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# Interior Estimates in Morrey Spaces for Strong Solutions to Nondivergence Form Equations with Discontinuous Coefficients

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Let us consider the nondivergence form elliptic equation $a_{ij}u_{x_ix_j}=f$. In this paper we show that if the known term $f$ belongs to the Morrey space $L^{p,\lambda}$ then the second derivatives of the $W^{2,p}$-solution $u$ belong to the same space. Next we derive a $C^{1,\alpha}_{\text{loc}}$-*regularity* result that is related to a recent work by Caffarelli. © 1993 Academic Press, Inc.

## INTRODUCTION

The aim of this note is to study the local regularity in the Morrey spaces $L^{p,\lambda}$ $(1<p<+\infty,\ 0<\lambda<n$; see Section 1 for definitions) of the second derivatives of the solution of an elliptic second order equation in nondivergence form.

Precisely let us consider the equation

$$\sum_{i,j=1}^{n} a_{ij}u_{x_ix_j}=f, \tag{*}$$

where $f$ is assumed to be in some $L^{p,\lambda}$ spaces and the coefficients are taken in the Sarason class $\mathscr{V.M.O.}$ introduced in [S] (see Section 1 for definitions).

In his recent paper [Ca$_1$] (which prepares the deep work [Ca$_2$]) in Theorem 1 of Section 3, L. Caffarelli proved that if $f$ belongs to the Morrey class $L^{n,n\alpha}$, with $0<\alpha<1$, then every $W^{2,p}$-viscosity solution $u$ of the equation (*) is of class $C^{1,\alpha}$.

We observe that it seems that the assumption on $f$ could not be replaced, in [Ca$_1$], by the weaker one $f\in L^{p,\sigma}$, $p<n$, $\sigma>0$, because of the method followed in the proof which relies on the maximum principle of Aleksandrov-Pucci.



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In this note a similar result can be obtained as an application of the $L^{p,\lambda}$ estimates for the second derivatives of a strong[1] solution under the assumptions

$$\frac{1}{p}-\frac{1}{n-\lambda}<0, \qquad 1<p<+\infty, \qquad 0<\lambda<n.$$

Our method is based on an integral representation formula of the second derivatives of the solution $u$ used in [CFL].

In order to derive the required estimates from such a representation formula, we need to study the action on Morrey spaces of some singular integral operators (of non-convolution type). This is done in Section 2 which is the main bulk of our work.

The study of some similar simpler operators has been made by us in [DR].

Here we have found it convenient to study previously such singular operators in some weighted Lebesgue classes and then deduce the $L^{p,\lambda}$ result using some special weights.

We believe that these results could be of some interest in themselves.

The application of these results to equation (*) is done in Section 3.

Finally we thank L. Tuccari for some useful talks.

## 1. Some Definitions

1.1. Let $\Omega$ be an open set in $\mathbb{R}^n$ $(n\geq 3)$. We denote $B_r(x)$ as the ball centered at $x$ with radius $r$. We set $B_r \equiv B_r(0)$. For any locally integrable function $f$ in $\Omega$ let us consider the quantity defined by

$$\eta(r)=\sup_{\substack{\rho\leq r\\ B_\rho\subset\subset\Omega}} \fint_{B_\rho(x)} |f(y)-f_{B_\rho(x)}|\,dy,$$

where $f_{B_\rho(x)}$ is the integral average of $f$ over $B_\rho(x)$. Call $\mathscr{B}.\mathscr{M}.\mathscr{O}.(\Omega)$ the set of those $f\in L^1_{\text{loc}}(\Omega)$ for which $\eta$ is bounded and $\mathscr{V}.\mathscr{M}.\mathscr{O}.(\Omega)$ the subset of $\mathscr{B}.\mathscr{M}.\mathscr{O}.(\Omega)$ consisting of those $f$ for which $\eta$ vanishes as $r$ decreases to zero.

$\mathscr{B}.\mathscr{M}.\mathscr{O}.(\Omega)$ is a Banach space with the norm (modulo constant functions) $\|f\|_* = \sup \eta(r)$.

1.2. As usual, for $p\in\,]1,+\infty[$, $\lambda\in\,]0,n[$, let

$$\|f\|^p_{L^{p,\lambda}(\Omega)}=\sup_{\substack{\rho>0\\ x\in\Omega}}\frac{1}{\rho^\lambda}\int_{B_\rho(x)\cap\Omega}|f(y)|^p\,dy$$

and call $L^{p,\lambda}(\Omega)$ the set of those measurable $f$ for which $\|f\|_{L^{p,\lambda}(\Omega)}$ is finite.

[1] See, e.g., [GT, Chap. 9] for a definition.

1.3. Let $k(x): \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$. We call $k(x)$ a *Calderon–Zygmund kernel* (C–Z kernel) if

(i) $k(x) \in C^\infty(\mathbb{R}^n \setminus \{0\})$;

(ii) $k(x)$ is homogeneous of degree $-n$;

(iii) $\int_\Sigma k(x)\, d\sigma_x = 0$, where $\Sigma = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere in $\mathbb{R}^n$.

1.4. Let $1 < p < +\infty$, $u(x)$ a non-negative locally integrable function on $\mathbb{R}^n$. We say that $u(x)$ belongs to class $A_p$ if and only if

$$\operatorname*{Sup}_Q \fint_Q u(x)\, dx \left( \fint_Q u(x)^{-1/(p-1)}\, dx \right)^{p-1} \equiv C_p < +\infty$$

the supremum being taken on the class of all cubes with the edges parallel to the coordinate axes. We call $C_p$ and $A_p$ constant of $u$.

