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Multiplicity results for a Neumann problem involving the p-Laplacian

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Abstract

In this paper, we establish some multiplicity results for the following Neumann problem:

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)+\lambda(x)|u|^{p-2}u=\alpha(x)f(u) & \text{in } \Omega, \\ \partial u/\partial\nu=0 & \text{on } \partial\Omega. \end{cases}$$

The multiple solutions are obtained by combining an existence theorem recently proved by G. Anello and G. Cordaro with well-known critical point theorems.

Keywords: Neumann problem; p-Laplacian; Multiple solutions

1. Introduction

Here and in the sequel, $\Omega\subset\mathbf{R}^N$ $(N\geqslant 1)$ is a bounded open set with boundary of class C^1, $f:\mathbf{R}\to\mathbf{R}$ is a continuous function, $\alpha\in L^1(\Omega)\setminus\{0\}$, $\alpha(x)\geqslant 0$ a.e. in Ω, $\lambda\in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega\lambda>0$, $p>N$.

In this paper we are interested in multiplicity results for the following Neumann problem:

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)+\lambda(x)|u|^{p-2}u=\alpha(x)f(u) & \text{in } \Omega, \\ \partial u/\partial\nu=0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where ν is the outward unit normal to $\partial\Omega$.

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Let us recall that a weak solution of (P) is any $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega \left|\nabla u(x)\right|^{p-2} \nabla u(x) \nabla v(x)\, dx + \int_\Omega \lambda(x) \left|u(x)\right|^{p-2} u(x) v(x)\, dx - \int_\Omega \alpha(x) f\big(u(x)\big) v(x)\, dx = 0$$

for all $v \in W^{1,p}(\Omega)$.

There is a wide literature that deals with multiplicity results for such a problem in the case $1 < p < N$. We refer, for instance, to [4] and references therein for details.

We find papers dealing with multiple solutions of nonlinear boundary value problem of the type

$$\begin{cases} -u'' = f(t,u), & t \in]0,\pi[,\ u \in \mathbf{R}, \\ u'(0) = \sigma_1, & u'(\pi) = \sigma_2. \end{cases}$$

In [5], the author assumes that f is a continuous "jumping" nonlinearity with nonnegative asymptotic limits. More specifically he requires that

$$\frac{f(t,u)}{u} \to \begin{cases} \mu & \text{as } u \to -\infty, \\ \nu & \text{as } u \to +\infty, \end{cases}$$

where μ and ν are nonnegative parameters and the above limits are uniform in t. We observe that Theorem 2.2 below is completely different from the ones mentioned above because of a superlinear behaviour of f.

In [6] the authors investigate the number of solutions of the one-dimensional homogeneous Neumann problem

$$\begin{cases} u'' + g(u) = s + h(t), & t \in]0,\pi[,\ u \in \mathbf{R}, \\ u'(0) = u'(\pi) = 0 \end{cases}$$

under the assumption of the existence of $\lim_{u\to+\infty} g'(u)$ or the requirement that $g'(u)$ increases from negative to positive as u increases. We show our different approach with an example where we stress the contrast between our conditions and those of Hart, Lazer and McKenna.

2. Results

Our first result is as follows:

Theorem 2.1. *Assume that*:

(1) *there exist* $\sigma > 0$, $\xi_0 \in \mathbf{R}$ *with* $|\xi_0| < \sigma/\|\lambda\|_{L^1(\Omega)}^{1/p}$ *such that*

$$F(\xi_0) = \sup_{|\xi| \leqslant c\sigma} F(\xi),$$

where

$$F(\xi)=\int\limits_0^{\xi} f(t)\,dt$$

and

$$c=\sup_{u\in W^{1,p}(\Omega)\setminus\{0\}}\frac{\sup_{x\in\Omega}|u(x)|}{(\int_\Omega|\nabla u(x)|^p\,dx+\int_\Omega\lambda(x)|u(x)|^p\,dx)^{1/p}};$$

(2)

$$\limsup_{|\xi|\to+\infty}\frac{F(\xi)}{|\xi|^p}<\frac{1}{pc^p\|\alpha\|_{L^1(\Omega)}};$$

(3) *there exists* $\bar{\xi}\in\mathbf{R}$ *such that*

$$F(\bar{\xi})-\sup_{|\xi|\leqslant c\sigma}F(\xi)>\frac{\|\lambda\|_{L^1(\Omega)}}{p\|\alpha\|_{L^1(\Omega)}}|\bar{\xi}|^p.$$

Then, (P) *admits at least three solutions in* $W^{1,p}(\Omega)$.

