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Mixed Morrey spaces and their applications to partial differential equations

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ABSTRACT

In this paper, new classes of functions are defined. These spaces generalize Morrey spaces and give a refinement of Lebesgue spaces. Some embeddings between these new classes are also proved. Finally, the authors apply these classes of functions to obtain regularity results for solutions of partial differential equations of parabolic type.

1. Introduction

This paper aims at defining new spaces and to study some embeddings between them. We will refer to them with the symbol $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. As applications we obtain some estimates, in these classes of functions, for the solutions of partial differential equations of parabolic type in nondivergence form. Preparatory to achieving these results is the study of the behavior of Hardy–Littlewood Maximal function, Riesz potential, Sharp and Fractional maximal functions, Singular integral operators with Calderón–Zygmund kernel and Commutators (see e.g. [22,23]).

We stress that are obtained results, known in L^p, in a new class of functions that can be view as an extension of the Morrey class introduced in 1966 in [17], and used by a lot of authors, see e.g. in [3], recently in [24,20,12,14,13] and others.

Let us point out that in doing this we need an extension to $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ of a celebrated inequality of Fefferman and Stein (see [10]) concerning the Sharp and the Maximal function (Theorem 4.5) and, also, we study the behavior of Riesz potential in the new class of functions, obtaining an extension of both a known estimate originally proved by Adams in [1] as well as of a result announced by Peetre in [19].

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2. Definitions and preliminary tools

In the sequel let $T > 0$ and Ω be a bounded open set of $\mathbb{R}^n$ such that $\exists A > 0 : \forall x \in \Omega$ and $0 \le \rho \le \; diam\,(\Omega)$, $|Q(x,\rho) \cap \Omega| \ge A\,\rho^n$, being $Q(x,\rho)$ a cube centered in x, having edges parallel to the coordinate axes and length 2ρ.

Definition 2.1. Let $1 < p < +\infty,\; 0 < \lambda < n$ and f be a real measurable function defined in $\Omega \subset \mathbb{R}^n$.

If $|f|^p$ is summable in Ω and the set described by the quantity

$$\frac{1}{\rho^\lambda} \int_{\Omega \cap B_\rho(x)} |f(y)|^p \, dy, \tag{2.1}$$

when changing of ρ in $]0,\; diam\,\Omega[$ and $x \in \Omega$, has an upper bound, then we say that f belongs to the *Morrey Space* $L^{p,\lambda}(\Omega)$.

If $f \in L^{p,\lambda}(\Omega)$, we define

$$\|f\|^p_{L^{p,\lambda}(\Omega)} := \sup_{\substack{x\in\Omega \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{\Omega \cap B_\rho(x)} |f(y)|^p \, dy \tag{2.2}$$

and the vector space naturally associated to the set of functions in $L^p(\Omega)$ such that (2.2) is finite, endowed with the norm (2.2), is a normed and complete space.

The exponent λ can take values that are not belonging to $]0,n[$ but the unique cases of real interest are that one for which $\lambda \in]0,n[$.

The above defined spaces are used, among others, in the theory of regular solutions to nonlinear partial differential equations and for the study of local behavior of solutions to nonlinear equations and systems (see e.g. [17,18]).

Remark 2.1. Similarly we can define the Morrey space in $L^{p,\lambda}(\mathbb{R}^n)$ as the space of functions such that is finite:

$$\|f\|^p_{L^{p,\lambda}(\mathbb{R}^n)} := \sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f(y)|^p \, dy. \tag{2.3}$$

Definition 2.2. Let $1 < p,q < +\infty,\; 0 < \lambda,\mu < n$. We define the set $L^{q,\mu}(0,T,L^{p,\lambda}(\Omega))$ as the class of functions f such that is finite:

$$\|f\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\Omega))} := \left(\sup_{\substack{t_0,t\in(0,T) \\ \rho>0}} \frac{1}{\rho^\mu} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\Omega \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{\Omega\cap B_\rho(x)} |f(y,t)|^p \, dy \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}, \tag{2.4}$$

with obvious modifications if $\Omega = \mathbb{R}^n$.

Definition 2.3. Let f be a locally integrable function defined on $\mathbb{R}^n$. We say that f is in the space $BMO(\mathbb{R}^n)$ (see [15]) if

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int_B |f(y) - f_B| dy < \infty$$

where B runs over the class of all balls in $\mathbb{R}^n$ and $f_B = \frac{1}{|B|}\int_B f(y)dy$.

Let $f \in BMO(\mathbb{R}^n)$ and $r > 0$. We define the VMO modulus of f by the rule

$$\eta(r) = \sup_{\rho \le r} \frac{1}{|B_\rho|} \int_{B_\rho} |f(y) - f_{B_\rho}| dy$$

where B_ρ is a ball with radius ρ, $\rho \le r$.

BMO is a Banach space with the norm $\|f\|_* = \sup_{r>0} \eta(r)$.

Definition 2.4. We say that a function $f \in BMO$ is in the Sarason class $VMO(\mathbb{R}^n)$ (see [25]) if

$$\lim_{r \to 0^+} \eta(r) = 0.$$

Definition 2.5. Let Σ the unit sphere: $\Sigma = \{x \in \mathbb{R}^{n+1}, |x| = 1\}$.

The function $k : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}$ is the classical Calderón–Zygmund kernel if:

(1) $k \in C^\infty(\mathbb{R}^{n+1} \backslash \{0\})$;
(2) $k(\mu x_1, \mu x_2, \ldots, \mu x_n, \mu^2 t) = \mu^{-(n+2)} k(x)$, for each $\mu > 0$;
(3) $\int_\Sigma |k(x)| \, d\sigma_x < \infty$ and $\int_\Sigma k(x) \, d\sigma_x = 0$.

The above definition, in particular condition (2), suggest to endow $\mathbb{R}^{n+1}$ with a metric, different to the standard Euclidean one. Thus let us consider, as Fabes and Riviére in the celebrated paper [9], the following distance $d(x, y) = \rho(x - y)$ between two generic points $x, y \in \mathbb{R}^{+1}n$ (used e.g. in [2]),

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + 4t^2}}{2}}, \quad x = (x', t) = (x'_1, \ldots, x'_n, t) \in \mathbb{R}^{n+1}. \tag{2.5}$$

Then $\mathbb{R}^{n+1}$, endowed with this metric, is a metric space.