We say that $u(x)$ belongs to $A_1$ if there exists a constant $C > 0$ such that

$$Mu(x) \leqslant Cu(x) \qquad \text{a.e. in } \mathbb{R}^n,$$

where $Mu$ is the usual Hardy–Littlewood maximal function of $u$. The infimum $C_1$ of the constants $C$ for which the previous inequality holds will be called the $A_1$ constant of $u$. Finally we let $A_\infty = \bigcup_{1 \leqslant p < +\infty} A_p$.

## 2. Singular Integrals and Commutators

In this section we prove the basic results about singular integrals and commutators that we need to get an a priori bound for second derivatives of the solution $u$ to equation (*).

We follow, as in [CFL], the usual technique of spherical harmonic expansion.

Because the constant in the weighted estimate for the Calderon Zygmund operator appears to depend on arguments which are not the same as in the unweighted case, we prefer, for the reader's convenience, to recall the outline of the proof of Theorem 2.10 in [CFL] giving all the details whenever needed.

We start recalling some well known definitions and properties of spherical harmonics (see, e.g., [N]).

Any homogeneous polynomial $p(x)$ of degree $m$, a solution of $\Delta u = 0$, will be called a solid (spherical) harmonic of degree $m$. Its restriction to the unit sphere $\Sigma$ will be called a spherical harmonic of degree $m$. Denote by

$H_m$ and $\mathscr{H}_m$ the spaces of solid and of spherical harmonics of degree $m$, respectively. Also set $H = \bigcup_{m \in N_0} H_m$ and $\mathscr{H} = \bigcup_{m \in N_0} H_m$. $\mathscr{H}_m$ is a finite dimensional vector space. Let $\dim \mathscr{H}_m = g_m$.

Call $\{Y_{lm}\}$, $l = 1, ..., g_m$, $m = 0, 1, 2, ...$, an orthonormal system of spherical harmonics complete in $L^2(\Sigma)$.

The following lemma holds.

LEMMA. *We have:*

(i) $g_m = \binom{m+n-1}{n-1} - \binom{m+n-3}{n-1} \leqslant C(n)\, m^{n-2}$;

(ii) $|D^\alpha(|x|^m Y_{km})| \leqslant C(n)\, |x|^{m-|\alpha|}\, m^{n/2+|\alpha|-1}$ $\forall x \in \mathbb{R}^n \setminus \{0\}$; $k = 1, ..., g_m$.

THEOREM 2.1. *Let $k$ be a real measurable function in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ such that*

(i) *$k(x, z)$ is a C.Z. kernel for a.a. $x \in \mathbb{R}^n$*

(ii) $\max_{|j| \leqslant 2n} \|(\partial^j/\partial z^j)\, k(x, z)\|_{L^\infty(\mathbb{R}^n \times \Sigma)} = M < +\infty$.

*Let $\omega \in A_p$ with $A_p$ constant $C_p$, $f \in L^p_\omega(\mathbb{R}^n)$, $1 < p < +\infty$, $\varphi \in \mathscr{B.M.O.}(\mathbb{R}^n)$, and for $\varepsilon > 0$ set*

$$K_\varepsilon f(x) = \int_{|x-y| > \varepsilon} k(x, x-y)\, f(y)\, dy$$

$$C_\varepsilon[\varphi, f](x) = \int_{|x-y| > \varepsilon} k(x, x-y)[\varphi(x) - \varphi(y)]\, f(y)\, dy.$$

*Then, there exist $Kf$, $C[\varphi, f] \in L^p_\omega(\mathbb{R}^n)$ such that*

$$\lim_{\varepsilon \to 0^+} \|K_\varepsilon f - Kf\|_{L^p_\omega(\mathbb{R}^n)} = \lim_{\varepsilon \to 0^+} \|C_\varepsilon[\varphi, f] - C[\varphi, f]\|_{L^p_\omega(\mathbb{R}^n)} = 0$$

*and moreover there exists a positive constant $C \equiv C(n, p, C_p, M)$ such that*

$$\|Kf\|_{L^p_\omega(\mathbb{R}^n)} \leqslant C\, \|f\|_{L^p_\omega(\mathbb{R}^n)}$$

*and*

$$\|C[\varphi, f]\|_{L^p_\omega(\mathbb{R}^n)} \leqslant C\, \|\varphi\|_*\, \|f\|_{L^p_\omega(\mathbb{R}^n)}.$$

*Proof.* By a density argument it is enough to prove the theorem for $f \in C_0^\infty(\mathbb{R}^n)$.

The proof will be given in several steps.

We define for each $m \in N$ and $k = 1, 2, ..., g_m$,

$$a_{km}(x) = \int_\Sigma k(x, z)\, Y_{km}(z)\, d\sigma_z.$$

*Step* 1. $\|a_{km}\|_{L^\infty} \leqslant C(n)\, Mm^{-2n}$.
See [CFL, Lemma 2.10, Step 1].