Proof. Let us introduce some notations. We endow $W^{1,p}(\Omega)$ with the norm

$$\|u\|=\left(\int\limits_\Omega\left|\nabla u(x)\right|^p dx+\int\limits_\Omega\lambda(x)\left|u(x)\right|^p dx\right)^{1/p}$$

which is equivalent to the usual one. For each $u\in W^{1,p}(\Omega)$ put

$$\Psi(u)=\frac{1}{p}\|u\|^p,$$
$$\Phi(u)=-\int\limits_\Omega\alpha(x)F\big(u(x)\big)\,dx.$$

Since $p>N$, $W^{1,p}(\Omega)$ is compactly embedded in $C^0(\bar{\Omega})$. This implies that the constant c is finite and that the functionals Ψ and Φ are well-defined, sequentially weakly lower semicontinuous and Gâteaux differentiable in $W^{1,p}(\Omega)$.

In particular we have

$$\Psi'(u)(v)=\int\limits_\Omega\left(\left|\nabla u(x)\right|^{p-2}\nabla u(x)\nabla v(x)+\lambda(x)\left|u(x)\right|^{p-2}u(x)v(x)\right)dx,$$
$$\Phi'(u)(v)=-\int\limits_\Omega\alpha(x)f\big(u(x)\big)v(x)\,dx$$

for all $v\in W^{1,p}(\Omega)$, and so each critical point of $\Psi+\Phi$ is a (weak) solution of (P).

The existence of a first solutions of problem (P) is proved by applying Theorem 2.1 of a recent paper by Anello and Cordaro (see [2]): by hypothesis (1) and the proof of that theorem in which the authors use the variational principle established by Ricceri in [8], the restriction of $\Psi + \Phi$ to the open ball of radius σ, centered at zero, has a minimum, say w_0. Consequently, w_0 is a critical point for $\Psi + \Phi$, and so a solution of (P) satisfying $\|w_0\| < \sigma$.

We claim now that the functional $\Psi + \Phi$ is coercive in $W^{1,p}(\Omega)$. Since (2) holds, there exist two constants $a, b \in \mathbf{R}$ with

$$0 < a < \frac{1}{pc^p\|\alpha\|_{L^1(\Omega)}} \tag{1}$$

such that $F(\xi) \leqslant a|\xi|^p + b$ for all $\xi \in \mathbf{R}$.

Fix $u \in W^{1,p}(\Omega)$. We have that $F(u(x)) \leqslant a|u(x)|^p + b$ for all $x \in \Omega$. Then, since $\alpha(x) \geqslant 0$ a.e. in Ω,

$$\int_\Omega \alpha(x)F\big(u(x)\big)\,dx \leqslant a\int_\Omega \alpha(x)\big|u(x)\big|^p\,dx + b\|\alpha\|_{L^1(\Omega)}$$
$$\leqslant a\|\alpha\|_{L^1(\Omega)}\|u\|^p_{C^0(\Omega)} + b\|\alpha\|_{L^1(\Omega)} \leqslant ac^p\|\alpha\|_{L^1(\Omega)}\|u\|^p + b\|\alpha\|_{L^1(\Omega)}.$$

So,

$$\Psi(u) + \Phi(u) \geqslant \left(\frac{1}{p} - ac^p\|\alpha\|_{L^1(\Omega)}\right)\|u\|^p - b\|\alpha\|_{L^1(\Omega)}.$$

By Eq. (1) the last member goes to infinity, as $\|u\| \to +\infty$.