Definition 2.6. The function $k(x, y) : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}$ is a *variable* Calderón–Zygmund kernel if:

(1) $k(x, \cdot)$ is a kernel in the sense of the above Definition 2.5, for a.e. $x \in \mathbb{R}^{n+1}$
(2) $\sup_{\rho(y)=1} \left| \left(\frac{\partial}{\partial y}\right)^\beta k(x, y) \right| \le c(\beta)$, for every multi-index β, independently of x.

Next proposition is proved in [21] (see also [4] or [16]), it is useful to recall the statement and the technique used in the proof, because it will inspire us to techniques contained therein, for subsequent results.

Proposition 2.1. *If* $1 < q < p < \infty$, $0 < \lambda < \mu < n$, $q = \frac{(n-\mu)\,p}{(n-\lambda)}$. *The following embedding is true*

$$L^{p,\lambda}(\Omega) \subset L^{q,\mu}(\Omega). \tag{2.6}$$

Proof. Applying Hölder inequality, we have

$$\int_{\Omega \cap B_\rho(x)} |f|^q(y) dy \le \left(\int_{\Omega \cap B_\rho(x)} |f|^{q \cdot \frac{p}{q}}(y) dy \right)^{\frac{q}{p}} \cdot |B_\rho|^{1-\frac{q}{p}} = C \left(\int_{\Omega \cap B_\rho(x)} |f|^p(y) dy \right)^{\frac{q}{p}} \cdot \rho^{n \cdot \left(1-\frac{q}{p}\right)} \tag{2.7}$$

$$= C \rho^{n \cdot \left(1-\frac{q}{p}\right)} \cdot \left(\frac{1}{\rho^\lambda} \int_{\Omega \cap B_\rho(x)} |f|^p(y) dy \right)^{\frac{q}{p}} \cdot \rho^{\lambda \cdot \frac{q}{p}} \tag{2.8}$$

$$\le C \rho^{n - n \cdot \frac{q}{p} + \lambda \cdot \frac{q}{p}} \cdot \|f\|_{L^{p,\lambda}(\Omega)} \tag{2.9}$$

$$= C \rho^\mu \cdot \|f\|^q_{L^{p,\lambda}(\Omega)}; \tag{2.10}$$

then we obtain

$$\frac{1}{\rho^{\mu}}\int_{\Omega\cap B_\rho(x)}|f|^q(y)dy \le C\cdot \|f\|^q_{L^{p,\lambda}(\Omega)}, \tag{2.11}$$

where

$$\mu = n - n\cdot\frac{q}{p} + \lambda\cdot\frac{q}{p}$$

and, obviously, we have

$$\frac{n-\mu}{n-\lambda} = \frac{q}{p}$$

and the conclusion follows.

Remark 2.2. It is possible to extend the previous result considering $1 \le q \le p < \infty$ and $0 \le \lambda, \mu < n$ such that $\frac{n-\mu}{q} \ge \frac{n-\lambda}{p}$.

3. Embedding results

Theorem 3.1. *Let* $1 < p < +\infty,\ 0 < \lambda < n,\ 1 < q < q_1 < \infty,\ 0 < \mu_1 < \mu < 1$ *and* $q = \frac{(1-\mu)\,q_1}{(1-\mu_1)}$, *we have*

$$L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\Omega)) \subset L^{q,\mu}(0,T,L^{p,\lambda}(\Omega)). \tag{3.1}$$

Proof. Let us suppose that $f \in L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\Omega))$, then is finite

$$\left(\sup_{\substack{t_0\in(0,T)\\ \rho>0}}\frac{1}{\rho^{\mu_1}}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\Omega\\ \rho>0}}\frac{1}{\rho^{\lambda}}\int_{\Omega\cap B_\rho(x)}|f(y,t)|^p\,dy\right)^{\frac{q_1}{p}}dt\right)^{\frac{1}{q_1}}. \tag{3.2}$$

Let us set $t \in (0,T)$ and apply Hölder inequality

$$\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\Omega\\ \rho>0}}\frac{1}{\rho^{\lambda}}\int_{\Omega\cap B_\rho(x)}|f(y,t)|^p\,dy\right)^{\frac{q}{p}}dt \tag{3.3}$$

$$\le \left(\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\Omega\\ \rho>0}}\frac{1}{\rho^{\lambda}}\int_{\Omega\cap B_\rho(x)}|f(y,t)|^p\,dy\right)^{\frac{q}{p}\cdot\frac{q_1}{q}}dt\right)^{\frac{q}{q_1}} |(0,T)\cap(t_0-\rho;t_0+\rho)|^{1-\frac{q}{q_1}} \tag{3.4}$$

$$= C\left(\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\Omega\\ \rho>0}}\frac{1}{\rho^{\lambda}}\int_{\Omega\cap B_\rho(x)}|f(y,t)|^p\,dy\right)^{\frac{q}{p}\cdot\frac{q_1}{q}}dt\right)^{\frac{q}{q_1}}\cdot\rho^{\left(1-\frac{q}{q_1}\right)} \tag{3.5}$$

$$= C\rho^{\left(1-\frac{q}{q_1}\right)}\left(\frac{1}{\rho^{\mu_1}}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\Omega\\ \rho>0}}\frac{1}{\rho^{\lambda}}\int_{\Omega\cap B_\rho(x)}|f(y,t)|^p\,dy\right)^{\frac{q_1}{p}}dt\right)^{\frac{q}{q_1}}\cdot\rho^{\mu_1\cdot\frac{q}{q_1}} \tag{3.6}$$

$$= C\|f\|^q_{L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\Omega))}\cdot\rho^{1-\frac{q}{q_1}+\mu_1\cdot\frac{q}{q_1}}. \tag{3.7}$$

Let

$$\mu = 1 - \frac{q}{q_1} + \mu_1\cdot\frac{q}{q_1} = 1-(1-\mu_1)\frac{q}{q_1}, \tag{3.8}$$

$$\frac{1-\mu}{1-\mu_1} = \frac{q}{q_1}; \tag{3.9}$$

it follows, as request, that

$$q = \frac{(1-\ \mu)q_1}{1-\mu_1}, \tag{3.10}$$

and the proof is complete.