*Step* 2. Series expansion of $K_\varepsilon f$, $C_\varepsilon[\varphi, f]$.
We have

$$|y|^n K(x, y) = \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} a_{km}(x)\, Y_{km}(y)$$

and, as in [CFL],

$$\begin{aligned} K_\varepsilon f(x) &= \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} a_{km}(x) \int_{|x-y|>\varepsilon} \frac{Y_{km}(x-y)}{|x-y|^n} f(y)\, dy \\ &\equiv \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} a_{km}(x)(R_{km\varepsilon} f)(x) \end{aligned}$$

and

$$\begin{aligned} C_\varepsilon[\varphi, f](x) &= \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} a_{km}(x) \int_{|x-y|>\varepsilon} \frac{Y_{km}(x-y)}{|x-y|^n} [\varphi(x)-\varphi(y)]\, f(y)\, dy \\ &\equiv \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} a_{km}(x)(S_{km\varepsilon} f)(x) \end{aligned}$$

*Step* 3. Fundamental inequality.

$$\left| \frac{Y_{km}(x-y)}{|x-y|^n} - \frac{Y_{km}(-y)}{|-y|^n} \right| \leqslant C(n)\, m^{n/2} \frac{|x|}{|y|^{n+1}}; \qquad |y| > 2\,|x| > 0$$

$k = 1, ..., g_m$; $m = 0, 1, ....$

*Proof.* We have

$$\begin{aligned} |x|^m \frac{\partial}{\partial x_i} \frac{Y_{km}(x)}{|x|^n} &= -n\, |x|^{m-n-2}\, x_i Y_{km}(x) + \left( |x|^m \frac{\partial}{\partial x_i} Y_{km}(x) \right) |x|^{-n} \\ &= -n\, |x|^{m-n-2}\, x_i Y_{km}(x) \\ &\quad + |x|^{-n} \left\{ \frac{\partial}{\partial x_i} (|x|^m\, Y_{km}(x)) - m\, |x|^{m-2}\, x_i Y_{km}(x) \right\}. \end{aligned}$$

Using the previous lemma we get

$$\left| \frac{\partial}{\partial x_i} \frac{Y_{km}(x)}{|x|^n} \right| \leqslant C(n)(n+2m)\, m^{n/2-1} \frac{1}{|x|^{n+1}}.$$

Now the inequality is an easy consequence of the mean value theorem.

In fact

$$\left|\frac{Y_{km}(x-y)}{|x-y|^n}-\frac{Y_{km}(-y)}{|-y|^n}\right| \leqslant C(n)(n+2m)\,m^{n/2-1}\frac{|x|}{|\theta x-y|^{n+1}}$$

$$\leqslant C(n)(n+2m)\,m^{n/2-1}2^{n+1}\frac{|x|}{|y|^{n+1}},$$

where $\theta$ is a convenient number in $]0,1[$.

*Step* 4. Let $R_{km}f$ be the singular integral operator with kernel $Y_{km}(x)/|x|^n$ and let $S_{km}f$ denote the singular commutator defined by

$$S_{km}[\varphi, f](x)=\int_{\mathbb{R}^n}\frac{Y_{km}(x-y)}{|x-y|^n}(\varphi(x)-\varphi(y))\,f(y)\,dy.$$

If $M_p f$ denotes the fractional Hardy–Littlewood maximal operator

$$(M_p f)(x)\equiv \operatorname*{Sup}_{r>0}\left(\fint_{B_r(x)}|f|^p\,dy\right)^{1/p}\qquad (1\leqslant p<+\infty)$$

we have, for a.a. $x\in\mathbb{R}^n$, $k=1,\ldots,g_m$

$$(R_{km}f)^{\#}(x)\leqslant C(n,p)\,m^{n/2}M_p f(x)$$

and

$$(S_{km}f)^{\#}(x)\leqslant C(n,p)\,\|\varphi\|_{*}\{m^{n/2}M_p f(x)+M_p(R_{km}f)(x)\}.$$

Here $f^{\#}$ denotes the usual sharp function of $f$ (see, e.g., [FS]).

*Proof.* By translation invariance we prove the inequalities at the point $x=0$. Let us begin proving the first of these.

By Step 3 we get

$$\int_{|y|>2|x|}\left|\frac{Y_{km}(x-y)}{|x-y|^n}-\frac{Y_{km}(-y)}{|-y|^n}\right||f(y)|\,dy$$

$$\leqslant C(n)\,m^{n/2}\,|x|\int_{|y|>2|x|}|f(y)|\,|y|^{-n-1}\,dy$$

$$\leqslant C(n)\,m^{n/2}\,|x|\sum_{k=1}^{+\infty}(2^k|x|)^{-n-1}\int_{2^k|x|\leqslant|y|<2^{k+1}|x|}|f(y)|\,dy$$

$$\leqslant C(n)\,m^{n/2}(M_p f)(0).$$

Now let $\varepsilon>0$ and write

$$f(x)=f(x)\,\chi_{B_\varepsilon}(x)+f(x)\,\chi_{\mathbb{R}^n\setminus B_\varepsilon}(x)\equiv f_1(x)+f_2(x).$$

Then

$$\begin{aligned}
&\varepsilon^{-n}\int_{|x|<\varepsilon/2}|R_{km}f(x)-R_{km\varepsilon}f(0)|\,dx\\
&\leqslant \varepsilon^{-n}\int_{|x|<\varepsilon/2}|R_{km}f_1(x)|\,dx\\
&\quad+\varepsilon^{-n}\int_{|x|<\varepsilon/2}|R_{km}f_2(x)-R_{km}f(0)|\,dx\\
&\leqslant C(n,p)\,\varepsilon^{-n/p}\left(\int_{\mathbb{R}^n}|f_1(x)|^p\,dx\right)^{1/p}+\varepsilon^{-n}\int_{|x|<\varepsilon/2}\int_{|y|>\varepsilon}|f(y)|\\
&\quad\times\left|\frac{Y_{km}(x-y)}{|x-y|^n}-\frac{Y_{km}(-y)}{|-y|^n}\right|\,dy\,dx\\
&\leqslant C(n,p)\,m^{n/2}(M_pf)(0),
\end{aligned}$$

where we have used the Calderon–Zygmund theorem and the previous estimate.

