J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Stable determination of a Lamé coefficient by one internal measurement of displacement

Giuseppe Di Fazio [a,*], Elisa Francini [b], Fabio Raciti [a], Sergio Vessella [b]

[a] *Dipartimento di Matematica e Informatica, Università di Catania, Viale A. Doria 6, 95125, Catania, Italy*
[b] *Dipartimento di Matematica, Università di Firenze, Viale Morgagni 67 A, Firenze, Italy*

ARTICLE INFO

Article history:
Received 7 February 2017
Available online xxxx
Submitted by C. Gutierrez

Keywords:
Elastography
Stability
Interior data

ABSTRACT

In this paper we show that the shear modulus μ of an isotropic elastic body can be stably recovered by the knowledge of one single displacement field whose boundary data can be assigned independently on the unknown elasticity tensor.

1. Introduction

In this paper we consider the following problem: let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain representing an elastic isotropic body with Lamé coefficients λ and μ. Assuming λ is known, we want to stably recover the shear modulus μ from the knowledge of one single displacement field in Ω, that is a solution $u \in (H^1(\Omega))^n$ to the elasticity system

$$\operatorname{div}\left(\mathbb{C}\hat{\nabla}u\right) = 0 \text{ in } \Omega$$

where

$$\mathbb{C}\hat{\nabla}u = \lambda \operatorname{div}(u) I_n + \mu\hat{\nabla}u,$$

with I_n being the $n \times n$ identity matrix and $\hat{\nabla}u = \frac{1}{2}(\nabla u + (\nabla u)^T)$.

This problem is connected to the imaging method usually called Elastography. The most common approach to Elastography consists in a 2-step reconstruction. In the first step the elastic displacement is

* Corresponding author.
E-mail addresses: difazio@dmi.unict.it (G. Di Fazio), francini@math.unifi.it (E. Francini), fraciti@dmi.unict.it (F. Raciti), sergio.vessella@unifi.it (S. Vessella).

http://dx.doi.org/10.1016/j.jmaa.2017.03.013
0022-247X/

recovered from either sound waves (Ultrasound Elastography) or protons' propagation (Magnetic Resonance Elastography) (see [9] for a general review on hybrid inverse problems). In the present analysis we study the second step, namely the quantitative reconstruction of Lamé parameters from the knowledge of elastic displacements.

This problem has recently been studied in [10] and [18] for isotropic elasticity tensors in the time harmonic regime, and in [11] for the anisotropic case at zero frequency. In these papers a key point in the proof of unique and stable reconstruction consists in looking for few displacement fields satisfying a rank maximality condition concerning their gradients. Existence of such non degenerate sets of solutions is usually proven by using density arguments (such as Runge approximation) or CGO solutions. Unfortunately, it is not possible to choose a-priori boundary data of these particular solutions, since they depend on the interior values of the unknown elasticity tensor. For an analysis of this issue in the scalar case we refer to [13] and [1]. For an optimization approach to the identification of Lamé coefficients and to the elastography inverse problem the interested reader can refer to [14] and [16].

For this reason, following the method used in [3] and [8], we propose here the choice of boundary values in such a way that the degeneracy of ∇u can be controlled by quantitative estimates of unique continuation. We point out that this stability estimate is obtained with only one internal measurement. On the other hand, here we focus our attention only on the shear modulus μ and assume that λ is known. This is not a big restriction for the possible application of our result, because the shear modulus μ is the parameter that changes more between healthy and damaged tissues (see [20]).

The paper is organized as follows: in section 2 we list the main notations and assumptions; in section 3 we formulate the problem and state our result. The main tools of our analysis are an integral stability estimate (section 4) and quantitative estimates of unique continuation (section 5). Finally, section 6 contains the proof of our result.

2. Preliminary assumptions

We denote points in $\mathbb{R}^n$ by $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Analogously, we denote by Ω' the set of points in $\mathbb{R}^{n-1}$ such that (x', x_n) belong to Ω for some x_n.

Assumption 2.1. We assume that Ω, a bounded domain in $\mathbb{R}^n$ with $n \geq 2$, is a domain with $C^{1,1}$ boundary, that is, for any $x_0 \in \partial\Omega$ there exists a rigid change of coordinates such that, $x_0 = 0$ and

$$\Omega \cap B_{r_0}(0) = \{x \in B_{r_0}(0) \ : \ x_n > \psi(x')\},$$

where ψ is a $C^{1,1}$ function defined in $B'_{r_0}(0)$ such that $\psi(0) = 0$ and

$$\|\psi\|_{C^{1,1}(B'_{r_0}(0))} \leq M_0.$$

As usual, for any $d > 0$, we set

$$\Omega_d = \{x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) > d\}.$$

In the sequel we deal with the Lamé coefficients λ and μ, on which we posit the following assumptions.