Remark 3.2. It is possible to extend the previous result considering $1 < q \le q_1 < \infty$, $0 < \mu_1 \le \mu < 1$ or $1 < \mu_1 \le \mu < n$ and

$$\frac{1-\mu}{q} \ge \frac{1-\mu_1}{q_1}. \tag{3.11}$$

Theorem 3.3. *Let* $1 < q < p < \infty$, $0 < \lambda < \mu < n$, $q = \frac{(n-\mu)\,p}{(n-\lambda)}$, $1 < q_2 < q_1 < \infty$, $0 < \mu_1 < \mu_2 < 1$ *or* $1 < \mu_1 < \mu_2 < n$ *and* $q_2 = \frac{(1-\mu_2)q_1}{(1-\mu_1)}$, *we have*

$$L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\Omega)) \subset L^{q_2,\mu_2}(0,T,L^{q,\mu}(\Omega)). \tag{3.12}$$

Proof. Let us set $t \in (0,T)$. If $1 < q < p < \infty$, $0 < \lambda < \mu < n$ and $q = \frac{(n-\mu)\,p}{(n-\lambda)}$, we have, from Proposition 2.1,

$$\frac{1}{\rho^\mu}\int_{\Omega\cap B_\rho(x)} |f|^q(y,t)dy \le C\left(\sup_{\substack{x\in\Omega\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{\Omega\cap B_\rho(x)} |f|^{q\cdot\frac{p}{q}}(y,t)dy\right)^{\frac{q}{p}}. \tag{3.13}$$

Let us fix $t_0 \in (0,T)$, then, integrating in $(0,T)\cap(t_0-\rho,t_0+\rho)$, we have

$$\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\Omega\\ \rho>0}} \frac{1}{\rho^\mu}\int_{\Omega\cap B_\rho(x)} |f|^q(y,t)dy\right)^{\frac{1}{q}\cdot q_2} dt \tag{3.14}$$

$$\le C\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\Omega\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{\Omega\cap B_\rho(x)} |f|^{q\cdot\frac{p}{q}}(y,t)dy\right)^{\frac{1}{p}\cdot q_2} dt \tag{3.15}$$

applying Hölder inequality, we have

$$\le C\left(\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\Omega\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{\Omega\cap B_\rho(x)} |f|^p(y,t)dy\right)^{\frac{q_2}{p}\cdot\frac{q_1}{q_2}} dt\right)^{\frac{q_2}{q_1}} \cdot \rho^{1-\frac{q_2}{q_1}} \tag{3.16}$$

$$= C\left(\frac{1}{\rho^{\mu_1}}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\Omega\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{\Omega\cap B_\rho(x)} |f|^p(y,t)dy\right)^{\frac{q_1}{p}} dt\right)^{\frac{q_2}{q_1}} \cdot \rho^{1-\frac{q_2}{q_1}+\mu_1\cdot\frac{q_2}{q_1}} \tag{3.17}$$

$$= C\|f\|^{q_2}_{L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\Omega))} \cdot \rho^{\mu_2} \tag{3.18}$$

where

$$\mu_2 = 1-(1-\mu_1)\cdot\frac{q_2}{q_1}, \tag{3.19}$$

then

$$\frac{1-\mu_2}{1-\mu_1} = \frac{q_2}{q_1}; \tag{3.20}$$

it follows

$$q_2 = \frac{(1- \mu_2)q_1}{(1-\mu_1)}. \tag{3.21}$$

Then, we obtain

$$\left(\frac{1}{\rho^{\mu_2}} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\Omega\\ \rho>0}} \frac{1}{\rho^{\mu}} \int_{\Omega\cap B_\rho(x)} |f|^q(y,t)dy \right)^{\frac{q_2}{q}} dt \right)^{\frac{1}{q_2}} \leq C\|f\|_{L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\Omega))} \tag{3.22}$$

and, finally

$$\|f\|_{L^{q_2,\mu_2}(0,T,L^{q,\mu}(\Omega))} \leq C \|f\|_{L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\Omega))}. \tag{3.23}$$

Remark 3.4. It is possible to extend the previous result considering $1 < q \leq p < \infty,\ 0 < \lambda \leq \mu < n,\ 1 < q_2 \leq q_1 < \infty,\ 0 < \mu_1 \leq \mu_2 < 1$ or $1 < \mu_2 \leq \mu_1 < n$ and

$$\frac{n-\mu}{q} \geq \frac{n-\lambda}{p}; \qquad \frac{1-\mu_2}{q_2} \geq \frac{1-\mu_1}{q_1}. \tag{3.24}$$

4. Main results

4.1. Estimate of some integral operators

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and recall the following Hardy–Littlewood maximal function

$$M\,f\,(x) = \sup_{\rho>0} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |f(y)|\,dy \tag{4.1}$$

where $B_\rho(x)$ is a ball centered at x and with radius ρ.

Proposition 4.1. *Let $1 < p < +\infty,\ 0 < \lambda < n$. Then*

$$\|M\,f\|_{L^{p,\lambda}(\mathbb{R}^n)} \leq C\ \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \tag{4.2}$$

where C is independent of f.

Let us now extend the previous result as follows.

Theorem 4.1. *Let $1 < p < +\infty,\ 0 < \lambda < n,\ 1 < q' < +\infty,\ 0 < \mu < 1$ or $1 < \mu < n$ and $f \in L^{q',\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. Then,*

$$\|M\,f\|_{L^{q',\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \leq C\ \|f\|_{L^{q',\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))}. \tag{4.3}$$