Taking the supremum with respect to $\varepsilon$ the inequality

$$(R_{km}f)^{\#}\,(0)\leqslant C(n,p)\,m^{n/2}M_pf(0)$$

follows.

Now we prove the second inequality.

If $x\in B_\varepsilon(0)$ we have, following [T, p. 418],

$$\begin{aligned}
(S_{km}f)(x)&=(\varphi(x)-\varphi_{B_\varepsilon})\,R_{km}f(x)-R_{km}(\varphi(y)-\varphi_{B_\varepsilon})\,f\chi_{B_{2\varepsilon}}(x)\\
&\quad-R_{km}(\varphi(y)-\varphi_{B_\varepsilon})\,f\chi_{\mathbb{R}^n\setminus B_{2\varepsilon}}(x)\\
&\equiv A(x)+B(x)+C(x).
\end{aligned}$$

Arguing in the same way as [T, p. 418] we get

$$⨍_{B_\varepsilon}|A(x)|\,dx\leqslant C(n,p)\,\|\varphi\|_*\,(M_p(R_{km}f))(0)$$

and

$$⨍_{B_\varepsilon}|B(x)|\,dx\leqslant C(n,p)\,\|\varphi\|_*\,(M_pf)(0).$$

Now, using the fundamental inequality (note that $\mathbb{R}^n\setminus B_{2\varepsilon}$ is contained in $\{|y|>2\,|x|>0\}$),

$$
\begin{aligned}
|C(x)-C(0)| &\leqslant \left|\int_{\mathbb{R}^n\setminus B_{2\varepsilon}} \left(\frac{Y_{km}(x-y)}{|x-y|^n}-\frac{Y_{km}(-y)}{|-y|^n}\right)(\varphi(y)-\varphi_{B_\varepsilon})\, f(y)\,dy\right| \\
&\leqslant C(n)\, m^{n/2}\, |x| \int_{\mathbb{R}^n\setminus B_{2\varepsilon}} |\varphi(y)-\varphi_{B_\varepsilon}|\, |f(y)|\, |y|^{-n-1}\,dy \\
&\leqslant C(n)\, m^{n/2}\, |x| \left(\int_{\mathbb{R}^n\setminus B_{2\varepsilon}} |f(y)|^p\, |y|^{-n-1}\,dy\right)^{1/p} \\
&\quad\times\left(\int_{\mathbb{R}^n\setminus B_{2\varepsilon}} |\varphi(y)-\varphi_{B_\varepsilon}|^p\, |y|^{-n-1}\,dy\right)^{1/p'} \\
&\leqslant C(n)\, m^{n/2}\, |x| \left(\sum_{k=1}^{+\infty} (2^k|x|)^{-n-1} \int_{|y|\leqslant 2^{k+1}|x|} |f(y)|^p\,dy\right)^{1/p} \\
&\quad\times\left(\sum_{k=1}^{+\infty} (2^k|x|)^{-n-1} \int_{|y|\leqslant 2^{k+1}\varepsilon} |\varphi-\varphi_\varepsilon|^{p'}\,dy\right)^{1/p'} \\
&\leqslant C(n,p)\, m^{n/2}(M_p f)(0)\, \|\varphi\|_*,
\end{aligned}
$$

where we have used Proposition 3.2 of [T, p. 206].

Then

$$
⨍_{B_\varepsilon} |C(x)-C(0)|\,dx \leqslant C(n,p)\, m^{n/2}(M_p f)(0)\, \|\varphi\|_*.
$$

Finally, for some $\varepsilon>0$,

$$
\begin{aligned}
(S_{km}f)^{\#}(0) &\leqslant 2 ⨍_{B_\varepsilon} |S_{km}f(x)-(S_{km}f)_{B_\varepsilon}|\,dx \\
&\leqslant C(n,p)\, \|\varphi\|_*\{m^{n/2} M_p f(0) + M_p(R_{km}f)(0)\}.
\end{aligned}
$$

*Step* 5. $\|R_{km}f\|_{p,\omega} \leqslant C m^{n/2}\, \|f\|_{p,\omega}$, $\|S_{km}f\|_{p,\omega} \leqslant C m^{n/2}\, \|\varphi\|_*\, \|f\|_{p,\omega}$ for every $f\in L^p_\omega(\mathbb{R}^n)$ with $C\equiv C(n,p,C_p)$.

*Proof.* From Step 4, if $1<q<p<+\infty$,

$$
\begin{aligned}
\int_{\mathbb{R}^n} |(R_{km}f)^{\#}|^p(x)\,\omega(x)\,dx &\leqslant C(n,p)\, m^{np/2} \int_{\mathbb{R}^n} |M_q f|^p(x)\,\omega(x)\,dx \\
&= C(n,p,C_p)\, m^{np/2} \int_{\mathbb{R}^n} |f(x)|^p\,\omega(x)\,dx
\end{aligned}
$$

and by a well known result of Fefferman and Stein (see [FS]), $\|R_{km}f\|_{p,\omega} \leqslant C(n,p,C_p)\, m^{n/2}\, \|f\|_{p,\omega}$. Similar arguments follow for the commutator $S_{km}$.

*Step* 6.

$$\|R_{km\varepsilon}f\|_{p,\omega} \leqslant C(n, p, C_p)\, m^{n/2}\, \|f\|_{p,\omega}$$
$$\|S_{km\varepsilon}f\|_{p,\omega} \leqslant C(n, p, C_p)\, m^{n/2}\, \|\varphi\|_*\, \|f\|_{p,\omega}$$

for every $f \in L^p_\omega(\mathbb{R}^n)$.

