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Existence and Location of Solutions to the Dirichlet Problem for a Class of Nonlinear Elliptic Equations

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Abstract—In a bounded open set $\Omega \subset \mathbf{R}^n$, we consider a Dirichlet problem of the type

$$-\Delta u = g(x,u) + h(x,u) + \alpha(x) + \frac{1}{\mu}(f(x,u) + l(x,u) + \beta(x)), \qquad \text{in } \Omega,$$
$$u_{|\partial\Omega} = 0,$$

where, in particular, $f(x,\cdot), g(x,\cdot)$ have a subcritical growth, and $h(x,\cdot), l(x,\cdot)$ are nonincreasing, with a critical growth. It is our aim to show that, for explicitly determined $\Psi : W_0^{1,2}(\Omega) \to \mathbf{R}$, and $\varphi :]r^*, +\infty[\to [0, +\infty[$, with $r^* = \inf_{W_0^{1,2}(\Omega)} \Psi$, for each $r > r^*$ and each $\mu > \varphi(r)$, the above problem has at least one weak solution that lies in $\Psi^{-1}(]-\infty, r[)$. A major novelty is just the precise determination of φ.

Keywords—Dirichlet problem, Weak solution, Location of solutions, Critical points.

Let $\Omega \subset \mathbf{R}^n$ $(n \geq 3)$ be an open bounded set, with smooth boundary. The object of this paper is to establish the following result.

THEOREM 1. *Let $\alpha, \beta \in L^{2n/(n+2)}(\Omega)$, and let $f, g, h, l : \Omega \times \mathbf{R} \to \mathbf{R}$ be four Carathéodory functions satisfying, in $\Omega \times \mathbf{R}$, the following conditions:*

$$\max\{|f(x,\xi)|, |g(x,\xi)|\} \leq a(1 + |\xi|^q),$$
$$\int_0^\xi (g(x,t) + h(x,t))\, dt \leq a(1 + |\xi|^s),$$
$$\max\{|h(x,\xi)|, |l(x,\xi)|\} \leq a\left(1 + |\xi|^{(n+2)/(n-2)}\right),$$

where a, q, s are three positive constants, with $s < 2$ and $q < (n+2)/(n-2)$. Finally, assume that, for each $x \in \Omega$, both the functions $h(x,\cdot)$ and $l(x,\cdot)$ are nonincreasing. For each $u \in$

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$W_0^{1,2}(\Omega)$, *put*

$$\Phi(u) = -\int_\Omega \left(\int_0^{u(x)} (f(x,\xi) + l(x,\xi))\,d\xi + \beta(x)u(x) \right) dx$$

and

$$\Psi(u) = \int_\Omega |\nabla u(x)|^2\,dx - 2\int_\Omega \left(\int_0^{u(x)} (g(x,\xi) + h(x,\xi))\,d\xi + \alpha(x)u(x) \right) dx.$$

Then, for each $r > \inf_{W_0^{1,2}(\Omega)} \Psi$ *and each* μ *satisfying*

$$\mu > \inf_{u\in\Psi^{-1}(]-\infty,r[)} \frac{\Phi(u) - \inf_{\overline{(\Psi^{-1}(]-\infty,r[))}_w} \Phi}{r - \Psi(u)},$$

where $\overline{(\Psi^{-1}(]-\infty,r[))}_w$ *is the closure of* $\Psi^{-1}(]-\infty,r[)$ *in the weak topology of* $W_0^{1,2}(\Omega)$, *the problem*

$$\begin{aligned} -\Delta u &= g(x,u) + h(x,u) + \alpha(x) + \frac{1}{2\mu}(f(x,u) + l(x,u) + \beta(x)), \qquad \text{in } \Omega,\\ u_{|\partial\Omega} &= 0 \end{aligned}$$

has at least one weak solution that lies in $\Psi^{-1}(]-\infty,r[)$.

As usual, if $\varphi : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function satisfying, in $\Omega\times\mathbf{R}$, a condition of the type

$$|\varphi(x,\xi)| \le a\left(1 + |\xi|^{(n+2)/(n-2)}\right),$$

a weak solution of the problem

$$\begin{aligned} -\Delta u &= \varphi(x,u), \qquad \text{in } \Omega,\\ u_{|\partial\Omega} &= 0 \end{aligned}$$

is any $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u(x)\nabla v(x)\,dx = \int_\Omega \varphi(x,u(x))v(x)\,dx,$$

for all $v \in W_0^{1,2}(\Omega)$.