Proof. Let $t \in (0,T)$. From [10] (Lemma 1, pg. 111), we have

$$\int_{\mathbb{R}^n} |M\,f(y,t)|^p \chi(y)\,dy \leq c \int_{\mathbb{R}^n} |\,f(y,t)|^p (M\,\chi)(y)\,dy \tag{4.4}$$

for any function f and χ the characteristic function of a ball $B_\rho(x) \subset \mathbb{R}^n$, being the constant c independent of f. Then

$$\int_{B_\rho(x)} |M\,f(y,t)|^p\,dy \leq \int_{B_{2\rho}(x)} |f(y,t)|^p (M\,\chi(y))dy + \sum_{k=1}^{+\infty} \int_{B_{2^{k+1}\rho} \setminus B_{2^k\rho}(x)} |f(y,t)|^p (M\,\chi(y))\,dy \tag{4.5}$$

it follows

$$\frac{1}{\rho^\lambda}\int_{B_\rho(x)}|M\,f(y,t)|^p\,dy$$
$$\leq C\frac{1}{(2\rho)^\lambda}\int_{B_{2\,\rho}(x)}|f(y,t)|^p(M\,\chi(y))\,dy+C\sum_{k=1}^{+\infty}\frac{1}{(2^{k+1}\rho)^\lambda}\int_{B_{2^{k+1}\,\rho}(x)}|f(y,t)|^p(M\,\chi(y))\,dy, \tag{4.6}$$

using the method applied in [5] and considering the supremum for $x\in\mathbb{R}^n$ and $\rho>0$. Let us fix $t_0\in(0,T)$, then, elevating to $\frac{q'}{p}$, integrating in $(0,T)\cap(t_0-\rho,t_0+\rho)$ and multiplying for $\rho^{-\mu}$, we obtain

$$\frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}}\frac{1}{\rho^\lambda}\int_{B_\rho(x)}|M\,f(y,t)|^p\,dy\right)^{\frac{q'}{p}}dt \tag{4.7}$$

$$\leq C\frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}}\frac{1}{\rho^\lambda}\int_{B_\rho(x)}|f(y,t)|^p\,dy\right)^{\frac{q'}{p}}dt \tag{4.8}$$

taking the supremum, in both sides, for $t_0\in(0,T)$ and $\rho>0$, we obtain

$$\left[\sup_{\substack{t_0\in(0,T)\\ \rho>0}}\frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}}\frac{1}{\rho^\lambda}\int_{B_\rho(x)}|M\,f(y,t)|^p\,dy\right)^{\frac{q'}{p}}dt\right]^{\frac{1}{q'}} \tag{4.9}$$

$$\leq\left[\sup_{\substack{t_0,t\in(0,T)\\ \rho>0}}\frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}}\frac{1}{\rho^\lambda}\int_{B_\rho(x)}|\,f(y,t)|^p\,dy\right)^{\frac{q'}{p}}dt\right]^{\frac{1}{q'}} \tag{4.10}$$

or, equivalently

$$\|M\,f\|_{L^{q',\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))}\leq C\,\|f\|_{L^{q',\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))}. \tag{4.11}$$

As application of this result we prove some estimates of the Riesz potential in $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ spaces.

Let us set $t\in(0,T)$ and consider, for $0<\alpha<n$, the fractional integral operator of order α,

$$I_\alpha f(x,t)=\int_{\mathbb{R}^n}\frac{f(y,t)}{|x-y|^{n-\alpha}}dy,\quad \text{a.e. in } \mathbb{R}^n. \tag{4.12}$$

Theorem 4.2. *Let* $0<\alpha<n,\ 1<p<\frac{n}{\alpha,},\ 0<\lambda<n-\alpha\,p,\ \frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n-\lambda}\ 1<q'<+\infty,\ 0<\mu'<1$ *and* $f\in L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. *Then,*

$$\|I_\alpha\,f\|_{L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))}\leq C\,\|f\|_{L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))}. \tag{4.13}$$

Proof. Let us fix $x\in\mathbb{R}^n,\ t_0\in(0,T)$ and $f\in L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. Then, set $t\in(0,T)$,

$$(I_\alpha f)(x,t)=\int_{|x-y|\leq\epsilon}\frac{f(y,t)}{|x-y|^{n-\alpha}}dy+\int_{|x-y|>\epsilon}\frac{f(y,t)}{|x-y|^{n-\alpha}}dy=I_1+I_2, \tag{4.14}$$

estimating separately each integral I_1 and I_2, as in [1] (Theorem 3.1) or [5] (Theorem 2), we obtain

$$|I_\alpha f|(x,t)\leq C(Mf)^{\frac{n-\lambda-\alpha p}{n-\lambda}}(x)\cdot\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}}\frac{1}{\rho^\lambda}\int_{B_\rho(x)}|f(y,t)|^p dy\right)^{\frac{1}{p}\cdot\frac{\alpha p}{n-\lambda}}, \tag{4.15}$$

recalling that $\frac{n-\lambda-\alpha p}{n-\lambda} = \frac{p}{q}$, elevating to the power q, integrating in $B_\rho(x)$ and multiplying to $\rho^{-\lambda}$, we have

$$\frac{1}{\rho^\lambda}\int_{B_\rho(x)} |I_\alpha f(y,t)|^q dy \le \frac{1}{\rho^\lambda}\int_{B_\rho(x)} (Mf)^p(y,t)\cdot\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p dy\right)^{\frac{\alpha q}{n-\lambda}} dy \tag{4.16}$$

applying Theorem 4.1, and observing that $\frac{\alpha q}{n-\lambda} + 1 = \frac{q}{p}$,

$$\le C\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p dy\right)^{\frac{\alpha q}{n-\lambda}}\cdot\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p dy\right) \tag{4.17}$$

$$= C\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p dy\right)^{\frac{q}{p}}, \tag{4.18}$$

then

$$\frac{1}{\rho^\lambda}\int_{B_\rho(x)} |I_\alpha f(y,t)|^q dy \le C\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p dy\right)^{\frac{q}{p}}, \tag{4.19}$$

considering the supremum for $x \in \mathbb{R}^n$ and $\rho > 0$ and elevating both members to $\frac{1}{q}$, we have

$$\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |I_\alpha f(y,t)|^q dy\right)^{\frac{1}{q}} \le C\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p dy\right)^{\frac{1}{p}}. \tag{4.20}$$

Now, elevating to q', integrating in $(0,T)\cap(t_0-\rho, t_0+\rho)$ and multiplying to $\rho^{-\mu}$, we have

$$\frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |I_\alpha f(y,t)|^q dy\right)^{\frac{q'}{q}} dt \tag{4.21}$$

$$\le C\frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p dy\right)^{\frac{q'}{p}} dt, \tag{4.22}$$

taking the supremum for $t_0 \in (0,T), \rho > 0$, we have

$$\sup_{\substack{t_0\in(0,T)\\ \rho>0}} \frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |I_\alpha f(y,t)|^q dy\right)^{\frac{q'}{q}} dt \tag{4.23}$$

$$\le C\sup_{\substack{t_0\in(0,T)\\ \rho>0}} \frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p dy\right)^{\frac{q'}{p}} dt. \tag{4.24}$$