*Proof.* It follows from the inequality, proven in Step 4,

$$\varepsilon^{-n} \int_{|x| < \varepsilon/2} |R_{km}f(x) - R_{km\varepsilon}f(0)|\, dx \leqslant C(n, p)\, m^{n/2} M_p f(0).$$

Then, we have

$$|R_{km\varepsilon}f(0)| \leqslant C(n)\, \varepsilon^{-n} \int_{|x| < \varepsilon/2} |R_{km}f(x)|\, dx + C(n, p)\, m^{n/2} M_p f(0)$$
$$\leqslant C(n, p)(M(R_{km}f)(0) + m^{n/2}(M_p f)(0))$$

and then by translation invariance

$$|R_{km\varepsilon}f(x)| \leqslant C(n, p)(M(R_{km}f)(x) + m^{n/2}(M_p f)(x)).$$

Taking $L^p_\omega$ norms of both sides it follows

$$\|R_{km\varepsilon}f\|_{p,\omega} \leqslant C(n, p, C_p)(\|R_{km}f\|_{p,\omega} + m^{n/2}\, \|f\|_{p,\omega})$$
$$\leqslant C(n, p, C_p)\, m^{n/2}\, \|f\|_{p,\omega},$$

where we have used the weighted version of the Hardy–Littlewood maximal function and the previous step.

Similar arguments follow for the commutator $S_{km\varepsilon}$.

*Step* 7.

$$\|K_\varepsilon f\|_{p,\omega} \leqslant C\, \|f\|_{p,\omega}, \qquad \|C_\varepsilon[\varphi, f]\|_{p,\omega} \leqslant C\, \|\varphi\|_*\, \|f\|_{p,\omega},$$

where $C \equiv C(n, p, C_p, M)$.

*Proof.*

$$\|K_\varepsilon f\|_{p,\omega} = \left\| \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} a_{km}(x)\, R_{km\varepsilon}f \right\|_{p,\omega}$$
$$\leqslant C(n, p, C_p)\, M \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} m^{-2n+n/2}\, \|f\|_{p,\omega}$$
$$= C(n, p, C_p)\, M \sum_{m=1}^{\infty} m^{-2n+n/2+n-2}\, \|f\|_{p,\omega}.$$

Similar arguments follow for the commutator $C_\varepsilon[\varphi, f]$.

*Step* 8. Letting

$$Kf(x) \equiv \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} a_{km}(x)\, R_{km} f(x)$$

and

$$C[\varphi, f](x) \equiv \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} a_{km}(x)\, S_{km} f(x)$$

we have

$$\|Kf\|_{p,\omega} \leqslant C\, \|f\|_{p,\omega}$$

and

$$\|C[\varphi, f]\|_{p,\omega} \leqslant C\, \|\varphi\|_* \, \|f\|_{p,\omega},$$

where $C \equiv C(n, p, C_p, M)$.

*Proof.* The proof is the same as in the previous step using step 5.

*Step* 9.

$$\lim_{\varepsilon \to 0^+} \|K_\varepsilon f - Kf\|_{p,\omega} = \lim_{\varepsilon \to 0^+} \|C_\varepsilon[\varphi, f] - C[\varphi, f]\|_{p,\omega} = 0.$$

Since the convergence of the series defining $K_\varepsilon f$ and $C_\varepsilon[\varphi, f]$ is uniform with respect to $\varepsilon$ we can take the limit under the series sign obtaining the thesis and so the theorem is proved.

THEOREM 2.2. *Let $B$ be an open ball of $\mathbb{R}^n$ and let $k(x, z)$ be a real measurable function in $B \times (\mathbb{R}^n \setminus \{0\})$ such that*

(i) *$k(x, z)$ is a C.Z. kernel for a.a. $x \in B$*

(ii) $\max_{|j| \leqslant 2n} \|(\partial^j/\partial z^j)\, k(x, z)\|_{L^\infty(B \times \Sigma)} = M < +\infty$.

*Let $\omega \in A_p$ with $A_p$ constant $C_p$, $f \in L^p_\omega(B)$, $1 < p < +\infty$, $\varphi \in \mathcal{B}.\mathcal{M}.\mathcal{O}.(B)$, and for $\varepsilon > 0$ and $x \in B$ set*

$$K_\varepsilon f(x) = \int_{|x-y| > \varepsilon,\, y \in B} k(x, x-y)\, f(y)\, dy$$

$$C_\varepsilon[\varphi, f](x) = \int_{|x-y| > \varepsilon,\, y \in B} k(x, x-y)[\varphi(x) - \varphi(y)]\, f(y)\, dy.$$

*Then, there exist $Kf$, $C[\varphi, f] \in L^p_\omega(B)$ such that*

$$\lim_{\varepsilon \to 0^+} \|K_\varepsilon f - Kf\|_{L^p_\omega(B)} = \lim_{\varepsilon \to 0^+} \|C_\varepsilon[\varphi, f] - C[\varphi, f]\|_{L^p_\omega(B)} = 0$$

*and moreover there exists a positive constant* $C \equiv C(n, p, C_p, M)$ *such that*

$$\|Kf\|_{L^p_\omega(B)} \leqslant C \|f\|_{L^p_\omega(B)}$$

*and*

$$\|C[\varphi, f]\|_{L^p_\omega(B)} \leqslant C \|\varphi\|_* \|f\|_{L^p_\omega(B)}.$$

*Proof.* The conclusion follows from Theorem 2.1 setting

$$\tilde{k}(x, z) = \begin{cases} k(x, z) & \text{if } x \in B, \quad z \neq 0 \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B, \quad z \neq 0 \end{cases}$$

$$\tilde{f}(x) = \begin{cases} f(x), & x \in B \\ 0, & x \in \mathbb{R}^n \setminus B \end{cases}$$

and $\tilde{\varphi}$ the $\mathscr{B}.\mathscr{M}.\mathscr{O}.(\mathbb{R}^n)$-extension of $\varphi$ given by Jones's theorem (see, e.g., [J, Theorem 1 and p. 54, (2.20)]).

