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A NEW EXISTENCE AND LOCALIZATION THEOREM FOR THE DIRICHLET PROBLEM

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Dedicated to the memory of Professor V. Lakshmikantham

ABSTRACT. Here is a very particular by-product of the main result of this paper: for each $h \in L^\infty(]0,1[) \setminus \{0\}$, with $h \geq 0$, and for each $\gamma \in]1,2[$, the only positive solution of the problem

$$\begin{cases} -u'' = h(t)u^{\gamma-1} & \text{in }]0,1[\\ u(0) = u(1) = 0 \end{cases},$$

satisfies the inequality

$$\int_0^1 |u'(t)|^2 dt \leq \int_0^1 h(t)dt \left(\frac{2(\text{ess sup}_{]0,1[} h)^{\frac{\gamma}{2}}}{\gamma \pi^\gamma} \right)^{\frac{2}{2-\gamma}}.$$

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1. RESULTS

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $p > 1$. On the Sobolev space $W_0^{1,p}(\Omega)$, we consider the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p dx \right)^{\frac{1}{p}}.$$

If $n \geq p$, we denote by $\mathcal{A}$ the class of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{\xi \in \mathbb{R}} \frac{|f(\xi)|}{1 + |\xi|^\gamma} < +\infty,$$

where $0 < \gamma < \frac{pn-n+p}{n-p}$ if $p < n$ and $0 < \gamma < +\infty$ if $p = n$. While, when $n < p$, $\mathcal{A}$ stands for the class of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Given $f \in \mathcal{A}$ and $h \in L^\infty(\Omega)$, consider the following Dirichlet problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) = h(x)f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (P_{hf})$$

Let us recall that a weak solution of (P_{hf}) is any $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx - \int_\Omega h(x)f(u(x))v(x)dx = 0$$

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for all $v \in W_0^{1,p}(\Omega)$.

The functionals $T, J_{hf} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$T(u) = \frac{1}{p}\|u\|^p$$

$$J_{hf}(u) = \int_\Omega h(x)F(u(x))dx,$$

where

$$F(\xi) = \int_0^\xi f(t)dt,$$

are C^1 with derivatives given by

$$T'(u)(v) = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx$$

$$J'_{hf}(u)(v) = \int_\Omega h(x)f(u(x))v(x)dx$$

for all $u, v \in W_0^{1,p}(\Omega)$. Consequently, the weak solutions of problem (P_{hf}) are exactly the critical points in $W_0^{1,p}(\Omega)$ of $T - J_{hf}$ which is called the energy functional of the problem. Moreover, J'_{hf} is compact, while T' is a homeomorphism between $W_0^{1,p}(\Omega)$ and its dual.

Let us now recall a consequence of the variational principle established in [5] (see also [1], [3], [4] and [7]):

THEOREM A. *Let X be a reflexive real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous functionals, with Ψ also coercive and $\Phi(0) = \Psi(0) = 0$.*

Then, for each $\sigma > \inf_X \Psi$ and each λ satisfying

$$\lambda > -\frac{\inf_{\Psi^{-1}(]-\infty,\sigma])} \Phi}{\sigma}$$

the restriction of $\lambda\Psi + \Phi$ to $\Psi^{-1}(]-\infty, \sigma[)$ has a global minimum.

If we want to apply Theorem A to get the existence of a weak solution for problem (P_{hf}), we need to know that, for some $\sigma > 0$, we have

$$\sup_{\|u\|^p \le \sigma} J_{hf}(u) < \frac{\sigma}{p}.$$

Such an estimate can be obtained under quite natural assumptions if $p > n$, due to the embedding of $W_0^{1,p}(\Omega)$ in $C^0(\overline{\Omega})$. Actually, in the almost totality of the very numerous papers dealing with applications of Theorem A and its consequences to problems of the type we are dealing with, we find the assumption $p > n$ (see the references in [7]).

The aim of the current paper is to highlight a class of nonlinearities f for which Theorem A can be applied without the assumption $p > n$.

Before stating the main result, we need to introduce some further notation.

Let $0 \leq a < b \leq +\infty$. For a generic pair of functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$, if $\lambda \in [a, b]$, we denote by $M(\varphi, \psi, \lambda)$ the set of all global minima of the function $\lambda\psi - \varphi$ or the empty set according to whether $\lambda < +\infty$ or $\lambda = +\infty$. We adopt the conventions $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$. We also put

$$\alpha(\varphi, \psi, b) = \max \left\{ \inf_{\mathbb{R}} \psi, \sup_{M(\varphi,\psi,b)} \psi \right\}$$

and

$$\beta(\varphi, \psi, a) = \min \left\{ \sup_{\mathbb{R}} \psi, \inf_{M(\varphi,\psi,a)} \psi \right\} .$$

Furthermore, let $q \in \left]0, \frac{pn}{n-p}\right]$ if $n > p$ or $q \in]0, +\infty[$ if $n \leq p$.