The proof of Theorem 1 depends on a critical point theorem we have recently established in [1]. We now recall it for the reader's convenience.

THEOREM A. *(See [1, Theorem 2.5].) Let* X *be a reflexive real Banach space, and let* $\Phi, \Psi : X \to \mathbf{R}$ *be two sequentially weakly lower semicontinuous and Gâteaux differentiable functionals. Assume also that* Ψ *is strongly continuous and satisfies* $\lim_{\|x\|\to+\infty} \Psi(x) = +\infty$.

Then, for each $r > \inf_X \Psi$ *and each* μ *satisfying*

$$\mu > \inf_{x\in\Psi^{-1}(]-\infty,r[)} \frac{\Phi(x) - \inf_{\overline{(\Psi^{-1}(]-\infty,r[))}_w} \Phi}{r - \Psi(x)},$$

where $\overline{(\Psi^{-1}(]-\infty,r[))}_w$ *is the closure of* $\Psi^{-1}(]-\infty,r[)$ *in the weak topology of* X, *the functional* $\Phi + \mu\Psi$ *has a critical point that lies in* $\Psi^{-1}(]-\infty,r[)$.

PROOF OF THEOREM 1. We apply Theorem A taking $X = W_0^{1,2}(\Omega)$, with the norm $\|u\| = (\int_\Omega |\nabla u(x)|^2\,dx)^{1/2}$, and Φ, Ψ as defined in the statement of Theorem 1. Let us show that these functionals satisfy the required conditions. First of all, by a classical result (see, for instance, [2, p. 248]), they are continuously Gâteaux differentiable and, for each $\mu \in \mathbf{R} \setminus \{0\}$, the critical points of $\Phi + \mu\Psi$ are precisely the weak solutions of the problem

$$\begin{aligned} -\Delta u &= g(x,u) + h(x,u) + \alpha(x) + \frac{1}{2\mu}(f(x,u) + l(x,u) + \beta(x)), \qquad \text{in } \Omega, \\ u_{|\partial\Omega} &= 0. \end{aligned}$$

Now, observe that, thanks to the Rellich-Kondrachov Theorem, the functional

$$u \to \int_\Omega \left(\int_0^{u(x)} f(x,\xi)\, d\xi + \beta(x)u(x) \right) dx$$

is sequentially weakly continuous. Moreover, since $l(x,\cdot)$ is nondecreasing, the functional $u \to -\int_\Omega (\int_0^{u(x)} l(x,\xi)\, d\xi)\, dx$ is convex, and so it is weakly lower semicontinuous. Consequently, the functional Φ is sequentially weakly lower semicontinuous. In a similar way, it is seen that also Ψ is so. Next, let us show that Ψ is coercive. Indeed, by the Sobolev embedding theorem and the Poincaré inequality, for suitable positive constants c, c_0, we have

$$\begin{aligned} \Psi(u) &= \|u\|^2 - 2\int_\Omega \left(\int_0^{u(x)} (g(x,\xi) + h(x,\xi))\, d\xi \right) dx - 2\int_\Omega \alpha(x)u(x)\, dx \\ &\geq \|u\|^2 - 2a\int_\Omega (|u(x)|^s + 1)\, dx - c\|u\| \geq \|u\|^2 - c_0(\|u\|^s + 1) - c\|u\| \end{aligned}$$

for all $u \in X$. Consequently, $\lim_{\|u\|\to\infty} \Psi(u) = +\infty$, as claimed. Now, the conclusion follows directly from Theorem A. ∎

Some remarks on Theorem 1 are now in order.

REMARK 1. Observe that the condition

$$\int_0^\xi (g(x,t) + h(x,t))\, dt \leq a(1 + |\xi|^s)$$

can be replaced by

$$\int_0^\xi g(x,t)\, dt \leq a(1 + |\xi|^s).$$

This follows from the above proof taking into account that, by convexity, there is a constant $c > 0$ such that

$$-\int_\Omega \left(\int_0^{u(x)} h(x,\xi)\, d\xi \right) dx \geq -c(\|u\| + 1)$$

for all $u \in W_0^{1,2}(\Omega)$.