Indeed we have the existence of a function $\tilde{K}\tilde{f} \in L^p_\omega(\mathbb{R}^n)$ by Theorem 2.1.

Let $Kf$ be the restriction to $B$ of $\tilde{K}\tilde{f}$. Then

$$\lim_{\varepsilon \to 0^+} \|K_\varepsilon f - Kf\|_{L^p_\omega(B)} = \lim_{\varepsilon \to 0^+} \|\tilde{K}_\varepsilon \tilde{f} - \tilde{K}\tilde{f}\|_{L^p_\omega(B)} = 0.$$

Moreover, by Theorem 2.1,

$$\|Kf\|_{L^p_\omega(B)} \leqslant \|\tilde{K}\tilde{f}\|_{L^p_\omega(\mathbb{R}^n)} \leqslant C \|\tilde{f}\|_{L^p_\omega(\mathbb{R}^n)} = C \|f\|_{L^p_\omega(B)}.$$

Similar arguments for the singular commutator lead to

$$\|C[\varphi, f]\|_{L^p_\omega(B)} \leqslant C \|\tilde{\varphi}\|_* \|\tilde{f}\|_{L^p_\omega(\mathbb{R}^n)}.$$

Applying Jones' theorem (see [J, p. 42 and p. 54, (2.20)]) we have

$$\|C[\varphi, f]\|_{L^p_\omega(B)} \leqslant C \|\varphi\|_* \|f\|_{L^p_\omega(B)}.$$

THEOREM 2.3. *Let $B$ an open ball of $\mathbb{R}^n$ and let $k(x, z)$ be a real function satisfying the hypotheses of Theorem 2.2.*

*If $1 < p < +\infty$, $0 < \lambda < n$ let $f \in L^{p,\lambda}(B)$ and $\varphi \in \mathscr{B}.\mathscr{M}.\mathscr{O}.(B)$ and $K_\varepsilon f$, $C_\varepsilon[\varphi, f]$, $Kf$, and $C[\varphi, f]$ have the same meaning as Theorem 2.2. Then $Kf$ and $C[\varphi, f]$ belong to $L^{p,\lambda}(B)$. Also*

$$\lim_{\varepsilon \to 0^+} \|K_\varepsilon f - Kf\|_{L^{p,\lambda}(B)} = \lim_{\varepsilon \to 0^+} \|C_\varepsilon[\varphi, f] - C[\varphi, f]\|_{L^{p,\lambda}(B)} = 0$$

*and there exists a constant $C$ such that*

$$\|Kf\|_{L^{p,\lambda}(B)} \leqslant C \|f\|_{L^{p,\lambda}(B)}$$

*and*

$$\|C[\varphi, f]\|_{L^{p,\lambda}(B)} \leqslant C\, \|\varphi\|_* \, \|f\|_{L^{p,\lambda}(B)},$$

*where* $C \equiv C(n, p, M, \lambda)$.

*Proof.* Fix any $\gamma \in ]\lambda/n, 1[$ we know that $(M\chi_{B_\rho(x_0)})^\gamma (x) \in A_1 \subseteq A_p$ with $A_1$ constant depending on $\gamma$ and $n$ only (see, e.g., [CR]).

For some $x_0 \in B$ and $\rho > 0$, we have

$$\begin{aligned}\|K_\varepsilon f - Kf\|^p_{L^{p,\lambda}(B)} &\leqslant 2\rho^{-\lambda} \int_{B \cap B_\rho(x_0)} |K_\varepsilon f - Kf|^p \, dx \\ &= 2\rho^{-\lambda} \int_B |K_\varepsilon f - Kf|^p \, \chi^\gamma_{B_\rho(x_0)}(x)\, dx \\ &\leqslant 2\rho^{-\lambda} \int_B |K_\varepsilon f - Kf|^p \, (M\chi_{B_\rho(x_0)})^\gamma (x)\, dx\end{aligned}$$

and, by Theorem 2.2,

$$\lim_{\varepsilon \to 0^+} \|K_\varepsilon f - Kf\|_{L^{p,\lambda}(B)} = 0.$$

Now for every $\sigma > 0$ and any $y \in B$ we get

$$\begin{aligned}\int_{B_\sigma(y) \cap B} |Kf(x)|^p \, dx &= \int_B |Kf(x)|^p \, (\chi_{B_\sigma(y)}(x))^\gamma \, dx \\ &\leqslant \int_B |Kf(x)|^p \, (M\chi_{B_\sigma(y)}(x))^\gamma \, dx \\ &\leqslant C\sigma^\lambda \, \|f\|^p_{L^{p,\lambda}(B)}{}^2\end{aligned}$$

so that

$$\|Kf\|_{L^{p,\lambda}(B)} \leqslant C\, \|f\|_{L^{p,\lambda}(B)}.$$

Similar arguments follow for the commutator $C[\varphi, f]$.