Finally, elevating to $\frac{1}{q'}$ we have

$$\|I_\alpha f\|_{L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))} \le C\,\|f\|_{L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))}. \tag{4.25}$$

Corollary 4.3. *Let $0 < \alpha < n,\ 1 < p < \frac{n}{\alpha,},\ 0 < \lambda < n - \alpha p$. Let us also set $1 < q < p$ such that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\lambda < \mu < n$ such that $\mu = \frac{n\lambda}{(n-\alpha p)},\ 1 < q' < +\infty,\ 0 < \mu' < 1$ and $f \in L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. Then,*

$$\|I_\alpha f\|_{L^{q',\mu'}(0,T,L^{q,\mu}(\mathbb{R}^n))} \le C\,\|f\|_{L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \tag{4.26}$$

where C is independent of f.

Proof. Let us fix $x \in \mathbb{R}^n$ and $t \in (0,T)$. From Corollary in [5], we have

$$\left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^{\mu}} \int_{B_\rho(x)} |I_\alpha f(y,t)|^q dy \right)^{\frac{1}{q}} \le C \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^{\lambda}} \int_{B_\rho(x)} |f(y,t)|^p dy \right)^{\frac{1}{p}}, \tag{4.27}$$

elevating to q', fixing $t_0 \in (0,T)$, integrating in $(0,T) \cap (t_0-\rho, t_0+\rho)$ and multiplying for $\rho^{-\mu'}$,

$$\left(\sup_{\substack{t_0\in(0,T) \\ \rho>0}} \frac{1}{\rho^{\mu'}} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^{\lambda}} \int_{B_\rho(x)} |I_\alpha f(y,t)|^q dy \right)^{\frac{q'}{q}} dt \right)^{\frac{1}{q'}} \tag{4.28}$$

$$\le C \left(\sup_{\substack{t_0\in(0,T) \\ \rho>0}} \frac{1}{\rho^{\mu'}} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^{\lambda}} \int_{B_\rho(x)} |f(y,t)|^p dy \right)^{\frac{q'}{p}} dt \right)^{\frac{1}{q'}} \tag{4.29}$$

that is

$$\|I_\alpha f\|_{L^{q',\mu'}(0,T,L^{q,\mu}(\mathbb{R}^n))} \le C\, \|f\|_{L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))}. \tag{4.30}$$

One more application of the technique used in the proof of Theorem 4.1 is the following result, where we set T a convolution singular integral operator $T = k * f$, where k is a usual Calderón–Zygmund kernel, studied by Coifman and Fefferman in [6].

Theorem 4.4. *Let* $1 < p < \infty,\ 0 < \lambda < n\ \ 1 < q' < +\infty,\ 0 < \mu' < 1$ *and* $f \in L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. *Then,*

$$\|T f\|_{L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))} \le C\, \|f\|_{L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))}. \tag{4.31}$$

Proof. Let us fix $x \in \mathbb{R}^n,\ t \in (0,T), f \in L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ and χ the characteristic function of a ball $B_\rho(x)$. Then, from a result by Coifman and Rochberg (see [7] pg. 251), $M(M\chi)^\gamma \le c\,(M\chi)^\gamma$, then $(M\chi)^\gamma$ is a A_1 weight.

It follows, from a result contained in [6], that

$$\int_{B_\rho(x)} |T f(y,t)|^p\, dy \le \int_{\mathbb{R}^n} |T f(y,t)|^p (M\,\chi(y))^\gamma dy \le C \int_{\mathbb{R}^n} |f(y,t)|^p (M\,\chi(y))^\gamma\, dy, \tag{4.32}$$

estimating the last term following the lines of the proof of Theorem 4.1, we get the conclusion.

Before we prove the next results we need to consider two variants of the Hardy–Littlewood maximal operator, that are the Sharp Maximal function and the Fractional maximal functions (see e.g. [8]).

Definition 4.1. Given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ let us define the following Sharp Maximal function

$$f^\sharp(x) = \sup_{B\supset\{x\}} \frac{1}{|B|} \int_B |f(y) - f_B|\, dy, \tag{4.33}$$

for a.e. $x \in \mathbb{R}^n$, where B is a generic ball in $\mathbb{R}^n$.

Definition 4.2. Set $t \in (0,T),\ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $0 < \eta < 1$. Let us define the Fractional Maximal function

$$(M_\eta f)(x) = \sup_{B\supset\{x\}} \frac{1}{|B|^{1-\eta}} \int_B |f(y,t) - f_B|\, dy, \tag{4.34}$$

for a.e. $x \in \mathbb{R}^n$, where B is a generic ball in $\mathbb{R}^n$.

The next theorem is a generalization of a well known inequality by Fefferman and Stein, see [10], pg. 153.

Theorem 4.5. *Let* $1 < p, q < \infty,\ 0 < \lambda, \mu < n$ *and* $f \in L^{q,\mu}(0, T, L^{p,\lambda}(\mathbb{R}^n))$.

Then, there exists a constant $C \geq 0$ *independent of* f *such that*

$$\|M\, f\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \leq C\, \|f^\sharp\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))}. \tag{4.35}$$

Proof. Let us fix $x \in \mathbb{R}^n, t \in (0, T)$. Let us also consider $\rho > 0, \gamma \in]\frac{\lambda}{n}; 1[,\ \ \chi = \chi_{B_\rho(x)},\ \forall x \in \mathbb{R}^n$ the characteristic function of a ball $B_\rho(x)$. We know that $(M\chi)^\gamma \in A_1$ and, from [11] pg. 410, we have

$$\int_{\mathbb{R}^n} (M\, f)^p(y, t)\omega(y)dy \leq C \int_{\mathbb{R}^n} |f^\sharp(y, t)|^p \omega(y)\, dy, \quad \forall \omega \in A_\infty, \forall f \in L^p_\omega(\mathbb{R}^n) \tag{4.36}$$

where $L^p_\omega(\mathbb{R}^n)$ is the L^p space with respect to the measure $d\mu = \omega\, dx$. We can use this inequality because $f \in L^{p,\lambda}(\mathbb{R}^n)$ implies $f \in L^p_{(M\chi)^\gamma}(\mathbb{R}^n)$ (see the calculation in [5] pg. 275).