Assumption 2.2.

$$\begin{cases} \mu, \lambda \in C^{0,1}(\bar{\Omega}) \\ \|\mu\|_{C^{0,1}(\bar{\Omega})} + \|\lambda\|_{C^{0,1}(\bar{\Omega})} \leq M \end{cases} \tag{2a}$$

$$\mu(x) \geq \alpha_0 > 0, \quad 2\mu(x) + n\lambda(x) \geq \beta_0 > 0 \text{ in } \bar{\Omega} \qquad \text{(strong convexity)} \tag{2b}$$

Please cite this article in press as: G. Di Fazio et al., Stable determination of a Lamé coefficient by one internal measurement of displacement, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.03.013

We also assume

Assumption 2.3. The function g belongs to $H^{3/2}(\partial\Omega)$, and

$$\|g\|_{H^{3/2}(\partial\Omega)} \leq L_0. \tag{1}$$

Moreover we assume that g is far from rigid movements, that is, given

$$\Theta(g) := \min\{\|g-(a+Wx)\|_{H^{1/2}(\partial\Omega)} \,:\, a\in\mathbb{R}^n,\ W\in\mathbb{R}^{n\times n}, W+W^T=0\}$$

we assume that

$$\Theta(g) \geq \delta_0 > 0. \tag{2}$$

In the sequel we will use the following frequency of function g:

Definition 2.1. For any $g \in H^1(\partial\Omega)$, we set

$$F[g] = \frac{\|g\|_{H^1(\partial\Omega)}}{\Theta(g)}$$

For any function g satisfying Assumption 2.3, we have $F[g] \leq L_0\delta_0^{-1}$.

3. Formulation of the problem and main result

Let us consider functions μ_1, μ_2 and λ such that (μ_1,λ) and (μ_2,λ) satisfy assumptions (2a) and (2b). Let u be the solution of the problem

$$\begin{cases} \operatorname{div}(\mathbb{C}_1\hat{\nabla}u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{3}$$

where

$$\mathbb{C}_1\hat{\nabla}u = \lambda \operatorname{div}(u) I_n + \mu_1\hat{\nabla}u,$$

and let v be the solution of the problem

$$\begin{cases} \operatorname{div}(\mathbb{C}_2\hat{\nabla}v) = 0 & \text{in } \Omega \\ v = k & \text{on } \partial\Omega \end{cases} \tag{4}$$

where

$$\mathbb{C}_2\hat{\nabla}v = \lambda \operatorname{div}(v) I_n + \mu_2\hat{\nabla}v.$$

Now we are ready to state our main result.

Theorem 3.1. *Let $d > 0$ be such that $\Omega_d \neq \emptyset$. Then*

$$\|\mu_1-\mu_2\|_{L^\infty(\Omega_d)} \leq C\left(\|\mu_1-\mu_2\|_{L^\infty(\partial\Omega)} + \|u-v\|_{L^2(\Omega)}^{1/4}\right)^{\delta} \tag{5}$$

where the constants $C > 0$ and $\delta \in (0,1)$ depend only on M_0, $|\Omega|$, r_0, d, M, α_0, β_0, L_0 and δ_0.

Please cite this article in press as: G. Di Fazio et al., Stable determination of a Lamé coefficient by one internal measurement of displacement, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.03.013

4. An integral estimate

Lemma 4.1. *Let u and v be as in (3) and (4). Moreover, let Assumptions 2.1 and 2.2 hold true. Then, there exists a positive constant C depending only on M_0, r_0, M, α_0, β_0, $\|g\|_{H^{3/2}(\partial\Omega)}$ and $\|k\|_{H^{3/2}(\partial\Omega)}$ such that*

$$\int_\Omega |\mu_1 - \mu_2||\hat{\nabla} u|^2 \, dx \leq C \left(\|\mu_1 - \mu_2\|_{L^\infty(\partial\Omega)} + \|u - v\|_{L^2(\Omega)}^{1/4} \right) . \tag{6}$$

In the sequel we will use the following notation. The dot between vectors is the scalar product while the dot between matrices is the product in the sense of Frobenius.

Proof of Lemma 4.1. Let $u, v \in W^{1,2}(\Omega)$ be a solutions of (3) and (4), respectively.

Let us set

$$\varphi := \mu_1 - \mu_2, \quad \eta := \max_{\partial\Omega} |\mu_1 - \mu_2|.$$

By comparing the weak formulations of problems (3) and (4) we easily get

$$\int_\Omega \varphi \hat{\nabla} u \cdot \hat{\nabla} \zeta \, dx = -\frac{1}{2} \int_\Omega \left(\lambda \operatorname{div}(u - v) \operatorname{div} \zeta + 2\mu_2 \hat{\nabla}(u - v) \cdot \hat{\nabla} \zeta \right) dx, \tag{7}$$

for every $\zeta \in W_0^{1,2}(\Omega)$.

To show the inequality we follow the route traced by [2] by choosing a suitable test function. For $h > 0$ set

$$\zeta(x) = \frac{\min\{(\varphi - \eta)^+, h)\}}{h} u(x) \tag{8}$$

as our test function. We can easily check that

$$\zeta(x) = \begin{cases} 0 & \text{if } \varphi \leq \eta \\ \dfrac{\varphi - \eta}{h} u & \text{if } \eta < \varphi \leq \eta + h \\ u & \text{if } \varphi > \eta + h. \end{cases}$$

Let us first consider the LHS of (7) with test function given by (8). We have

$$\begin{aligned} \int_\Omega \varphi \hat{\nabla} u \cdot \hat{\nabla} \zeta \, dx &= \int_{\eta < \varphi < \eta + h} \varphi \hat{\nabla} u \cdot \hat{\nabla} \left(\frac{\varphi - \eta}{h} u \right) dx + \int_{\varphi > \eta + h} \varphi |\hat{\nabla} u|^2 dx \\ &= \frac{1}{2h} \int_{\eta < \varphi < \eta + h} \varphi \hat{\nabla} u \cdot [\nabla \varphi \otimes u + u \otimes \nabla \varphi] \, dx \\ &\quad + \frac{1}{h} \int_{\eta < \varphi < \eta + h} \varphi (\varphi - \eta) |\hat{\nabla} u|^2 \, dx + \int_{\varphi > \eta + h} \varphi |\hat{\nabla} u|^2 dx \end{aligned} \tag{9}$$