THEOREM 2.4. *Let* $\varphi \in \mathcal{V.M.O.}(\mathbb{R}^n)$. *Then, for any* $\varepsilon > 0$, *there exists* $\rho_0 > 0$ *such that, if* $B_r$ *is a ball with radius* $r$ *such that* $0 < r < \rho_0$, $k(x, z)$ *satisfies the hypotheses of Theorem* 2.3 *in* $B_r$ *and* $f$ *belongs to* $L^{p,\lambda}(B_r)$ $(1 < p < +\infty,\ 0 < \lambda < n)$ *we have*

$$\|C[a, f]\|_{L^{p,\lambda}(B_r)} \leqslant C\varepsilon \, \|f\|_{L^{p,\lambda}(B_r)} \qquad \forall f \in L^p(B_r),$$

*for some constant* $C = C(n, p, \lambda, M)$.

[2] For the last inequality see the proof of Theorem 1 in [CF, p. 275].

*Proof.* Fix $\rho_0$ such that $\eta(\rho_0) < \varepsilon$. Then apply Theorem 2.3.

*Remark* 2.5. We observe that the result in Theorem 2.4 is true if we assume $\varphi$ is defined only in some ball $B$ with $\|\varphi\|_* < \varepsilon$.

This clearly follows from Jones's extension theorem.

## 3. An a Priori Estimate and Some Regularity Results

Now as an application of the results in the previous section we get an a priori estimate for strong solutions to the nondivergence form elliptic equations.

Precisely let $n \geqslant 3$, $B$ be an open ball of $\mathbb{R}^n$, and $L \equiv \sum_{i,j=1}^n a_{ij}(x)(\partial^2/\partial x_i\, \partial x_j)$, where

(i) $a_{ij}(x) = a_{ji}(x)\ \forall i, j = 1, ..., n$, a.a. $x \in B$

(ii) $\exists \nu > 0$, $\nu^{-1} |\xi|^2 \leqslant \sum_{i,j=1}^n a_{ij}(x)\, \xi_i \xi_j \leqslant \nu\, |\xi|^2$ a.e. in $B$ $\forall \xi \in \mathbb{R}^n$.

In the following we set

$$\Gamma(x, t) = \{(n-2)\, \omega_n (\det a_{ij}(x))^{1/2}\}^{-1} \left( \sum_{i,j=1}^n A_{ij}(x)\, t_i t_j \right)^{1-n/2}$$

for a.e. $x \in B$, for $t \in \mathbb{R}^n \setminus \{0\}$, where, as usual, we denote by $A_{ij}(x)$ the entries of the inverse of the matrix $(a_{ij}(x))_{i,j=1,2,...,n}$. Observe that, for fixed $x_0 \in \tilde{B}$ ($\tilde{B}$ being the subset of $\Omega$ in which (i) and (ii) hold everywhere) $\Gamma(x_0, t)$ is a fundamental solution for the operator

$$L_0 u(x) \equiv \sum_{i,j=1}^n a_{ij}(x_0)\, u_{x_i x_j}(x).$$

Also we set

$$\Gamma_i(x, t) = \frac{\partial}{\partial t_i} \Gamma(x, t) \qquad \text{and} \qquad \Gamma_{ij}(x, t) = \frac{\partial^2}{\partial t_i\, \partial t_j} \Gamma(x, t).$$

We recall an integral representation formula for the second derivative of a function in the class $W_0^{2,p}(B)$ (see [CFL, Sect. 3]).

THEOREM 3.1. *Let* $n \geqslant 3$, $B$, *and* $(a_{ij})_{i,j=1,...,n}$ *be as above and* $u \in W_0^{2,p}(B)$. *Then, for a.a.* $x \in B$,

$$\begin{aligned} u_{x_i x_j}(x) = P.V. \int_B & \Gamma_{ij}(x, x-y) \\ & \times \left[ \sum_{h,k=1}^n (a_{hk}(x) - a_{hk}(y))\, u_{x_h x_k}(y) + Lu(y) \right] dy \\ & + Lu(x) \int_{|t|=1} \Gamma_i(x, t)\, t_j\, d\sigma_t. \end{aligned}$$

In the following we let $n \geqslant 3$, $\Omega$ be an open subset of $\mathbb{R}^n$, and

$$Lu(x) \equiv \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j}(x),$$

where

$$\begin{aligned}
&\text{(i)}\quad a_{ij}(x) \in \mathcal{V.M.O.}(\Omega) \\
&\text{(ii)}\quad a_{ij}(x) = a_{ji}(x) \qquad \forall i, j = 1, ..., n, \qquad \text{a.a. } x \in \Omega \qquad\qquad \text{(H)} \\
&\text{(iii)}\quad \exists \nu > 0, \qquad \nu^{-1} |\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqslant \nu\, |\xi|^2 \qquad \text{a.e. in } \Omega, \qquad \forall \xi \in \mathbb{R}^n.
\end{aligned}$$

Furthermore call $\eta_{ij}(r)$ the quantity associated to $a_{ij}(x)$ in Definition 1.1 and set $\eta(r) = (\sum_{i,j=1}^{n} \eta_{ij}^2(r))^{1/2}$.

Also we call

$$\max_{i,j=1,...,n}\ \max_{|\alpha| \leqslant 2n} \left\| \frac{\partial^\alpha \Gamma_{ij}(x, t)}{\partial t^\alpha} \right\|_{L^r(\Omega \times \Sigma)} = M.$$

We have the following

THEOREM 3.2. *Under assumption* (H), $p \in ]1, +\infty[$, $\lambda \in ]0, n[$, *there exist positive numbers* $C = C(n, p, \lambda, M)$ *and* $\rho_0 = \rho_0(C, n)$ *such that for any ball* $B_r \subset\subset \Omega$, $r < \rho_0$, *and any* $u \in W_0^{2,p}(B_r)$ *such that* $u_{x_i x_j} \in L^{p,\lambda}(B_r)$ *we have*

$$\|u_{x_i x_j}\|_{L^{p,\lambda}(B_r)} \leqslant C\, \|Lu\|_{L^{p,\lambda}(B_r)} \qquad \forall i, j = 1, ..., n.$$

*Proof.* This follows immediately from the representation formula in Theorem 3.1 and Theorems 2.3 and 2.4.