Choosing $\omega(y) = (M\chi)^\gamma(y)$, we have, from [8] pg. 327,

$$\int_{B_\rho(x)} (M\, f)^p(y, t)dy \leq \int_{\mathbb{R}^n} (M\, f)^p(y, t)(M\, \chi)^\gamma(y)dy \tag{4.37}$$

$$\leq C \cdot \int_{\mathbb{R}^n} |f^\sharp(y, t)|^p (M\, \chi)^\gamma(y)\, dy \leq C\rho^\lambda \sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f^\sharp(y, t)|^p\, dy, \quad \forall f \in L^p_\omega(\mathbb{R}^n) \tag{4.38}$$

then

$$\frac{1}{\rho^\lambda} \int_{B_\rho(x)} (M\, f)^p(y, t)dy \leq C \sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f^\sharp(y, t)|^p\, dy, \quad \forall f \in L^p_\omega(\mathbb{R}^n) \tag{4.39}$$

and, taking the supremum for $x \in \mathbb{R}^n$ and $\rho > 0$ we have

$$\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} (M\, f)^p(y, t)dy \leq C \sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f^\sharp(y, t)|^p\, dy, \tag{4.40}$$

set $t_0 \in (0, T)$, elevating to $\frac{q}{p}$, integrating in $(0, T) \cap (t_0 - \rho, t_0 + \rho)$ and multiplying for $\rho^{-\mu}$, we have

$$\frac{1}{\rho^\mu} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} (M\, f(y, t))^p(y, t)dy \right)^{\frac{q}{p}} dt \tag{4.41}$$

$$\leq C \frac{1}{\rho^\mu} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f^\sharp(y, t)|^p\, dy \right)^{\frac{q}{p}} dt \tag{4.42}$$

then, we obtain

$$\left(\sup_{\substack{t_0\in(0,T) \\ \rho>0}} \frac{1}{\rho^\mu} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} (M\, f(y, t))^p(y, t)dy \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} \tag{4.43}$$

$$\leq C \left(\sup_{\substack{t_0\in(0,T) \\ \rho>0}} \frac{1}{\rho^\mu} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f^\sharp(y, t)|^p\, dy \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} \tag{4.44}$$

and we get the conclusion.

Theorem 4.6. *Let $1 < p, q, q_1 < \infty,\ 0 < \lambda, \mu_1 < n$ and $f \in L^{q_1,\mu_1}(0, T, L^{p,\lambda}(\mathbb{R}^n))$.*

Then, for every $\eta \in]0, (1-\frac{\lambda}{n})\frac{1}{p}[$, there exists a constant $C \geq 0$ independent of f such that

$$\|M_\eta f\|_{L^{q_1,\mu_1}(0,T,L^{q,\lambda}(\mathbb{R}^n))} \leq C\ \|f\|_{L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \tag{4.45}$$

where

$$\frac{1}{q} = \frac{1}{p} - \frac{n\,\eta}{n-\lambda}. \tag{4.46}$$

Proof. Let $x \in \mathbb{R}^n$ and $t_0 \in (0, T)$.

Let us fix $1 < r < p$ and

$$\varepsilon = \frac{\left(1-\frac{\lambda}{n}\right)\cdot\frac{p}{n} - \eta}{\left(1-\frac{\lambda}{n}\right)\frac{1}{p}}. \tag{4.47}$$

Set $t \in (0, T)$, for a generic ball B of $\mathbb{R}^n$, we have

$$\frac{1}{|B|^{1-\eta}} \int_B |f(y,t)|dy \tag{4.48}$$

$$\leq \left(\frac{1}{|B|}\int_B |f(y,t)|^r dy\right)^{\frac{\varepsilon}{r}} \cdot \left(\frac{1}{|B|^{\frac{\lambda}{n}}}\int_B |f(y,t)|^p dy\right)^{\frac{(1-\varepsilon)}{p}} \tag{4.49}$$

then

$$\frac{1}{|B|^{1-\eta}}\int_B |f(y,t)|dy \leq\ [M(|f|^r)]^{\frac{\varepsilon}{r}}(y,t)\cdot\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_B |f(y,t)|^p dy\right)^{1-\varepsilon} \tag{4.50}$$

from which it follows

$$(M_\eta(f))^{\frac{p}{\varepsilon}}(y,t) \leq (M(|f|^r))^{\frac{p}{r}}(y,t)\cdot\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}^{\frac{(1-\varepsilon)}{\varepsilon}\cdot p} \quad \text{a.e. } y \in \mathbb{R}^n,\ t \in (0,T). \tag{4.51}$$

Denoting by $\chi(y) = \chi_{B_\rho(x)}(y)$ we have

$$\int_{\mathbb{R}^n} (M_\eta(f))^{\frac{p}{\varepsilon}}(y,t)\cdot\chi(y)\,dy \leq \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}^{\frac{(1-\varepsilon)}{\varepsilon}\cdot p} \int_{\mathbb{R}^n} (M(|f|^r))^{\frac{p}{r}}(y,t)\cdot\chi(y)\,dy \tag{4.52}$$

$$\leq \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}^{\frac{(1-\varepsilon)}{\varepsilon}\cdot p} \int_{\mathbb{R}^n} |f|^p(y,t)\ \cdot (M\,\chi(y))\ dy. \tag{4.53}$$

Then, we obtain

$$\int_{B_\rho(x)} (M_\eta(f))^{\frac{p}{\varepsilon}}(y,t)\,dy \leq C\,\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}^{\frac{p}{\varepsilon}}\ \cdot \rho^\lambda. \tag{4.54}$$

Let us observe that

$$\frac{p}{\varepsilon} = q \tag{4.55}$$

indeed, using (4.46), we have

$$\varepsilon = \frac{n-\lambda-n\,\eta\,p}{n-\lambda}, \tag{4.56}$$

dividing by $n\cdot p$, we deduce exactly (4.47).