Let us now focus on the integral

$$I = \frac{1}{2h} \int_{\eta < \varphi < \eta + h} \varphi \hat{\nabla} u \cdot [\nabla \varphi \otimes u + u \otimes \nabla \varphi] \, dx. \tag{10}$$

By using the fact that, for any symmetric matrix A and for any vector b and c $A\cdot[b\otimes c+c\otimes b]=2b^TAc$,

$$\begin{aligned}\frac{1}{2}\varphi\hat{\nabla}u\cdot[\nabla\varphi\otimes u+u\otimes\nabla\varphi]&=(\nabla\varphi)^T\varphi u\hat{\nabla}u=\frac{1}{2}(\nabla\varphi^2)^T\hat{\nabla}u\,u\\&=\frac{1}{2}\operatorname{div}[\varphi^2\hat{\nabla}u\,u]-\frac{1}{2}\varphi^2\operatorname{div}(\hat{\nabla}u\,u)\\&=\frac{1}{2}\operatorname{div}[\varphi^2u\hat{\nabla}u]-\frac{1}{2}\varphi^2\operatorname{div}(u\hat{\nabla}u).\end{aligned}\tag{11}$$

Let us denote by ν_c the unit outer normal to the set $\{\varphi>c\}$ and let us apply twice Green formula to obtain

$$\begin{aligned}I&=\frac{1}{2h}\left\{\eta^2\int\limits_{\varphi=\eta}(u\hat{\nabla}u)\cdot\nu_\eta\,ds_x-(\eta+h)^2\int\limits_{\varphi=\eta+h}(u\hat{\nabla}u)\cdot\nu_{\eta+h}\,ds_x\right\}\\&\qquad-\frac{1}{2h}\int\limits_{\eta<\varphi<\eta+h}\varphi^2\operatorname{div}(u\hat{\nabla}u)\,dx\\&=\frac{1}{2h}\left\{\eta^2\int\limits_{\varphi>\eta}\operatorname{div}(u\hat{\nabla}u)\,dx-(\eta+h)^2\int\limits_{\varphi>\eta+h}\operatorname{div}(u\hat{\nabla}u)\,dx\right\}\\&\qquad-\frac{1}{2h}\int\limits_{\eta<\varphi<\eta+h}\varphi^2\operatorname{div}(u\hat{\nabla}u)\,dx\\&=-\frac{1}{2h}\int\limits_{\eta<\varphi<\eta+h}(\varphi^2-\eta^2)\operatorname{div}(u\hat{\nabla}u)\,dx-(\eta+h/2)\int\limits_{\varphi>\eta+h}\operatorname{div}(u\hat{\nabla}u)\,dx.\end{aligned}$$

It is easy to check that in the set $\{\eta<\varphi<\eta+h\}$ we have $0\le\frac{\varphi^2-\eta^2}{2h}\le\eta+\frac{h}{2}$, and, hence,

$$|I|\le(\eta+h/2)\int\limits_{\varphi>\eta}|\operatorname{div}(u\hat{\nabla}u)|\,dx\tag{12}$$

By putting together (9), (10) and (12), we have that, for ζ given by (8)

$$\begin{aligned}\int\limits_{\Omega}\varphi\hat{\nabla}u\cdot\hat{\nabla}\zeta\,dx&\ge-|I|+\frac{1}{h}\int\limits_{\eta<\varphi<\eta+h}\varphi(\varphi-\eta)|\hat{\nabla}u|^2\,dx+\int\limits_{\varphi>\eta+h}\varphi|\hat{\nabla}u|^2dx\\&\ge-(\eta+h/2)\int\limits_{\varphi>\eta}|\operatorname{div}(u\hat{\nabla}u)|\,dx+\int\limits_{\varphi>\eta+h}\varphi|\hat{\nabla}u|^2dx.\end{aligned}\tag{13}$$

Let us now estimate the RHS of (7) for ζ given by (8).

We have

$$\begin{aligned}\int\limits_{\Omega}\lambda\operatorname{div}(u-v)\operatorname{div}\zeta\,dx&=\int\limits_{\eta<\varphi<\eta+h}\lambda\operatorname{div}(u-v)\left[\frac{\varphi-\eta}{h}\operatorname{div}(u)+\frac{\nabla\varphi\cdot u}{h}\right]dx+\\&\quad+\int\limits_{\varphi>\eta+h}\lambda\operatorname{div}(u-v)\operatorname{div}(u)\,dx\end{aligned}$$

then, by Assumption 2.2,

$$\left|\int_\Omega \lambda \operatorname{div}(u-v) \operatorname{div}\zeta dx\right| \le M\left(\int_{\varphi>\eta} |\operatorname{div}(u-v)\operatorname{div} u|\, dx + \frac{1}{h}\int_{\varphi>\eta+h} |\operatorname{div}(u-v)||\nabla\varphi||u|\, dx\right) \tag{14}$$