We give now a lower estimate of the functional $\Psi + \Phi$ on the closed ball centered at zero of radius σ, $\bar{B}(0,\sigma)$. Let us choose $u \in W^{1,p}(\Omega)$, $\|u\| \leqslant \sigma$. We deduce that $\|u\|_{C^0(\bar{\Omega})} \leqslant c\sigma$, and

$$\Phi(u) = -\int_\Omega \alpha(x)F\big(u(x)\big)\,dx \geqslant -\|\alpha\|_{L^1(\Omega)} \sup_{|\xi|\leqslant c\sigma} F(\xi) = -\|\alpha\|_{L^1(\Omega)}F(\xi_0).$$

Then,

$$\inf_{u\in\bar{B}(0,\sigma)} (\Psi + \Phi)(u) \geqslant \inf_{u\in\bar{B}(0,\sigma)} \Psi(u) + \inf_{u\in\bar{B}(0,\sigma)} \Phi(u) \geqslant -\|\alpha\|_{L^1(\Omega)}F(\xi_0).$$

The functional $\Psi + \Phi$ is weakly sequentially lower semicontinuous and coercive in a reflexive Banach space. Hence it has a global minimum, say w_1. If we denote by $\bar{w}$ the constant function on Ω taking the value $\bar{\xi}$, by (3) we have

$$\Psi(\bar{w}) + \Phi(\bar{w}) = \frac{1}{p}\|\lambda\|_{L^1(\Omega)}|\bar{\xi}|^p - \int_\Omega \alpha(x)F(\bar{\xi})\,dx$$
$$= \frac{1}{p}\|\lambda\|_{L^1(\Omega)}|\bar{\xi}|^p - \|\alpha\|_{L^1(\Omega)}F(\bar{\xi}) < -\|\alpha\|_{L^1(\Omega)}F(\xi_0).$$

So, $w_1 \neq w_0$. Since w_1 is a minimum of $\Psi + \Phi$, then it is a critical point of the same functional, i.e., a solution of problem (P).

Another critical point of the functional $\Psi + \Phi$ is obtained applying a well-known result due to Pucci and Serrin (see [7]).

Let us check the Palais–Smale condition (PS) for $\Psi+\Phi$. We have that $\Psi+\Phi$ is coercive and it is easily seen that Φ' is compact. Proposition 1 of [3] ensures that Ψ' admits a continuous inverse on $(W^{1,p}(\Omega))^\star$. Therefore, by Example 38.25 of [11], we deduce that $\Psi+\Phi$ has the (PS) property.

Our conclusion follows by Corollary 1 of [7] which gives a third solution of problem (P). □

Example 1. Let $\gamma > 1$, $p > \max\{\gamma, N\}$, $\lambda(x) = 1$ and let Ω be a convex set whose diameter is less than or equal to $N^{1/p}((p-N)/(p-1))^{1-1/p}$.

Then, there exists a constant depending on γ, say c_γ, such that for each $\alpha \in L^1(\Omega)$, with $\alpha(x) \geqslant 0$, a.e. in Ω, and $\|\alpha\|_{L^1(\Omega)} > c_\gamma$, the problem

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + |u|^{p-2}u = \alpha(x)|u+1|^\gamma \sin(u+1) & \text{in } \Omega, \\ \partial u/\partial \nu = 0 \quad \text{on } \partial\Omega \end{cases}$$

has at least three solutions.

First of all we observe that under the hypotheses above we have the following estimate (see, for instance, [1]):

$$c \leqslant \left(\frac{2^{p-1}}{\operatorname{meas}(\Omega)}\right)^{1/p},$$

where $\operatorname{meas}(\Omega)$ denotes the measure of Ω.

Put $f(t) = |t+1|^\gamma \sin(t+1)$ and $F(\xi) = \int_0^\xi f(t)\,dt$.