REMARK 2. When $q < 1$, for each $\mu > 0$, the functional $\Phi + \mu\Psi$ is coercive, and so it admits a global minimum on $W_0^{1,2}(\Omega)$ which is a weak solution of our problem. Nevertheless, even in that case, Theorem 1 gives an additional information: the location of a weak solution, expressed by the fact that it lies in $\Psi^{-1}(]-\infty, r[)$. Consequently, if for r and μ as in the statement one also has

$$\inf_{u \in W_0^{1,2}(\Omega)} (\Phi(u) + \mu\Psi(u)) < \inf_{u \in \Psi^{-1}(]-\infty,r[)} (\Phi(u) + \mu\Psi(u)),$$

the considered Dirichlet problem has at least two distinct weak solutions.

REMARK 3. Stressing again the information given about the location of a weak solution, observe that when $\inf_{W_0^{1,2}(\Omega)} \Psi < 0$, if we choose $r \in]\inf_{W_0^{1,2}(\Omega)} \Psi, 0[$ and μ as in the statement, we get a solution u of the problem such that $\Psi(u) < r < 0$. Since $\Psi(0) = 0$, we then have $u \neq 0$. In other words, Theorem 1 can also be used to obtain the existence of a nontrivial solution in cases where zero is a solution.

REMARK 4. When $q > 1$, the functional $\Phi + \mu\Psi$ can be unbounded below for all $\mu > 0$. This happens, for instance, if there exist $c, p > 0$, with $p < q$, such that, for all $(x, \xi) \in \Omega \times [0, +\infty[$, one has

$$f(x,\xi) \geq c(1 + |\xi|^q),$$
$$g(x,\xi) \geq -c(1 + |\xi|^p),$$

and

$$h(x,\xi) = l(x,\xi) = 0.$$

Among the consequences of Theorem 1, it is worth noticing the following.

THEOREM 2. *Let $\beta \in L^{2n/(n+2)}(\Omega)$, and let $f, g, h : \Omega \times \mathbf{R} \to \mathbf{R}$ be three Carathéodory functions satisfying, in $\Omega \times \mathbf{R}$, the following conditions:*

$$|f(x,\xi)| \leq a|\xi|^p + b,$$
$$|g(x,\xi)| \leq d(1 + |\xi|^q),$$
$$\int_0^\xi (g(x,t) + h(x,t))\,dt \leq 0,$$
$$|h(x,\xi)| \leq d\left(1 + |\xi|^{(n+2)/(n-2)}\right),$$

where $a, b, d > 0$, and $1 < p, q < (n+2)/(n-2)$. Finally, assume that, for each $x \in \Omega$, the function $h(x,\cdot)$ is nonincreasing. Put

$$c = \left(\sup_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\left(\int_\Omega |u(x)|^{p+1}\,dx\right)^{1/(p+1)}}{\left(\int_\Omega |\nabla u(x)|^2\,dx\right)^{1/2}} \right)^{p+1},$$
$$c_1 = \sup_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\left|\int_\Omega u(x)\,dx\right|}{\left(\int_\Omega |\nabla u(x)|^2\,dx\right)^{1/2}},$$

and

$$c_2 = \sup_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\left|\int_\Omega \beta(x)u(x)\,dx\right|}{\left(\int_\Omega |\nabla u(x)|^2\,dx\right)^{1/2}}.$$

Then, for each μ satisfying

$$\mu > p\left[\frac{ac}{p+1}\left(\frac{bc_1 + c_2}{p-1}\right)^{p-1}\right]^{1/p}$$

the problem

$$-\Delta u = g(x,u) + h(x,u) + \frac{1}{2\mu}(f(x,u) + \beta(x)), \qquad \text{in } \Omega,$$
$$u_{|\partial\Omega} = 0$$

has at least one weak solution u_0 satisfying

$$\int_\Omega |\nabla u_0(x)|^2\,dx - 2\int_\Omega \left(\int_0^{u_0(x)} (g(x,\xi) + h(x,\xi))\,d\xi\right)dx < \left(\frac{(p+1)(bc_1 + c_2)}{ac(p-1)}\right)^{2/p}.$$