We proceed in the same way to estimate

$$\begin{aligned}
\int_\Omega 2\mu_2 \hat\nabla(u-v)\cdot\hat\nabla\zeta\, dx &= \frac{1}{h}\int_{\eta<\varphi<\eta+h} \mu_2\hat\nabla(u-v)\cdot[\nabla\varphi\otimes u + u\otimes\nabla\varphi + 2(\varphi-\eta)\hat\nabla u]\, dx \\
&\quad + \int_{\varphi>\eta+h} 2\mu_2\hat\nabla(u-v)\cdot\hat\nabla u\, dx \\
&= \frac{1}{h}\int_{\eta<\varphi<\eta+h} 2\mu_2(\nabla\varphi)^T\hat\nabla(u-v)\, u\, dx + \\
&\quad + \frac{1}{h}\int_{\eta<\varphi<\eta+h} 2\mu_2(\varphi-\eta)\hat\nabla(u-v)\cdot\hat\nabla u\, dx + \\
&\quad + \int_{\varphi>\eta+h} 2\mu_2\hat\nabla(u-v)\cdot\hat\nabla u\, dx.
\end{aligned}$$

By Assumption 2.2, we get

$$\begin{aligned}
\left|\int_\Omega 2\mu_2\hat\nabla(u-v)\cdot\hat\nabla\zeta\, dx\right| &\le \frac{2M}{h}\int_{\eta<\varphi<\eta+h} |(\nabla\varphi)||\hat\nabla(u-v)\, u|\, dx \\
&\quad + 2M\int_{\varphi>\eta} |\hat\nabla(u-v)\cdot\hat\nabla u|\, dx
\end{aligned} \tag{15}$$

Finally, by putting together (13), (7), (14) and (15) we get

$$\begin{aligned}
\int_{\varphi>\eta+h} \varphi|\hat\nabla u|^2\, dx &\le \left(\eta+\frac{h}{2}\right)\int_{\varphi>\eta} |\operatorname{div}(u\hat\nabla u)|\, dx + \\
&\quad + \frac{M}{2}\left(\int_{\varphi>\eta} |\operatorname{div}(u-v)\operatorname{div} u|\, dx + \frac{1}{h}\int_{\varphi>\eta+h} |\operatorname{div}(u-v)||\nabla\varphi||u|\, dx\right) \\
&\quad + M\left(\int_{\varphi>\eta} |\hat\nabla(u-v)\cdot\hat\nabla u|\, dx + \frac{1}{h}\int_{\eta<\varphi<\eta+h} |(\nabla\varphi)||\hat\nabla(u-v)\, u|\, dx\right).
\end{aligned}$$

If we use $-\varphi$ instead of φ we find a similar estimate. Then, merging the two, we get

Please cite this article in press as: G. Di Fazio et al., Stable determination of a Lamé coefficient by one internal measurement of displacement, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.03.013

$$\int_{|\varphi|>\eta+h} |\varphi||\hat{\nabla}u|^2\,dx \le \left(\eta+\frac{h}{2}\right)\int_{|\varphi|>\eta} |\operatorname{div}(u\hat{\nabla}u)|\,dx + \\ + \frac{M}{2}\left(\int_{|\varphi|>\eta} |\operatorname{div}(u-v)\operatorname{div}u|\,dx + \frac{1}{h}\int_{|\varphi|>\eta+h} |\operatorname{div}(u-v)||\nabla\varphi||u|\,dx\right) \\ + M\left(\int_{|\varphi|>\eta} |\hat{\nabla}(u-v)\cdot\hat{\nabla}u|\,dx + \frac{1}{h}\int_{\eta<|\varphi|<\eta+h} |(\nabla\varphi)||\hat{\nabla}(u-v)\,u|\,dx\right) \tag{16}$$

Since

$$\int_{|\varphi|<\eta+h} |\varphi||\hat{\nabla}u|^2\,dx \le (\eta+h)\int_{|\varphi|<\eta+h} |\hat{\nabla}u|^2\,dx \le (\eta+h)\int_{\Omega} |\hat{\nabla}u|^2\,dx$$

we obtain, from (16)

$$\int_{\Omega} |\varphi||\hat{\nabla}u|^2\,dx \le (\eta+h)\int_{\Omega} |\hat{\nabla}u|^2\,dx + \left(\eta+\frac{h}{2}\right)\int_{\Omega} |\operatorname{div}(u\hat{\nabla}u)|\,dx + \\ + \frac{M}{2}\left(\int_{\Omega} |\operatorname{div}(u-v)\operatorname{div}u|\,dx + \frac{1}{h}\int_{\Omega} |\operatorname{div}(u-v)||\nabla\varphi||u|\,dx\right) \\ + M\left(\int_{\Omega} |\hat{\nabla}(u-v)\cdot\hat{\nabla}u|\,dx + \frac{1}{h}\int_{\Omega} |\nabla\varphi||\hat{\nabla}(u-v)\,u|\,dx\right) \tag{17}$$

By [21, Theorem 7.1, chap. 3], we have

$$\|u\|_{H^2(\Omega)} \le C\|g\|_{H^{3/2}(\partial\Omega)}$$

and, since $|\nabla\varphi| \le 2M$ by (2.2), by Hölder inequality, we can easily get from (17) that

$$\int_{\Omega} |\varphi||\hat{\nabla}u|^2\,dx \le C\|g\|_{3/2}\left\{(3\eta+2h)\|g\|_{3/2}\right. \\ \left. + 2M\left(1+\frac{2M}{h}\right)\left(\|\operatorname{div}(u-v)\|_{L^2(\Omega)} + \|\hat{\nabla}(u-v)\|_{L^2(\Omega)}\right)\right\} \tag{18}$$

where $\|g\|_{3/2} = \|g\|_{H^{3/2}(\partial\Omega)}$.