(1) is satisfied with $\sigma = (\pi - 1/2)(\operatorname{meas}(\Omega))^{1/p}$, $\xi_0 = \pi - 1$. It is easy to check that

$$\limsup_{|\xi|\to+\infty} \frac{F(\xi)}{|\xi|^p} = 0.$$

(3) holds with $\bar{\xi} = (3\pi - 1)$, and $\|\alpha\|_{L^1(\Omega)} > c_\gamma$ where

$$c_\gamma = \frac{1}{p(F(3\pi-1) - F(\pi-1))}(3\pi-1)^p \operatorname{meas}(\Omega).$$

Remark 2.1. In [6] the one-dimensional homogeneous Neumann problem

$$\begin{cases} u'' + g(u) = s + h(t), & t \in\,]0,\pi[,\ u \in \mathbf{R}, \\ u'(0) = u'(\pi) = 0 \end{cases}$$

is studied under convenient hypotheses on g'. If we put $h = 0$ and $g(u) = \alpha f(u) - u + s$, with $\alpha > 0$ sufficiently large, we obtain the following Neumann problem:

$$\begin{cases} -u'' + u = \alpha f(u), & u \in \mathbf{R}, \\ u'(0) = u'(\pi) = 0. \end{cases}$$

Choose $f(u) = (u+1)\sin(u+1)$. Hence, $F(\xi) = -(\xi+1)\cos(\xi+1) + \sin(\xi+1) + \cos 1 - \sin 1$. The hypotheses of Theorem 2.1 are satisfied with $N = 1$, $p > 1$.

Since $g'(u) = \alpha f'(u) - 1 = -(u+1)\cos(u+1) + \sin(u+1) - 1$, we have

$$\limsup_{u\to\pm\infty} g'(u) = +\infty, \qquad \liminf_{u\to\pm\infty} g'(u) = -\infty,$$

and the results of [6], where the authors assume that $\lim_{u\to+\infty} g'(u)$ exists or that $g'(u)$ increases from negative to positive as u increases are not applicable.

In the following result we require that the functional $\Psi + \Phi$ has the (PS) property, but it is not coercive.

It is possible to apply again the result by Pucci and Serrin but we cannot assure the existence of the global minimum for our functional.

Theorem 2.2. *Assume that*:

(1) *there exist* $\sigma > 0$, $\xi_0 \in \mathbf{R}$ *with* $|\xi_0| < \sigma/\|\lambda\|^{1/p}_{L^1(\Omega)}$ *such that*

$$F(\xi_0) = \sup_{|\xi|\leqslant c\sigma} F(\xi);$$

(2) *there exist* $q > p$, $R_0 > 0$ *such that*

$$0 < qF(\xi) \leqslant f(\xi)\xi$$

for all ξ, $|\xi| \geqslant R_0$.

Then, (P) *admits at least two solutions in* $W^{1,p}(\Omega)$.

Proof. As in the proof of Theorem 2.1, there is a solution $w_0 \in W^{1,p}(\Omega)$ of (P), with $\|w_0\| < \sigma$.

We want to prove now that the functional $\Psi + \Phi$ satisfies (PS) condition and that it is unbounded below.

Let $\{u_n\} \subseteq W^{1,p}(\Omega)$ satisfy the following conditions:

(i) $\sup_{n\in\mathbf{N}} |\Psi(u_n) + \Phi(u_n)| < M$,
(ii) $\lim_{n\in\mathbf{N}} \|\Psi'(u_n) + \Phi'(u_n)\|_{(W^{1,p}(\Omega))^\star} = 0$.

