\tau) \in \text{qbdry}Z$, $-F[x^*(\tau)] \notin S_Z(x^*(\tau))$. Then, by Theorem 1, $-F[x^*(\tau)] \in N_Z(x^*(\tau))$, which is equivalent to say that $x^*(\tau)$ solves (1.1). In the other cases, $-F[x^*(\tau)] \in S_Z(x^*(\tau))$, which implies $P_{S_Z(x^*(\tau))}(-F[x^*(\tau)]) = F[x^*(\tau)] = 0$, and the equivalence is completely proved.

3. APPLICATIONS AND EXAMPLES

In this section, we apply our result to the *time dependent* complementary problem, which is encounterd in several applications, for instance, in the theory of spatially distributed market models [4]. In this case, the relevant set is

$$Z = \{ x(\tau) \in \mathcal{L} : x_j(\tau) \geq 0, \ \forall\, j = 1 \dots n, \ \text{a.e. } \tau \in [0,T] \}.$$

One can show that

$$\text{qi}Z = \{ x(\tau) \in \mathcal{L} : x_j(\tau) > 0, \ \forall\, j = 1 \dots n, \ \text{a.e. } \tau \in [0,T] \}.$$

Let us recall our calculation scheme.

(1) Construction of the normal cone in points of the quasiboundary of Z (let us recall that, for points of qi$Z$, the normal cone contains only the zero vector).

(2) $\forall\, x(\tau) \in Z$, calculate

$$u[(x(\tau))] = \operatorname{argmin} \left\{ \int_{[0,T]} |F[x(\tau)] + \xi(\tau)|^2 \, d\tau, \ -F[x(\tau)] - \xi(\tau) \in -F[x(\tau)] - N_Z(x(\tau)) \right\}. \tag{3.12}$$

(3) Solve, in Z, the equation $u[x(\tau)] = 0$.

Let us observe that this method slightly improves (and generalize to infinite dimension) the direct method proposed by Maugeri [9].

Let us describe the *quasiboundary* of Z. The elements of qbdry$Z$ are the vectors $(x_1(t), \dots, x_n(t))$ such that $\forall\, j \in I = \{1, 2, \dots, n\}$, $\exists\, E_j \subset [0,T]$ with $|E_j| \geq 0$ and $|CE_j| \geq 0$:

$$x_j(t) : \begin{cases} > 0, & \text{on } E_j, \\ = 0, & \text{on } CE_j, \end{cases}$$

with the following warnings.

(1) $|CE_j| \neq 0$ for at list one index, otherwise, we obtain the *quasi-interior*.

(2) $|E_j| = 0$ and $|CE_j| = 0$ cannot hold simultaneously.

For the sake of clarity, we prefer to split the analysis in four different cases.

CASE (i). $|E_j| = 0$, $\forall j \in I$. In this case, we obtain the null vector (almost everywhere on $[0,T]$).

CASE (ii). $\exists A, B \subset I, A \cup B = I$, $B \neq \{\phi\}$: $|CE_j| = 0$ if $j \in B$, $|E_j| = 0$, if $j \in A = I \setminus B$. Hence,

$$x_j(t) : \begin{cases} > 0, & \text{on } [0,T], \quad j \in B, \\ = 0, & \text{on } [0,T], \quad j \in A. \end{cases}$$

If $B = \phi$, we reobtain the previous case.

CASE (iii). $\forall j \in I$, $|E_j| > 0$, $|CE_j| > 0$. In this case, $\forall j \in I$, we have

$$x_j(t) : \begin{cases} > 0, & \text{on } E_j, \\ = 0, & \text{on } CE_j. \end{cases}$$

CASE (iv). For some index $j \in C \subset I$, $C \neq \{\phi\}$, $C \neq \{I\}$, we have the previous situation, i.e.,

$$x_j(t) : \begin{cases} > 0, & \text{on } E_j, \qquad |E_j| > 0, \\ = 0, & \text{on } CE_j, \quad Q|CE_j| > 0, \end{cases}$$

while the other components can be almost everywhere zero or almost everywhere positive on $[0,T]$. More formally, $\exists B', A' \subset (I \setminus C)$: $|CE_j| = 0$ if $j \in B'$ (eventually, $B' = \{\phi\}$). $|E_j| = 0$ if $j \in A' = (I \setminus C)\ B'$.