Let us notice that, by using an interpolation inequality (see, for example [15, Theorem 7.28]), if we denote by D any partial derivative of first order, we have

$$\|D(u-v)\|_{L^2(\Omega)} \le \varepsilon\|u-v\|_{H^2(\Omega)} + \frac{c}{\varepsilon}\|u-v\|_{L^2(\Omega)} \\ \le C\left[\varepsilon(\|g\|_{3/2}+\|k\|_{3/2}) + \frac{1}{\varepsilon}\|u-v\|_{L^2(\Omega)}\right] \tag{19}$$

Now choose $\varepsilon = \|u-v\|^{1/2}_{L^2(\Omega)}$ and we get

Please cite this article in press as: G. Di Fazio et al., Stable determination of a Lamé coefficient by one internal measurement of displacement, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.03.013

$$\|D(u-v)\|_{L^2(\Omega)} \leq C\left(\|g\|_{3/2}+\|k\|_{3/2}+1\right)\|u-v\|_{L^2(\Omega)}^{1/2}, \tag{20}$$

with C depending only on M_0, r_0, M, α_0, β_0.

By (18) and (20) we have

$$\begin{aligned}\int_\Omega |\varphi||\hat{\nabla} u|^2\,dx \leq C\|g\|_{3/2}\Big\{&(3\eta+2h)\|g\|_{3/2} \\ &+3M\left(1+\frac{M}{h}\right)\left(\|g\|_{3/2}+\|k\|_{3/2}+1\right)\|u-v\|_{L^2(\Omega)}^{1/2}\Big\}\end{aligned} \tag{21}$$

We finally choose $h=\|u-v\|_{L^2(\Omega)}^{1/4}$ and get (6). □

5. Quantitative estimates of unique continuation

Most of the results that we state and prove in this section are already known for solutions of the Lamé system, but they are usually stated in terms of Neumann boundary conditions. We want here to use Dirichlet boundary conditions that are better related to the internal measurements we are going to use.

Theorem 5.1 *(Three sphere inequality for $|\hat{\nabla} u|$). Under Assumption 2.2, there exists $\theta \in (0,1]$ depending only on α_0, β_0 and M such that for every $u \in H^1(B_R)$ solution to the equation*

$$\operatorname{div}(\mathbb{C}\hat{\nabla} u)=0$$

and for every r_1, r_2, r_3 such that $0<r_1<r_2<r_3<\theta R$ we have

$$\int_{B_{r_2}} |\hat{\nabla} u|^2 dx \leq C\left(\int_{B_{r_1}} |\hat{\nabla} u|^2 dx\right)^{\delta}\left(\int_{B_{r_3}} |\hat{\nabla} u|^2 dx\right)^{1-\delta}, \tag{22}$$

where $C>0$ and $\delta \in (0,1)$ depend only on α_0, β_0, M, r_1/r_3 and r_2/r_3.

Proof. The proof of this estimates goes along the same lines of the proof of Corollary 3.3 in [6]. The regularity of the Lamé coefficients can be lowered by starting from the three spheres inequality for the solution proved in [19, Theorem 1.1]. □

Theorem 5.2 *(Lipschitz Propagation of Smallness). Under Assumptions 2.1, 2.2 and 2.3, let $u \in H^1(\Omega)$ be solution to (3). Then, for every $\rho>0$ and for every $x \in \Omega_{5\rho}$, we have*

$$\int_{B_\rho(x)} |\hat{\nabla} u|^2 dx \geq C_\rho \int_\Omega |\hat{\nabla} u|^2 dx, \tag{23}$$

where C_ρ depends on α_0, β_0, M, r_0, M_0, $|\Omega|$, $F[g]$, and ρ.

Proof. The proof follows essentially the same lines of the proof of Proposition 4.1 in [6]. First of all, as in Lemma 4.2 in [6], by Hölder inequality and Sobolev inequality we can estimate

$$\int_{\Omega\setminus\Omega_{5\rho/8}} |\hat{\nabla} u|^2 dx \leq C\rho^{1/n}\|g\|_{H^1(\partial\Omega)}^2. \tag{24}$$

Please cite this article in press as: G. Di Fazio et al., Stable determination of a Lamé coefficient by one internal measurement of displacement, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.03.013

The only difference consists in substituting inequality (4.6) in [6] with the trace estimate

$$\|u\|_{H^{3/2}(\Omega)} \leq C\|g\|_{H^1(\partial\Omega)}.$$

As in (4.12) in [6], by using a suitable chain of balls and the three spheres inequality (Theorem 5.1) we get

$$\frac{\|\hat{\nabla} u\|_{L^2(\Omega_{5\rho/8})}}{\|\hat{\nabla} u\|_{L^2(\Omega)}} \leq \frac{C}{\rho^{n/2}} \left(\frac{\|\hat{\nabla} u\|_{L^2(B_\rho(x))}}{\|\hat{\nabla} u\|_{L^2(\Omega)}} \right)^{\delta^L} \tag{25}$$

where C and δ depends only on α_0, β_0, M and $|\Omega|$, whereas $L \leq \frac{|\Omega|}{\omega_n \rho^n}$.