Set

$$c_q = \sup_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |u(x)|^q dx}{\left(\int_\Omega |\nabla u(x)|^p dx\right)^{\frac{q}{p}}} .$$

Finally, denote by $\mathcal{F}_q$ the family of all lower semicontinuous functions $\psi : \mathbb{R} \to \mathbb{R}$, with $\sup_{\mathbb{R}} \psi > 0$, such that

$$\inf_{\xi \in \mathbb{R}} \frac{\psi(\xi)}{|\xi|^q + 1} > -\infty$$

and

$$\gamma_\psi := \sup_{\xi \in \mathbb{R} \setminus \{0\}} \frac{\psi(\xi)}{|\xi|^q} < +\infty .$$

After introducing these notations, we can state our main result:

Theorem 1.1. *Let $f \in \mathcal{A}$ and $h \in L^\infty(\Omega) \setminus \{0\}$, with $h \geq 0$. Moreover, assume that there exists $\psi \in \mathcal{F}_q$ such that, for each $\lambda \in]a, b[$, the function $\lambda\psi - F$ is coercive and has a unique global minimum in $\mathbb{R}$. Finally, suppose that there exists a number $r > 0$ satisfying*

$$\alpha(F, \psi, b) < r < \beta(F, \psi, a)$$

and

$$\sup_{\psi^{-1}(r)} F < \frac{r^{\frac{p}{q}}}{p(\gamma_\psi \text{ess sup}_\Omega h c_q)^{\frac{p}{q}} \left(\int_\Omega h(x) dx\right)^{\frac{q-p}{q}}} . \tag{1}$$

Under such hypotheses, the problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2} \nabla u) = h(x) f(u) & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega \end{cases}$$

has a weak solution which is a local minimum of the energy functional and satisfies

$$\int_\Omega |\nabla u(x)|^p dx < \left(\frac{r \int_\Omega h(x) dx}{\gamma_\psi \text{ess sup}_\Omega h c_q} \right)^{\frac{p}{q}} .$$

Our proof of Theorem 1.1 is based on a joint application of Theorem A with the following proposition which is nothing else than a very particular case of Theorem 1 of [6].

PROPOSITION A. *Let* $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ *be two functions such that, for each* $\lambda \in]a, b[$*, the function* $\lambda\psi - \varphi$ *is lower semicontinuous, coercive and has a unique global minimum in* $\mathbb{R}$*. Assume that*

$$\alpha(\varphi, \psi, b) < \beta(\varphi, \psi, a).$$

Then, for each $s \in]\alpha(\varphi, \psi, b), \beta(\varphi, \psi, a)[$*, there exists* $\lambda_s \in]a, b[$*, such that the unique global minimum of the function* $\lambda_s\psi - \varphi$ *lies in* $\psi^{-1}(s)$*.*

Proof of Theorem 1.1. Set

$$\sigma = \left(\frac{r \int_\Omega h(x)dx}{\gamma_\psi \text{ess sup}_\Omega h c_q} \right)^{\frac{p}{q}}. \tag{2}$$

Hence

$$r = \frac{\gamma_\psi \text{ess sup}_\Omega h c_q \sigma^{\frac{q}{p}}}{\int_\Omega h(x)dx}. \tag{3}$$

By the Sobolev embedding theorem, we have

$$\{u \in W_0^{1,p}(\Omega) : \|u\|^p \leq \sigma\} \subseteq \left\{ u \in L^q(\Omega) : \int_\Omega |u(x)|^q dx \leq c_q \sigma^{\frac{q}{p}} \right\}. \tag{4}$$

In turn, as $\psi \in \mathcal{F}_q$, we have

$$\int_\Omega h(x)\psi(u(x))dx \leq \text{ess sup}_\Omega h \gamma_\psi \int_\Omega |u(x)|^q dx \tag{5}$$

for all $u \in L^q(\Omega)$. Therefore, from (4) and (5), it follows that

$$\{u \in W_0^{1,p}(\Omega) : \|u\|^p \leq \sigma\} \subseteq \left\{ u \in L^q(\Omega) : \int_\Omega h(x)\psi(u(x))dx \leq \text{ess sup}_\Omega h \gamma_\psi c_q \sigma^{\frac{q}{p}} \right\}. \tag{6}$$