Then, we obtain

$$\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} \left(M_\eta(f)\right)^q(y,t)\,dy\right)^{\frac{1}{q}} \le C\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f|^p(y,t)\,dy\right)^{\frac{1}{p}}, \tag{4.57}$$

elevating to q_1 integrating both sides in $(0,T)\cap(t_0-\rho;t_0+\rho)$ and multiplying for $\frac{1}{\rho^{\mu_1}}$, we have

$$\frac{1}{\rho^{\mu_1}}\int_{(0,T)\cap(t_0-\rho;t_0+\rho)}\left[\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} \left(M_\eta(f)\right)^{\frac{p}{\varepsilon}}(y,t)\,dy\right]^{\frac{q_1}{q}} dt \tag{4.58}$$

$$\le C\,\frac{1}{\rho^{\mu_1}}\int_{(0,T)\cap(t_0-\rho;t_0+\rho)}\left[\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f|^p(y,t)\,dy\right]^{\frac{q_1}{p}} dt \tag{4.59}$$

the last term is less or equal than

$$C\sup_{\substack{t_0\in(0,T)\\ \rho>0}}\frac{1}{\rho^{\mu_1}}\int_{(0,T)\cap(t_0-\rho;t_0+\rho)}\left[\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f|^p(y,t)\,dy\right]^{\frac{q_1}{p}} dt. \tag{4.60}$$

Finally, we have

$$\sup_{\substack{t_0\in(0,T)\\ \rho>0}}\frac{1}{\rho^{\mu_1}}\int_{(0,T)\cap(t_0-\rho;t_0+\rho)}\left[\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} (M_\eta f)^q(y,t)\,dy\right]^{\frac{q_1}{q}} dt \tag{4.61}$$

$$\le C\sup_{\substack{t_0\in(0,T)\\ \rho>0}}\frac{1}{\rho^{\mu_1}}\int_{(0,T)\cap(t_0-\rho;t_0+\rho)}\left[\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f|^p(y,t)\,dy\right]^{\frac{q_1}{p}} dt. \tag{4.62}$$

Elevating both sides to $\frac{1}{q_1}$, we have

$$\|M_\eta f\|_{L^{q_1,\mu_1}(0,T,L^{q,\lambda}(\mathbb{R}^n))} \le C\,\|f\|_{L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\mathbb{R}^n))}. \tag{4.63}$$

4.2. Estimates of singular integral operators and commutators

Let $k(x,y)$ be a variable Calderón–Zygmund kernel for a.e. $x\in\mathbb{R}^{n+1}$, $f\in L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ with $1<p,q<\infty$ $0<\lambda,\mu<n, a\in BMO(\mathbb{R}^{n+1})$. For $\varepsilon>0$ let us define the operator K_ε and the commutator $C_\varepsilon[a,f]$, as follows

$$K_\varepsilon f(x)=\int_{\rho(x-y)>\varepsilon} k(x,x-y)f(y)dy \tag{4.64}$$

$$C_\varepsilon[a,f]=K_\varepsilon(af)(x)-a(x)K_\varepsilon f(x)=\int_{\rho(x-y)>\varepsilon} k(x,x-y)[a(x)-a(y)]f(y)dy. \tag{4.65}$$

In the next theorem we prove that $K_\varepsilon f$ and $C_\varepsilon[a,f]$ are, uniformly in ε, bounded from $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ into itself. This fact allows us to let $\varepsilon\to0$ obtaining as limits in $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ the following singular integral and commutator

$$Kf(x)=P.V.\int_{\mathbb{R}^n} k(x,x-y)f(y)dy=\lim_{\varepsilon\to0}K_\varepsilon f(x) \tag{4.66}$$

$$C[a,f](x)=P.V.\int_{\mathbb{R}^n} k(x,x-y)[a(x)-a(y)]f(y)dy=\lim_{\varepsilon\to0}C_\varepsilon[a,f](x). \tag{4.67}$$

These operators are bounded in the class $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$.

Theorem 4.7. *Let $k(x,y)$ be a variable Calderón–Zygmund kernel, for a.e. $x \in \mathbb{R}^{n+1}$, $1 < p,q < \infty, 0 < \lambda, \mu < n$ and $a \in VMO(\mathbb{R}^{n+1})$.*

For any $f \in L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ the singular integrals Kf, $C[a,f] \in L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ exist as limits in $L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$, for $\varepsilon \to 0$, of $K_\varepsilon f$ and $C_\varepsilon[a,f]$, respectively. Then, the operators $Kf, C[a,f] : L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n)) \to L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ are bounded and satisfy the following inequalities

$$\|Kf\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \le c\|f\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \tag{4.68}$$

$$\|C[a,f]\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \le c\|a\|_*\|f\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \tag{4.69}$$

where $c = c(n,p,\lambda,\alpha,K)$, the dependence on K is through the constant $c(\beta)$ in Definition 2.6 part (2), for suitable β.

Moreover, for every $\epsilon > 0$ there exists $\rho_0 > 0$ such that, if B_r is a ball with radius r such that $0 < r < \rho_0$, $k(x,y)$ satisfies the above assumptions and $f \in L^{q,\mu}(0,T,L^{p,\lambda}(B_r))$, we have

$$\|C[a,f]\|_{L^{q,\mu}(0,T,L^{p,\lambda}(B_r))} \le c\,\epsilon\,\|f\|_{L^{q,\mu}(0,T,L^{p,\lambda}(B_r))} \tag{4.70}$$

for some constant c independent of f.

Proof. For every $t \in (0,T)$, from the known inequality (see e.g. [5])

$$\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |(Kf)(y,t)|^p dy \le c \sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p\, dy, \tag{4.71}$$

fixing $t_0 \in (0,T)$, elevating to $\frac{q}{p}$, integrating in $(0,T)\cap(t_0-\rho,t_0+\rho)$, multiplying for $\rho^{-\mu}$, we have

$$\frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |(Kf)(y,t)|^p dy\right)^{\frac{q}{p}} dt \tag{4.72}$$

$$\le c\frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p\, dy\right)^{\frac{q}{p}} dt \tag{4.73}$$

then, we have

$$\sup_{\substack{t_0\in(0,T)\\ \rho>0}} \frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |(Kf)(y,t)|^p dy\right)^{\frac{q}{p}} dt \tag{4.74}$$

$$\le c \sup_{\substack{t_0\in(0,T)\\ \rho>0}} \frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p dy\right)^{\frac{q}{p}} dt \tag{4.75}$$

elevating to $\frac{1}{q}$, we get the conclusion for Kf. Similar is the proof of (4.69), starting from the inequality

$$\|C[a,f]\|_{L^{p,\lambda}(\mathbb{R}^n)} \le c\|a\|_*\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}. \tag{4.76}$$

Finally, using the VMO assumption, if we fix ρ_0 such that $\eta(\rho_0) < \epsilon$, we get the conclusion. Let us remark that the result is also true if we assume a defined only in some ball with $\|a\|_* < \epsilon$.