Now we derive our regularity results. The first one is a Morrey-type regularity theorem, while the second is a $C^{1,\alpha}$ result which extends, in some sense, a result of L. Caffarelli (cf., e.g., [$\mathrm{Ca}_1$, Sect. 3, Theorem 1; $\mathrm{Ca}_2$, Theorem 2]).

THEOREM 3.3. *Assume* (H), $1 < p < +\infty$, $0 < \lambda < n$. *Then if* $u \in W^{2,p}(\Omega)$, $Lu \in L^{p,\lambda}(\Omega)$, *and given* $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ *we have* $D^2u \in L^{p,\lambda}(\Omega')$ *and there exists a positive constant* $C \equiv C(n, p, \lambda, \nu, d(\Omega', \partial\Omega''), \eta)$ *such that*

$$\|D^2 u\|_{L^{p,\lambda}(\Omega')} \leqslant C\{\|u\|_{L^{p,\lambda}(\Omega'')} + \|Lu\|_{L^{p,\lambda}(\Omega'')}\}.$$

*Proof.* Let $B \subset \Omega$ be a ball whose radius will be chosen later.

For $i, j, h, k = 1, ..., n$ and $f \in L^{p,\lambda}(B_\rho)$ let us define

$$S_{ijhk}(f)(x) = P.V. \int_{B_\rho} \Gamma_{ij}(x, x-y)[a_{hk}(x) - a_{hk}(y)]\, f(y)\, dy.$$

By Theorem 2.4 we can take $\rho > 0$ so small that $\sum_{ijhk} \|S_{ijhk}\| < 1$. Next, we consider a ball $B$ with radius not greater then $\rho$ and a cut-off function $\beta \in C_0^\infty(B)$.

Since $v \equiv \beta u$ belongs to $W_0^{2,p}(B)$ we can apply Theorem 3.1 to get

$$v_{x_i x_j}(x) = \sum_{hk} S_{ijhk}(v_{x_h x_k}) + P.V. \int_B \Gamma_{ij}(x, x-y)\, Lv(y)\, dy + C_{ij}(x)\, Lv(x), \qquad (**)$$

where the $C_{ij}(x)$ are bounded functions arising from the formula in Theorem 3.1.

By the hypothesis $Lu \in L^{p,\lambda}(B)$ and from Sobolev's lemma it follows that $Lv \in L^{p,\lambda_1}(B)$ with $0 < \lambda_1 \leqslant \lambda$.

For the sake of simplicity let us now define

$$h \equiv (h_{ij})_{i,j=1,\ldots,n} \equiv P.V. \int_B \Gamma_{ij}(x, x-y)\, Lv(y)\, dy + C_{ij}(x)\, Lv(x)$$

from which it follows that

$$v_{x_i x_j} = \sum_{hk} S_{ijhk}(v_{x_h x_k}) + h_{ij}, \qquad i, j = 1, \ldots, n.$$

Finally let us define

$$Tw \equiv ((Tw)_{ij})_{i,j=1,\ldots,n} \equiv \Big(\sum_{hk} S_{ijhk}(w) + h_{ij}\Big)_{i,j=1,\ldots,n}$$

for every $w \in L^{p,\lambda_1}(B)$ and observe that clearly $T$ is a contraction in $[L^{p,\lambda_1}(B)]^{n^2}$.

Then we know that $T$ has a unique fixed point $\tilde{w}$ in $[L^{p,\lambda_1}(B)]^{n^2}$ and also in $[L^p(B)]^{n^2}$ (cf. [CFL]).

By formula (**), $(v_{x_i x_j})_{i,j=1,\ldots,n}$ is also a fixed point in $[L^p(B)]^{n^2}$ and then by the uniqueness $v_{x_i x_j} = \tilde{w}_{ij}$ $\forall i, j = 1, \ldots, n$. If $\lambda_1 = \lambda$ then we are done. Otherwise we get the conclusion iterating this argument a finite number of times. The estimate now follows by standard covering arguments and Theorem 3.2.

THEOREM 3.4 (cf. [$Ca_1$, Sect. 3, Theorem 1]). *Assume* (H), $1 < p < \infty$, $n - p < \lambda < n$. *Then if* $u \in W^{2,p}(\Omega)$, $Lu \in L^{p,\lambda}(\Omega)$ *given* $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ *we have* $Du \in C^{0,\alpha}(\bar{\Omega}')$ *where* $\alpha = 1 - n/p + \lambda/p$ *and moreover*

$$\|Du\|_{C^{0,\alpha}(\bar{\Omega}')} \leqslant C\{\|u\|_{L^{p,\lambda}(\Omega'')} + \|Lu\|_{L^{p,\lambda}(\Omega'')}\}.$$

*Here* $C$ *has the same meaning as in Theorem* 3.3.

*Proof.* Using Theorem 3.3 and the Sobolev–Morrey embedding (see, e.g., [Sta, Theorem 3.2]) we have at once the conclusion.

*Remark* 3.5. Let us observe that in the previous theorems we could weaken hypothesis (H) by substituting (i) with a requirement of sufficient smallness of $\eta(r)$.

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