PROOF. Let Φ, Ψ be defined as in the statement of Theorem 1, with $\alpha = l = 0$. Since $\int_0^\xi (g(x,t) + h(x,t))\, dt \le 0$, one has

$$\inf_{u \in W_0^{1,2}(\Omega)} \Psi(u) = \Psi(0) = 0$$

as well as, for each $r > 0$,

$$\inf_{u\in\Psi^{-1}(]-\infty,r[)} \frac{\Phi(u) - \inf_{\overline{(\Psi^{-1}(]-\infty,r[))}_w} \Phi}{r - \Psi(u)} \le \inf_{u\in\Psi^{-1}(]-\infty,r[)} \frac{\Phi(u) - \inf_{\Psi^{-1}(]-\infty,r])} \Phi}{r - \Psi(u)}$$
$$\le \frac{-\inf_{\Psi^{-1}(]-\infty,r])} \Phi}{r} \le \frac{\sup_{\int_\Omega |\nabla u(x)|^2\, dx \le r} -\Phi(u)}{r}$$
$$\le \frac{\sup_{\int_\Omega |\nabla u(x)|^2\, dx \le r} \left((a/(p+1)) \int_\Omega |u(x)|^{p+1}\, dx + b\int_\Omega |u(x)|\, dx + \int_\Omega \beta(x) u(x)\, dx\right)}{r}$$
$$\le \frac{(ac/(p+1))r^{(p+1)/2} + (bc_1 + c_2) r^{1/2}}{r}.$$

Now, an immediate calculation shows that

$$\inf_{r>0} \frac{(ac/(p+1))r^{(p+1)/2} + (bc_1 + c_2)r^{1/2}}{r} = p\left[\frac{ac}{p+1}\left(\frac{bc_1 + c_2}{p-1}\right)^{p-1}\right]^{1/p}$$

and that the infimum is attained at the point

$$\left(\frac{(p+1)(bc_1+c_2)}{ac(p-1)}\right)^{2/p}.$$

Now, the conclusion follows directly from Theorem 1. ∎

For instance, from Theorem 2 one can get propositions as follows.

PROPOSITION 1. *Let $n = 3$, let λ_1 be the first eigenvalue of the problem*

$$-\Delta u = \lambda u, \qquad \text{in } \Omega,$$
$$u_{|\partial\Omega} = 0$$

and set

$$c = \left(\sup_{u\in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\left(\int_\Omega |u(x)|^4\, dx\right)^{1/4}}{\left(\int_\Omega |\nabla u(x)|^2\, dx\right)^{1/2}}\right)^4.$$

Then, for each $\beta \in L^2(\Omega) \setminus \{0\}$ and each a satisfying

$$0 < a < \frac{2\lambda_1}{27c\|\beta\|^2_{L^2(\Omega)}}$$

the (unique) weak solution u_0 of the problem

$$-\Delta u = -u^5 + \beta(x), \qquad \text{in } \Omega,$$
$$u_{|\partial\Omega} = 0$$

satisfies the inequality

$$\int_\Omega |\nabla u_0(x)|^2\, dx + \frac{1}{3}\int_\Omega |u_0(x)|^6\, dx + \frac{a}{2}\int_\Omega |u_0(x)|^4\, dx < \left(\frac{2\|\beta\|_{L^2(\Omega)}}{ac\lambda_1^{1/2}}\right)^{2/3}.$$

PROOF. Apply Theorem 2 taking $p = 3$, and

$$f(x,\xi) = a\xi^3,$$
$$g(x,\xi) = -a\xi^3,$$
$$h(x,\xi) = -\xi^5.$$

To conclude, take into account that

$$c_2 \le \frac{\|\beta\|_{L^2(\Omega)}}{\lambda_1^{1/2}}$$

and that the choice of a gives

$$\frac{1}{2} > 3\left(\frac{acc_2^2}{16}\right)^{1/3}.$$

The uniqueness of the weak solution of the considered problem follows from the fact the functional

$$u \to \int_\Omega \left(\frac{1}{2}|\nabla u(x)|^2 + \frac{1}{6}|u(x)|^6 - \beta(x)u(x)\right) dx$$

is coercive and strictly convex, and so it has a unique global minimum. ∎

Finally, concerning the comparison of Theorem 1 with other known results, we believe that the closest one, in the spirit, is the following.

THEOREM B. *(See [3, Theorem 4].) Let $f : \Omega \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function satisfying, in $\Omega \times \mathbf{R}$, the condition*

$$|f(x,\xi)| \le a(1 + |\xi|^q),$$

where $a > 0$ and $0 < q < (n+2)/(n-2)$.

Then, the following alternative holds: either the problem

$$-\Delta u = f(x,u), \qquad \text{in } \Omega,$$
$$u_{|\partial\Omega} = 0$$

has at least one weak solution, or for each $r > 0$ there is $\lambda \in]0,1[$ such that the problem

$$-\Delta u = \lambda f(x,u), \qquad \text{in } \Omega,$$
$$u_{|\partial\Omega} = 0$$

has at least one weak solution u_0 satisfying $\|u_0\| = r$.

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