The last condition implies that there exists $\nu \in \mathbf{N}$ such that for all $v \in W^{1,p}(\Omega)$ and $n > \nu$

$$\left|\Psi'(u_n)(v) + \Phi'(u_n)(v)\right| < \|v\|.$$

Hence,

$$\begin{aligned} qM + \|u_n\| &\geqslant q\big(\Psi(u_n) + \Phi(u_n)\big) - \Psi'(u_n)(u_n) - \Phi'(u_n)(u_n) \\ &= \frac{q}{p}\|u_n\|^p - q\int_\Omega \alpha(x)F\big(u_n(x)\big)\,dx - \|u_n\|^p \\ &\quad + \int_\Omega \alpha(x)f\big(u_n(x)\big)u_n(x)\,dx \\ &= \left(\frac{q}{p} - 1\right)\|u_n\|^p - \int_\Omega \alpha(x)\big[qF\big(u_n(x)\big) - f\big(u_n(x)\big)u_n(x)\big]\,dx. \end{aligned}$$

By hypothesis (2) we have that, if $n > \nu$,

$$\begin{aligned}
&\int_\Omega \alpha(x)\big[qF\big(u_n(x)\big) - f\big(u_n(x)\big)u_n(x)\big]\,dx \\
&\quad = \int_{\{x\in\Omega:\ |u_n(x)|\leqslant R_0\}} \alpha(x)\big[qF\big(u_n(x)\big) - f\big(u_n(x)\big)u_n(x)\big]\,dx \\
&\qquad + \int_{\{x\in\Omega:\ |u_n(x)|> R_0\}} \alpha(x)\big[qF\big(u_n(x)\big) - f\big(u_n(x)\big)u_n(x)\big]\,dx \\
&\quad \leqslant \max_{|\xi|\leqslant R_0}\big(qF(\xi) - f(\xi)\xi\big)\|\alpha\|_{L^1(\Omega)}.
\end{aligned}$$

Put $c_0 = \max_{|\xi|\leqslant R_0}(qF(\xi) - f(\xi)\xi)$.

Hence, if $n > \nu$,

$$\left(\frac{q}{p} - 1\right)\|u_n\|^p \leqslant \|u_n\| + qM + c_0\|\alpha\|_{L^1(\Omega)},$$

from which $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$. So, there exists a subsequence that we still denote by $\{u_n\}$, weakly convergent to some $u \in W^{1,p}(\Omega)$.

Hence, there exists a constant c_1 such that $\|u_n - u\| \leqslant c_1$ for each $n \in \mathbf{N}$.

By (ii), in correspondence of an arbitrary $\varepsilon > 0$, there exists $\nu \in \mathbf{N}$ such that for all $v \in W^{1,p}(\Omega)$ and $n > \nu$,

$$\big|\Psi'(u_n)(v) + \Phi'(u_n)(v)\big| < \frac{\varepsilon}{c_1}\|v\|.$$

Hence, $|\Psi'(u_n)(u_n - u) + \Phi'(u_n)(u_n - u)| < \varepsilon$ for $n > \nu$, that is

$$\lim_{n\to+\infty}\big[\Psi'(u_n)(u_n - u) + \Phi'(u_n)(u_n - u)\big] = 0.$$

Since $W^{1,p}(\Omega)$ is compactly embedded in $C^0(\bar{\Omega})$, $\{u_n\}$ converges to u in $C^0(\bar{\Omega})$. It is easy to show that the Lebesgue dominated convergence theorem implies that

$$\lim_{n\to+\infty}\Phi'(u_n)(u_n - u) = \lim_{n\to+\infty}\left(-\int_\Omega \alpha(x)f\big(u_n(x)\big)\big(u_n(x) - u(x)\big)\,dx\right) = 0,$$

and so

$$\lim_{n\to+\infty}\Psi'(u_n)(u_n - u) = 0. \tag{2}$$

Since $\Psi'(u)$ is a linear and continuous functional on $W^{1,p}(\Omega)$, we have also

$$\lim_{n\to+\infty}\Psi'(u)(u_n - u) = 0. \tag{3}$$

We claim that $\{u_n\}$ converges strongly to u in $W^{1,p}(\Omega)$. Observe that if $x, y \in \mathbf{R}^N$ and $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbf{R}^N$, there exist two positive constants c_p and $\bar{c}_p$ such that

$$\big\langle |x|^{p-2}x - |y|^{p-2}y,\, x - y\big\rangle \geqslant \begin{cases} c_p|x - y|^p & \text{if } p \geqslant 2, \\ c_p\frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } p \leqslant 2. \end{cases}$$