Now, for each $\lambda \in]a, b[$, denote by $\hat{\xi}_\lambda$ the unique global minimum in $\mathbb{R}$ of the function $\lambda\psi - F$. In view of Proposition A, since $r \in]\alpha(F, \psi, b), \beta(F, \psi, a)[$, there exists $\lambda_r \in]a, b[$ such that

$$\psi(\hat{\xi}_{\lambda_r}) = r.$$

So, we have

$$\lambda_r r - F(\hat{\xi}_{\lambda_r}) \leq \lambda_r \psi(\xi) - F(\xi)$$

for all $\xi \in \mathbb{R}$. From this, it clearly follows that

$$F(\hat{\xi}_{\lambda_r}) = \sup_{\psi^{-1}(r)} F. \tag{7}$$

Likewise, for each $u \in L^q(\Omega)$, it follows that

$$(\lambda_r r - F(\hat{\xi}_{\lambda_r})) \int_\Omega h(x)dx \leq \int_\Omega h(x)(\lambda_r \psi(u(x)) - F(u(x)))dx.$$

Therefore, taking (7) into account, for each $u \in L^q(\Omega)$ satisfying

$$\int_\Omega h(x)\psi(u(x))dx \leq r \int_\Omega h(x)dx,$$

we have

$$\int_\Omega h(x)F(u(x))dx \leq \sup_{\psi^{-1}(r)} F \int_\Omega h(x)dx.$$

In view of (3) and (6), this implies that

$$\sup_{\|u\|^p \leq \sigma} \int_\Omega h(x)F(u(x))dx \leq \sup_{\psi^{-1}(r)} F \int_\Omega h(x)dx. \tag{8}$$

At this point, from (1), (2) and (8), it follows that

$$\frac{\sup_{\|u\|^p \leq \sigma} \int_\Omega h(x)F(u(x))dx}{\sigma} < \frac{1}{p}.$$

This allows us to apply Theorem A, with $X = W_0^{1,p}(\Omega)$, $\Phi = -J_{hf}$ and $\Psi = T$. Consequently, the functional $T - J_{hf}$ has a local minimum in $W_0^{1,p}(\Omega)$ with norm less than $\sigma^{\frac{1}{p}}$, and the proof is complete. □

We conclude by presenting an application of Theorem 1.1.

Theorem 1.2. *Let $h \in L^\infty(\Omega) \setminus \{0\}$, with $h \geq 0$, and let $f : [0, +\infty[\to [0, +\infty[$ be a continuous function satisfying the following conditions:*

(i_1) *F has no global maximum in $[0, +\infty[$;*
(i_2) *the function $t \to \frac{f(t)}{t^{q-1}}$ is decreasing in $]0, +\infty[$ and tends to 0 as $t \to +\infty$;*
(i_3) *either*

$$\lim_{t \to 0^+} \frac{F(t)}{t^p} = +\infty$$

or $\text{ess inf}_\Omega h > 0$ *and*

$$\liminf_{t \to 0^+} \frac{F(t)}{t^p} > \frac{1}{p \ \text{ess inf}_\Omega h c_p} ;$$

(i_4) *there is $s > 0$ such that*

$$F(s) < \frac{s^p}{p(\text{ess sup}_\Omega h c_q)^{\frac{p}{q}} (\int_\Omega h(x)dx)^{\frac{q-p}{q}}}.$$

Then, the problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) = h(x)f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a positive weak solution satisfying

$$\int_\Omega |\nabla u(x)|^p dx < \left(\frac{\int_\Omega h(x)dx}{\text{ess sup}_\Omega h c_q} \right)^{\frac{p}{q}} s^p. \tag{9}$$

Proof. Extend f to the whole $\mathbb{R}$ by putting $f(t) = f(0)$ for all $t < 0$. Clearly, in view of (i_2), we have $f \in \mathcal{A}$. Moreover, as $hf \geq 0$, each non-zero weak solution of (P_{hf}) is strictly positive in Ω ([2], [8]). We are going to apply Theorem 1.1 with $\psi(t) = |t|^q$ and $a = 0$, $b = +\infty$. So, let $\lambda > 0$. Of course, from (i_2) it follows that

$$\lim_{|t| \to +\infty} (\lambda |t|^q - F(t)) = +\infty.$$

Let us show that the function $t \to \lambda|t|^q - F(t)$ has a unique global minimum. Arguing by contradiction, assume that this function has two distinct global minima t_1, t_2. We can suppose that $t_1 < t_2$. Since $\lambda|t|^q - F(t) > 0$ for all $t < 0$, it would follow that $t_1 \geq 0$. By the Rolle theorem, there would be $t_3 \in]t_1, t_2[$ such that

$$q\lambda t_3^{p-1} = f(t_3).$$

As a consequence, we would have

$$\frac{f(t_2)}{t_2^{q-1}} = \frac{f(t_3)}{t_3^{q-1}},$$

contrary to (i_1). Clearly, $\alpha(F, |\cdot|^q, +\infty) = 0$ and, in view of (i_1), $\beta(F, |\cdot|^q, 0) = +\infty$. As a consequence, from (i_4) it follows that (1) is satisfied with $r = s^q$. Hence, Theorem 1.1 ensures the existence of a weak solution of problem (P_{hf}) which is a local minimum of the energy functional and satisfies (9). Finally, let us show that 0 is not a local mininum of the energy functional, so that the above solution is not zero. By a classical result, there is a bounded and positive function $v \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |v(x)|^p dx = c_p \int_\Omega |\nabla v(x)|^p dx.$$