By (24), we have

$$\frac{\|\hat{\nabla} u\|_{L^2(\Omega_{5\rho/8})}}{\|\hat{\nabla} u\|_{L^2(\Omega)}} = 1 - \frac{\|\hat{\nabla} u\|_{L^2(\Omega \setminus \Omega_{5\rho/8})}}{\|\hat{\nabla} u\|_{L^2(\Omega)}} \geq 1 - \frac{C\rho^{1/n}\|g\|_{H^1(\partial\Omega)}}{\|\hat{\nabla} u\|_{L^2(\Omega)}}.$$

Now, we need to estimate $\|\hat{\nabla} u\|_{L^2(\Omega)}$ from below. Let us set

$$\overline{a} = \frac{1}{|\Omega|} \int_\Omega u\, dx, \quad \overline{W} = \frac{1}{|\Omega|} \int_\Omega \hat{\nabla} u\, dx. \tag{26}$$

By trace inequality and Korn inequality we have

$$\|g - (\overline{a} + \overline{W}x)\|_{H^{1/2}(\partial\Omega)} \leq C\|u - (\overline{a} + \overline{W}x)\|_{H^1(\Omega)} \leq C\|\hat{\nabla} u\|_{L^2(\Omega)}, \tag{27}$$

and, hence,

$$\frac{\|\hat{\nabla} u\|_{L^2(\Omega_{5\rho/8})}}{\|\hat{\nabla} u\|_{L^2(\Omega)}} \geq 1 - \frac{C\rho^{1/n}\|g\|_{H^1(\partial\Omega)}}{\|g - (\overline{a} + \overline{W}x)\|_{H^{1/2}(\partial\Omega)}} \geq 1 - C\rho^{1/n}F[g]. \tag{28}$$

Let us take $\overline{\rho}$ such that

$$1 - C\overline{\rho}^{1/n}F[g] \geq \frac{1}{2}$$

so that, by (25) and (28) the thesis (23) follows for $\rho \leq \overline{\rho}$. For larger values of ρ inequality (23) is trivial. □

Now, we need a doubling inequality for $\hat{\nabla} u$. We start with recalling a doubling inequality for u that corresponds to [17, Theorem 1.2] (see also [7] for similar results).

Theorem 5.3. *Under Assumption 2.2, there exists a positive constant C such that for every $v \in H^1(B_{2R})$ solution to $\mathrm{div}(\mathbb{C}\hat{\nabla} v) = 0$ we have*

$$\int_{B_{2r}(x)} |v|^2 dx \leq C \int_{B_r(x)} |v|^2 dx \tag{29}$$

for every $B_{2r}(x) \subset B_{R/2}$ and with C depending on α_0, β_0, M and increasingly on

$$F_{loc} = \frac{\|v\|_{L^2(B_{2R} \setminus B_R)}}{\|v\|_{L^2(B_R \setminus B_{R/2})}}$$

Theorem 5.4. *Under Assumptions 2.1, 2.2 and 2.3, let u be a solution to (3). Then, for every $x_0 \in \Omega_d$ and $0 < r \leq d$,*

$$\int_{B_r(x_0)} |\hat{\nabla} u|^2 \, dx \geq C_d \left(\frac{r}{d}\right)^K \|g\|^2_{H^{1/2}(\partial\Omega)}, \tag{30}$$

where C_d and K depend on α_0, β_0, M, r_0, M_0, $|\Omega|$ and K depends also on $F[g]$.

Proof. Let

$$v = u - c_r - W_r(x - x_0)$$

where

$$c_r = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u \, dx \text{ and } W_r = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \hat{\nabla} u \, dx.$$

Since function v is still a solution of equation $\operatorname{div}(\mathbb{C}_1 \hat{\nabla} v) = 0$ in Ω, by Caccioppoli inequality (see [6, Lemma 3.4]) we have

$$\int_{B_{3r/2}(x_0)} |\nabla v|^2 dx \leq \frac{C}{r^2} \int_{B_{2r}(x_0)} |v|^2 dx,$$

where C depends only on α_0, β_0 and M, hence, trivially,

$$\int_{B_{3r/2}(x_0)} |\hat{\nabla} u|^2 dx = \int_{B_{3r/2}(x_0)} |\hat{\nabla} v|^2 dx \leq \frac{C}{r^2} \int_{B_{2r}(x_0)} |v|^2 dx. \tag{31}$$

By Korn inequality (see [6, Lemma 3.5])

$$\int_{B_r(x_0)} |v|^2 dx = \int_{B_r(x_0)} |u - c_r - W_r(x - x_0)|^2 dx \leq C r^2 \int_{B_r(x_0)} |\hat{\nabla} u|^2 dx. \tag{32}$$

By (29), (31) and (32) we have

$$\int_{B_{3r/2}(x_0)} |\hat{\nabla} u|^2 dx \leq C \int_{B_r(x_0)} |\hat{\nabla} u|^2 dx \tag{33}$$

where C depends on α_0, β_0, M, r_0, M_0, $|\Omega|$ and increasingly on

$$F_{r,loc} = \frac{\|u - c_r - W_r(x - x_0)\|_{L^2(B_{2R} \setminus B_R)}}{\|u - c_r - W_r(x - x_0)\|_{L^2(B_R \setminus B_{R/2})}}.$$