In each point of the *quasiboundary*, we want to calculate the normal cone at Z, which is defined by

$$N_Z(x(\tau)) := \left\{ \xi(\tau) \in \mathcal{L} : \int_{[0,T]} \langle \xi(\tau), z(\tau) - x(\tau) \rangle \, d\tau \leq 0, \ \forall z(\tau) \in Z \right\}. \tag{3.13}$$

We shall carry out calculations in each of the previous four cases and find the following results.

CASE (i). In this case, the normal cone is made up by the vectors whose components are almost everywhere nonpositive on $[0,T]$.

CASE (ii). Here the elements of the normal cone are $\xi_j(\tau) \leq 0$, $\forall j \in A$, a.e. $t \in [0,T]$ and $\xi_j(\tau) = 0$, $\forall j \in B$, a.e. $t \in [0,T]$.

CASE (iii). In this case, we have

$$\xi_j(t) : \begin{cases} = 0, & \text{a.e. on } E_j, \\ \leq 0, & \text{a.e. on } CE_j, \end{cases}$$

CASE (iv). The previous analysis allows us to write down immediately the elements of the normal cone in the last case:

$$\forall j \in C, \ \xi_j(t) : \begin{cases} = 0, & \text{a.e. on } E_j, \\ \leq 0, & \text{a.e. on } CE_j, \end{cases}$$

$$\begin{aligned} &\forall j \in A' \, \xi_j(\tau) \leq 0, && \text{a.e. } \tau \in [0,T], \\ &\forall j \in B' \, \xi_j(\tau) = 0, && \text{a.e.} \tau \in [0,T]. \end{aligned}$$

Let us calculate $u[x(\tau)]$ for $x(\tau) \in \text{qbdry} Z$.

(i) Let $x(\tau) = (0, \dots, 0)$, a.e. $\tau \in [0,T]$, so that $\xi_j(\tau) \leq 0$, a.e. $\tau \in [0,T]$, $\forall j \in I$.
If $F_j[x(\tau)] \leq 0$ on $D_j \subset [0,T]$, $|D_j| > 0$, the minimum is realized if we choose

$$\xi_j(\tau) = 0, \qquad \text{a.e. } \tau \in D_j.$$

If $F_j[x(\tau)] > 0$ on $D'_j \subset [0,T]$, $|D'_j| > 0$, the minimum is realized by choosing

$$\xi_j(\tau) = -F_j[x(\tau)], \qquad \text{a.e. } \tau \in D'_j.$$

Hence,

$$u_j[x(t)] = \begin{cases} -F_j[x(\tau)], & \text{on } D_j, \\ 0, & \text{on } D'_j. \end{cases}$$

(ii) Let

$$x_j(t) : \begin{cases} > 0, & \text{on } [0,T], \quad j \in B, \\ = 0, & \text{on } [0,T], \quad j \in A. \end{cases}$$

Thus, $\xi_j(\tau) \leq 0$, $\forall j \in A$, a.e. $t \in [0,T]$, $\xi_j(\tau) = 0\, \forall j \in B$, a.e. $t \in [0,T]$. For $j \in A$, we have to distinguish two different subcases, as before. For $j \in B$, the j component of the minimum point is: $-F_j[x(\tau)]$. The analysis in the other two cases follows immediately from (i) and (ii).

Let us consider the following example.

$F_1[x_1(\tau), x_2(\tau)] = a(\tau)x_1(\tau) - x_2(\tau)$ $F_2[x_1(\tau), x_2(\tau)] = x_1(\tau) - x_2(\tau) - c(\tau)$, where $c(\tau)$ is positive on $[0,T]$ and $a(\tau)$ is the null function in $[0, T/2]$ and is positive and less than one in $(T/2, T)$. Following our calculation procedure, we find the solutions

$$x_1(t) = \begin{cases} \dfrac{c(\tau)}{(1-a(\tau))}, & \text{on } \left(\dfrac{T}{2}, T\right], \\ x_1(\tau) > c(\tau), & \text{on } \left[0, \dfrac{T}{2}\right], \end{cases}$$

$$x_2(t) = \begin{cases} \dfrac{a(\tau)c(\tau)}{(1-a(\tau))}, & \text{on } \left(\dfrac{T}{2}, T\right], \\ 0, & \text{on } \left[0, \dfrac{T}{2}\right]. \end{cases}$$

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