Thus, referring to [10] for details,

$$\Psi'(u_n)(u_n-u)-\Psi'(u)(u_n-u)\geqslant\begin{cases}c_p\|u_n-u\|^p & \text{if } p\geqslant 2,\\ \frac{\bar{c}_p}{M^{2-p}}\|u_n-u\|^2 & \text{if } p\leqslant 2,\end{cases}$$

for each $n\in\mathbf{N}$, where

$$M=\max\Big\{\|u\|,\sup_{n\in\mathbf{N}}\|u_n\|\Big\}.$$

Our claim follows immediately by Eqs. (2) and (3).

Let us check that $\Psi+\Phi$ is unbounded below. By our assumption

$$\xi|\xi|^q\frac{\partial}{\partial\xi}\big(|\xi|^{-q}F(\xi)\big)=-qF(\xi)+f(\xi)\xi\geqslant 0$$

for all ξ, $|\xi|\geqslant R_0$; and so, if $\xi\geqslant R_0$, $F(\xi)\geqslant R_0^{-q}F(R_0)|\xi|^q$; if $\xi\leqslant -R_0$, $F(\xi)\geqslant R_0^{-q}\times F(-R_0)|\xi|^q$, that is

$$F(\xi)\geqslant R_0^{-q}\min\big\{F(-R_0),F(R_0)\big\}|\xi|^q=c_2|\xi|^q,$$

with $c_2=R_0^{-q}\min\{F(-R_0),F(R_0)\}>0$ (on the strength of (2)), $|\xi|\geqslant R_0$.

Choose now a function $u\in W^{1,p}(\Omega)\setminus\{0\}$, with $\inf_\Omega u>0$, $\mu>R_0/(\inf_\Omega u)$.

$$\begin{aligned}\Psi(\mu u)+\Phi(\mu u)&=\frac{\mu^p}{p}\|u\|^p-\int_\Omega\alpha(x)F\big(\mu u(x)\big)\,dx\\&\leqslant\frac{\mu^p}{p}\|u\|^p-c_2\mu^q\int_\Omega\alpha(x)\big|u(x)\big|^q\,dx.\end{aligned}$$

Since $q>p$, the last member goes to $-\infty$, as $\mu\to+\infty$.

Our conclusion follows by the Theorem 1 of [7]: there exists a point $w_2\in W^{1,p}(\Omega)$ different from w_0, that is a critical point of $\Psi+\Phi$, i.e., a solution of (P). □

Remark 2.2. We remark that hypothesis (2) implies that the function f is superlinear in both directions. Namely, (2) implies that

$$\lim_{\xi\to\pm\infty}\frac{f(\xi)}{\xi}=+\infty.$$

The following theorem is a multiplicity result in the special case $N=1$, $p=2$. In the Hilbert space $H^1(\,]0,1[\,)$ it is possible to use an existence result by Schechter and Tintarev. For the reader's convenience, we recall its statement.

Proposition [9, Corollary of Theorem 2.5]. *Let g be a weakly continuous Frechét differentiable map from an infinite dimensional real Hilbert space H to* $\mathbf{R}$. *Define, for $t\geqslant 0$,*

$$S_t=\big\{x\in H:\ \|x\|^2=t\big\},\qquad \gamma(t)=\sup_{u\in S_t}g(u).$$

Let $t_0>0$ such that $\gamma(t_0)$ is not a maximum for γ. Then, there exists $\rho>0$ such that $g'(u)=\rho u$ has a solution u with $\|u\|^2>t_0$.