Now, we need to bound $F_{r,loc}$ from above independently of r. First of all we notice that,

$$|c_r| \leq \|u\|_{L^\infty(B_r(x_0))} \leq \|u\|_{L^\infty(B_{2R})}, \text{ and } |W_r| \leq \|\nabla u\|_{L^\infty(B_r(x_0))} \leq \|\nabla u\|_{L^\infty(B_{2R})},$$

hence, by internal regularity estimates (see, for example [12]) and [21, Theorem 4.2, chap. 3],

$$\begin{aligned}\|u - c_r - W_r(x - x_0)\|_{L^2(B_{2R}\setminus B_R)} &\le C\left(\|u\|_{L^\infty(B_{2R})} + \|\nabla u\|_{L^\infty(B_{2R})}\right)\\ &\le C\|u\|_{H^1(\Omega)} \le C\|g\|_{H^{1/2}(\partial\Omega)}.\end{aligned} \tag{34}$$

Let us now consider a ball $B_{r_1}(\overline{x}) \subset B_R \setminus B_{R/2}$ with $r_1 = \max\{d/5, R/4\}$ and notice that, by Caccioppoli inequality,

$$\begin{aligned}\int\limits_{B_R\setminus B_{R/2}} |u - c_r - W_r(x - x_0)|^2 dx &\ge \int\limits_{B_{r_1}(\overline{x})} |u - c_r - W_r(x - x_0)|^2 dx\\ &\ge \frac{r_1^2}{C} \int\limits_{B_{\frac{r_1}{2}}(\overline{x})} |\nabla(u - c_r - W_r(x - x_0))|^2 dx\\ &\ge \frac{r_1^2}{C} \int\limits_{B_{\frac{r_1}{2}}(\overline{x})} |\hat{\nabla} u|^2 dx.\end{aligned} \tag{35}$$

By (23),

$$\int\limits_{B_{\frac{r_1}{2}}(\overline{x})} |\hat{\nabla} u|^2 dx \ge C_{r_1} \int\limits_{\Omega} |\hat{\nabla} u|^2 dx.$$

Arguing as in the proof of Theorem 5.2, by (27), we have

$$\int\limits_{B_{\frac{r_1}{2}}(\overline{x})} |\hat{\nabla} u|^2 dx \ge C_{r_1} \|g - (\overline{a} + \overline{W}x)\|^2_{H^{1/2}(\partial\Omega)}, \tag{36}$$

and, hence, by (34), (35) and (36)

$$F_{r,loc} \le CF[g] \tag{37}$$

where C does not depend on r.

Then, by (33) and (37),

$$\int\limits_{B_{3r/2}(x_0)} |\hat{\nabla} u|^2 dx \le C \int\limits_{B_r(x_0)} |\hat{\nabla} u|^2 dx \tag{38}$$

where C depends on α_0, β_0, M, r_0, M_0, $|\Omega|$ and increasingly on $F[g]$.

Once the doubling inequality (38) for $\hat{\nabla} u$ is obtained, the polynomial rate of $\int_{B_r(x_0)} |\hat{\nabla} u|^2 dx$ can be easily obtained by iteration (see [5, Remark 4.11] for a similar procedure). □

6. Proof of Theorem 3.1

Here we follow an argument already used in [8] (see e.g. proof of Theorem 3.1). Let us set again $\varphi = \mu_1 - \mu_2$. By (6) we obtain

$$\int\limits_{\Omega_d} |\varphi||\widehat{\nabla u}|^2\, dx \le \varepsilon^2,$$

where

$$\varepsilon^2 = C\left(\|\mu_1 - \mu_2\|_{L^\infty(\partial\Omega)} + \|u - v\|^{1/4}_{L^2(\Omega)}\right)$$

Now, let $x_0 \in \Omega_d$ be such that $|\varphi(x_0)| = \max_{\bar{\Omega}_d} |\varphi|$. Using the Lipschitz assumption on μ_1, μ_2 we obtain

$$|\varphi(x_0)| \le |\varphi(x)| + 2Mr \qquad \forall x \in B_r(x_0),\ r \in (0, d].$$

Multiplying both sides by $|\hat{\nabla} u|^2$ and integrating over $B_r(x_0)$ we get

$$\begin{aligned}|\varphi(x_0)| \int\limits_{B_r(x_0)} |\hat{\nabla} u|^2\, dx &\le \int\limits_{B_r(x_0)} |\varphi(x)||\hat{\nabla} u|^2\, dx + 2Mr \int\limits_{B_r(x_0)} |\hat{\nabla} u|^2\, dx \\ &\le \int\limits_{\Omega} |\hat{\nabla} u|^2\, dx + 2Mr \int\limits_{B_r(x_0)} |\hat{\nabla} u|^2\, dx \\ &\le \varepsilon^2 + 2Mr \int\limits_{B_r(x_0)} |\hat{\nabla} u|^2\, dx\,.\end{aligned}$$

Then,

$$|\varphi(x_0)| \le \frac{\varepsilon^2}{\int_{B_r(x_0)} |\hat{\nabla} u|^2\, dx} + 2Mr.$$

Now we use (30) and set $N_1 = \left(C_d \|g\|^2_{H^{1/2}(\partial\Omega)}\right)^{-1}$, $N_2 = \log_2 K$, as to get

$$|\varphi(x_0)| \le N_1 \left(\frac{d}{r}\right)^{N_2} \varepsilon^2 + 2Mr \qquad \forall r \in (0, d]\,.$$