5. Applications to partial differential equations

As application of the previous results we obtain a regularity result for strong solutions to the nondivergence form parabolic equations.

Precisely, let $n \geq 3, Q_T = \Omega' \times (0,T)$ be a cylinder of $\mathbb{R}^{n+1}$ of base $\Omega' \subset \mathbb{R}^n$. In the sequel let us set $x = (x',t) = (x_1', x_2, \ldots, x_n', t)$ a generic point in Q_T, $f \in L^{q,\mu}(0,T,L^{p,\lambda}(\Omega'))$, $1 < p,q < \infty$, $0 < \lambda, \mu < n$ and

$$L\,u = u_t - \Sigma_{i,j=1}^n a_{ij}(x',t)\frac{\partial^2 u}{\partial x_i' \partial x_j'} \tag{5.1}$$

where

$$a_{ij}(x',t) = a_{ji}(x',t), \quad \forall i,j = 1,\ldots,n, \text{ a.e. } x \in Q_T \tag{5.2}$$

$$\exists \nu > 0 : \nu^{-1}|\xi|^2 \leq \Sigma_{i,j=1}^n a_{ij}(x',t)\,\xi_i\xi_j \leq \nu|\xi|^2, \quad \text{a.e. in } Q_T, \forall \xi \in \mathbb{R}^n \tag{5.3}$$

$$a_{ij}(x',t) \in VMO(Q_T) \cap L^\infty(Q_T), \quad \forall i,j = 1,\ldots,n. \tag{5.4}$$

Let us consider

$$Lu(x',t) = f(x',t). \tag{5.5}$$

A strong solution to (5.5) is a function $u(x) \in L^{q,\mu}(0,T,L^{p,\lambda}(\Omega'))$ with all its weak derivatives $D_{x_i'}u$, $D_{x_i'x_j'}u$, $i,j = 1,\ldots,n$ and $D_t u$, satisfying (5.5), $\forall x \in Q_T$.

Let us now fix the coefficient $x_0 = (x_0', t_0) \in Q_T$ and consider the fundamental solution of $L_0 = L(x_0)$, is given, for $\tau > 0$, by

$$\Gamma(x_0;\theta) = \Gamma(x_0', t_0; \zeta, \tau) = \frac{(4\pi\tau)^{\frac{1-n}{2}}}{\sqrt{a^{ij}(x_0)}} \exp\left(-\frac{A^{ij}(x_0)\zeta_i\zeta_j}{4\tau}\right) \tag{5.6}$$

that is equal to zero if $\tau \leq 0$, being $A^{ij}(x_0)$ the entries of the inverse matrix $\{a^{ij}(x_0)\}^{-1}$.

The second order derivatives with respect to ζ_i and ζ_j, denoted by $\Gamma_{ij}(x_0, t_0; \zeta, \tau)$, $i,j = 1,\ldots,n$, and $\Gamma_{ij}(x;\theta)$, are kernels of mixed homogeneity in the sense that $\alpha_1 = \cdots, \alpha_{n-1} = 1$, $\alpha_n = 2$ (it follows that $\alpha = n+1$).

Theorem 5.1. *Let $n \geq 3, a_{ij} \in VMO(Q_T) \cap L^\infty(Q_T)$, $B_r \subset\subset \Omega'$ a ball in $\mathbb{R}^n$.*

Then, for every u having compact support in $B_r \times (0,T)$, solution of $Lu = f$ such that $D_{x_i'x_j'}u \in L^{q,\mu}(0,T,L^{p,\lambda}(B_r))$ $\forall i,j = 1,\ldots,n$, there exists $r_0 = r_0(n.p,\nu,\eta)$ such that, if $r < r_0$, then

$$\|D_{x_i'x_j'}\,u\|_{L^{q,\mu}(0,T,L^{p,\lambda}(B_r))} \leq C\|Lu\|_{L^{q,\mu}(0,T,L^{p,\lambda}(B_r))}, \quad i,j = 1,\ldots,n \tag{5.7}$$

$$\|\,u_t\|_{L^{q,\mu}(0,T,L^{p,\lambda}(B_r))} \leq C\|Lu\|_{L^{q,\mu}(0,T,L^{p,\lambda}(B_r))}. \tag{5.8}$$

Proof. Let $C_t = \{v \in C_0^\infty(\mathcal{A}) : v(x',0) = 0, \mathcal{A} = \mathbb{R}^{n+1} \cap \{t \geq 0\}\}$ and $u \in C_t$. The local representation formula for the second order spatial derivatives of u (see [2]), is the following

$$D_{x_i'x_j'}u(x) = \lim_{\varepsilon\to 0}\int_{\rho(x-y)>\varepsilon} \Gamma_{ij}(x;x-y)Lu(y)dy$$
$$+ \lim_{\varepsilon\to 0}\int_{\rho(x-y)>\varepsilon} \Gamma_{ij}(x;x-y)\Sigma_{h,k=1}^n[a^{hk}(y) - a^{hk}(x)] \cdot D_{y_h'\,y_k'}u(y)dy + L(x)\int_\Sigma \nu_i(y)\Gamma_j(x;y)d\sigma, \tag{5.9}$$

for all $i,j = 1,\ldots,n$, and for x in the support of u, being $\nu_i(y)$ the ith component of the unit outward normal to Σ at $y \in \Sigma$.

From (4.68) and (4.69) we get the first inequality. Let us now observe that

$$u_t = Lu + \Sigma_{i,j=1}^n a_{ij}(x',t)\frac{\partial^2 u}{\partial x_i'\partial x_j'} \tag{5.10}$$

and the second inequality (5.8) is proved.

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