We deduce the following result:

Theorem 2.3. *Let* $\Omega =]0,1[,\ p = 2$. *Suppose that* $f : \mathbf{R} \to \mathbf{R}$ *is a continuous function,* $\alpha \in L^1(]0,1[) \setminus \{0\}$, $\alpha(t) \geqslant 0$ *a.e. in* $]0,1[$, $\lambda \in L^\infty(]0,1[)$ *with* $\operatorname{ess\,inf}_{]0,1[} \lambda > 0$. *Assume that*:

(1) *there exists* $\sigma > 0$, $\xi_0 \in \mathbf{R}$ *with* $|\xi_0| < \sigma / \|\lambda\|^{1/2}_{L^1(]0,1[)}$ *such that*

$$F(\xi_0) = \sup_{|\xi| \leqslant c\sigma} F(\xi);$$

(2) *there exists* $\bar{\xi} \in \mathbf{R}$, *such that*

$$F(\bar{\xi}) > \sup_{|\xi| \leqslant c\sigma} F(\xi).$$

Then, there exists $\nu > 0$ *such that the Neumann problem*

$$\begin{cases} -u'' + \lambda(t)u = \nu\alpha(t)f(u) & \text{in }]0,1[, \\ u'(0) = u'(1) = 0 \end{cases} \tag{P_ν}$$

admits at least two solutions in $H^1(]0,1[)$.

Proof. Put $X = H^1(]0,1[)$, $\Psi(u) = (1/2)\int_0^1 (|u'(t)|^2 + \lambda(t)|u(t)|^2)\,dt = (1/2)\|u\|^2$, $\Phi(u) = -\int_0^1 \alpha(t)F(u(t))\,dt$. We remark that critical points of $\Psi + \nu\Phi$ are exactly the solutions of $u + \nu\Phi'(u) = 0$.

By Theorem 2.1 of [2], there exists a solution $w_0 \in X$ of (P_ν) with $\|w_0\| < \sigma$. Let us apply the Proposition above with $g = -\Phi$, $t_0 = \sigma^2$, $t_1 = |\bar{\xi}|^2 \|\lambda\|^2_{L^1(]0,1[)}$. We claim that $\gamma(t_0) < \gamma(t_1)$.

Fix $u \in X$ such that $\|u\| = \sigma$. Then $\|u\|_{C^0([0,1])} \leqslant c\sigma$, where c is defined as in Theorem 2.1. By (2) it follows that

$$\sup_{\|u\|=\sigma} \int_0^1 \alpha(t)F\big(u(t)\big)\,dt \leqslant \sup_{|\xi| \leqslant c\sigma} \int_0^1 \alpha(t)F(\xi)\,dt$$

$$= \|\alpha\|_{L^1(]0,1[)} \sup_{|\xi| \leqslant c\sigma} F(\xi) < \|\alpha\|_{L^1(]0,1[)} F(\bar{\xi}) \leqslant \sup_{\|u\|=\sqrt{t_1}} \int_0^1 \alpha(t)F\big(u(t)\big)\,dt,$$

that is

$$\sup_{\|u\|^2 = t_0} -\Phi(u) < \sup_{\|u\|^2 = t_1} -\Phi(u).$$

Hence, we deduce the existence of $\rho > 0$ such that $\Phi'(u) + \rho u = 0$ has a solution $w_1 \in H^1(]0,1[)$, with $\|w_1\| > \sigma$. Put $\nu = 1/\rho$. Thus, w_1 is a second solution of (P_ν). □

Example 2. Let $\gamma > 2$. Then, for each $\alpha \in L^1(]0,1[) \setminus \{0\}$, $\alpha(t) \geqslant 0$ a.e. in $]0,1[$, there exists $\nu > 0$ such that the Neumann problem

$$\begin{cases} -u'' + u = \nu\alpha(t)|u+1|^\gamma \sin(u+1) & \text{in }]0,1[, \\ u'(0) = u'(1) = 0 \end{cases}$$

admits at least two solutions.

Put $f(t) = |t+1|^\gamma \sin(t+1)$ and $F(\xi) = \int_0^\xi f(t)\,dt$. Hypothesis (1) is fulfilled with $\sigma = \pi - 1/2$, $\xi_0 = \pi - 1$ (see Example 1); hypothesis (2) holds with $\bar{\xi} = 5\pi - 1$.

We observe that

$$\limsup_{|\xi| \to +\infty} \frac{F(\xi)}{\xi^2} = +\infty.$$

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