Finally, by setting $\lambda = \dfrac{r}{d}$ we obtain

$$|\varphi(x_0)| \le N_1 \lambda^{-N_2} \varepsilon^2 + 2Md\lambda \qquad \forall \lambda \in (0, 1]\,.$$

Let

$$\bar{\lambda} = \left(\frac{N_1 \varepsilon^2}{2Md}\right)^{1/(N_2+1)}.$$

If $\bar{\lambda} \le 1$ we choose $\lambda = \bar{\lambda}$ and get

$$|\varphi(x_0)| \le 2\left(N_1 \varepsilon^2\right)^{1/(N_2+1)} (2Md)^{N_2/(N_2+1)}\,. \tag{39}$$

If $\bar{\lambda} > 1$ we immediately get

$$|\varphi(x_0)| \le 2M \le 2M \left(\frac{N_1 \varepsilon^2}{2Md}\right)^{1/(N_2+1)}$$

from which together with (39) we obtain (5). □

References

[1] G.S. Alberti, G. Bal, M. Di Cristo, Critical points for elliptic equations with prescribed boundary conditions, arXiv:1611.06989, 2016.

[2] G. Alessandrini, An identification problem for an elliptic equation in two variables, Ann. Mat. Pura Appl. (4) 145 (1986) 265–295.

[3] G. Alessandrini, Global stability for a coupled physics inverse problem, Inverse Probl. 30 (2014) 075008, 10 pp.

[4] G. Alessandrini, A. Morassi, E. Rosset, Detecting an inclusion in an elastic body by boundary measurements, SIAM J. Math. Anal. 33 (6) (2002) 1247–1268.

[5] G. Alessandrini, A. Morassi, E. Rosset, Size estimates, in: G. Alessandrini, G. Uhlmann (Eds.), Inverse Problems: Theory and Applications, in: Contemporary Math., vol. 333, AMS, Providence, RI, 2003, pp. 1–33.

[6] G. Alessandrini, A. Morassi, E. Rosset, Detecting an inclusion in an elastic body by boundary measurements, SIAM Rev. 46 (1) (2004) 477–498 (revised and updated version of [4]).

[7] G. Alessandrini, A. Morassi, E. Rosset, S. Vessella, On doubling inequalities for elliptic systems, J. Math. Anal. Appl. 357 (2009) 349–355.

[8] G. Alessandrini, M. Di Cristo, E. Francini, S. Vessella, Stability for quantitative photoacoustic tomography with well chosen illuminations, Ann. Mat. Pura Appl. (2017), http://dx.doi.org/10.1007/s10231-016-0577-4, in press.

[9] G. Bal, Hybrid inverse problems and internal functionals, in: Inverse Problems and Applications: Inside Out. II, in: Math. Sci. Res. Inst. Publ., vol. 60, Cambridge University Press, Cambridge, 2013, pp. 325–368.

[10] G. Bal, C. Bellis, S. Imperiale, F. Monard, Reconstruction of constitutive parameters in isotropic linear elasticity from noisy full-field measurements, Inverse Probl. 30 (2014) 125004, 22 pp.

[11] G. Bal, F. Monard, G. Uhlmann, Reconstruction of a fully anisotropic elasticity tensor from knowledge of displacement fields, SIAM J. Appl. Math. 75 (5) (2015) 2214–2231.

[12] S. Campanato, Sistemi Ellittici in Forma Divergenza: Regolaritá all'Interno, Quaderni della Scuola Normale Superiore, 1980.

[13] Y. Capdeboscq, On a counter-example to quantitative Jacobian bounds, J. Éc. Polytech. Math. 2 (2015) 171–178.

[14] A. Gibali, B. Jadamba, A.A. Khan, F. Raciti, B. Winkler, Gradient and extragradient methods for the elasticity imaging inverse problem using an equation error formulation: a comparative numerical study, in: Nonlinear Analysis and Optimization, in: Contemporary Mathematics, vol. 659, 2016, pp. 65–89.

[15] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd ed., Springer-Verlag, Berlin, 1983.

[16] B. Jadamba, A.A. Khan, F. Raciti, On the inverse problem of identifying Lamé coefficients in linear elasticity, Comput. Math. Appl. 56 (2008) 431–443.

[17] H. Koch, C.L. Lin, J.N. Wang, Doubling inequalities for the Lamé system with rough coefficients, Proc. Amer. Math. Soc. (June 2016) (electronic version).

[18] R.Y. Lai, Uniqueness and stability of Lamé parameters in elastography, J. Spectr. Theory 4 (2014) 841–877.

[19] C.L. Lin, G. Nakamura, J.N. Wang, Optimal three-ball inequalities and quantitative uniqueness for the Lamé system with Lipschitz coefficients, Duke Math. J. 155 (1) (2010) 189–204.

[20] A.P. Sarvazyan, A.R. Skovoroda, S.Y. Emelianov, L.B. Fowlkes, J.G. Pipe, R.S. Adler, R.B. Buxton, P.L. Carson, Biophysical bases of elasticity imaging, Acoust. Imag. 21 (1995) 223–240.

[21] T. Valent, Boundary Value Problems of Finite Elasticity, Springer-Verlag, Berlin, 1988.

Please cite this article in press as: G. Di Fazio et al., Stable determination of a Lamé coefficient by one internal measurement of displacement, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.03.013