Now, if ess $\inf_\Omega h = 0$, we denote by η a positive number such that the set $h^{-1}([\eta, +\infty[)$ has a positive measure. While, we set

$$\eta = \text{ess } \inf_\Omega h$$

if ess $\inf_\Omega h > 0$. Then, thanks to (i_3), there is $\delta > 0$ such that

$$F(t) > \frac{\int_\Omega |v(x)|^p dx}{p\eta c_p \int_{h^{-1}([\eta,+\infty[)} |v(x)|^p dx} t^p$$

for all $t \in]0, \delta[$. Hence, for each $\mu \in \left]0, \frac{\delta}{\sup_\Omega v}\right[$, we have

$$\begin{aligned}
\int_\Omega h(x)F(\mu v(x))dx &\geq \int_{h^{-1}([\eta,+\infty[)} h(x)F(\mu v(x))dx \\
&> \frac{\int_\Omega |v(x)|^p dx}{p\eta c_p \int_{h^{-1}([\eta,+\infty[)} |v(x)|^p dx} \int_{h^{-1}([\eta,+\infty[)} h(x)|\mu v(x)|^p dx \\
&\geq \frac{\int_\Omega |\mu v(x)|^p dx}{p\eta c_p \int_{h^{-1}([\eta,+\infty[)} |\mu v(x)|^p dx} \eta \int_{h^{-1}([\eta,+\infty[)} |\mu v(x)|^p dx
\end{aligned}$$

$$= \frac{\int_\Omega |\mu v(x)|^p dx}{pc_p}$$
$$= \frac{1}{p}\int_\Omega |\nabla \mu v(x)|^p dx.$$

This shows that the energy functional takes negative values in each ball of $W_0^{1,p}(\Omega)$ centered at 0 and so 0 is not a local minimum for it. The proof is complete. □

Notice the following remarkable corollary of Theorem 1.2.

Corollary 1.3. *Let $q > 1$, let $h \in L^\infty(\Omega) \setminus \{0\}$, with $h \geq 0$, let $a_1, \dots, a_k$ $(k \geq 3)$ be k non-negative numbers and let $q_1, ..., q_k$ be k numbers lying in $]1, q[$. Assume that either there is some $i \in \{1, \dots, k\}$ for which $q_i < p$ and $a_i > 0$, or* ess $\inf_\Omega h > 0$ *and, if we put*

$$I = \{i \in \{1, \dots, k\} : q_i = p\},$$

we have $I \neq \emptyset$ and

$$\sum_{i \in I} a_i > \frac{1}{\text{ess inf}_\Omega h c_p}.$$

Finally, let $s > 0$ be such that

$$\frac{a_1}{q_1}\log(1 + s^{q_1}) + \frac{a_2}{q_2}\text{arctg} s^{q_2} + \sum_{i=3}^{k} \frac{a_i}{q_i} s^{q_i} < \frac{s^p}{p(\text{ess sup}_\Omega h c_q)^{\frac{p}{q}} (\int_\Omega h(x)dx)^{\frac{q-p}{q}}}.$$

Then, the problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) = h(x)\left(a_1 \frac{u^{q_1-1}}{1+u^{q_1}} + a_2 \frac{u^{q_2-1}}{1+u^{2q_2}} + \sum_{i=3}^k a_i u^{q_i-1}\right) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a positive weak solution satisfying

$$\int_\Omega |\nabla u(x)|^p dx < \left(\frac{\int_\Omega h(x)dx}{\text{ess sup}_\Omega h c_q}\right)^{\frac{p}{q}} s^p.$$

Remark 1.4. For instance, taking into account that $c_2 = \pi^{-2}$ when $\Omega =]0, 1[$, from Corollary 1.3 it follows that that, for each $h \in L^\infty(]0, 1[) \setminus \{0\}$, with $h \geq 0$, and for each $\gamma \in]1, 2[$, the only positive solution of the problem

$$\begin{cases} -u'' = h(t)u^{\gamma-1} & \text{in }]0, 1[\\ u(0) = u(1) = 0 \end{cases}$$

satisfies the inequality

$$\int_0^1 |u'(t)|^2 dt \leq \int_0^1 h(t)dt \left(\frac{2(\text{ess sup}_{]0,1[} h)^{\frac{\gamma}{2}}}{\gamma \pi^\gamma}\right)^{\frac{2}{2-\gamma}}.$$

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