Algebraic Geometry 4 (1) (2017) 79–103
doi:10.14231/AG-2017-004

Irreducible components of Hilbert schemes of rational curves with given normal bundle

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To Rosario Strano, on his 70th birthday

Abstract

We develop a new general method for computing the decomposition type of the normal bundle to a projective rational curve. This method is then used to detect and explain an example of a reducible Hilbert scheme parametrizing all the rational curves in $\mathbb{P}^s$ with a given decomposition type of the normal bundle. We also characterize smooth non-degenerate rational curves contained in rational normal scrolls in terms of the splitting type of their restricted tangent bundles and compute their normal bundles.

1. Introduction

The projective rational curves $C \subset \mathbb{P}^s$ of degree d form a quasi-projective irreducible subscheme $\mathcal{H}^{\mathrm{rat}}_{d,s}$ of the Hilbert scheme of $\mathbb{P}^s$. Any of these curves is the image of a birational map $f\colon \mathbb{P}^1 \to \mathbb{P}^s$, defined up an automorphism of $\mathbb{P}^1$. If one restricts oneself to rational curves with ordinary singularities, one may classify these curves by considering the splitting types as a direct sum of line bundles of the vector bundles $f^*\mathcal{T}_{\mathbb{P}^s}$ and $\mathcal{N}_f = f^*\mathcal{T}_{\mathbb{P}^s}/\mathcal{T}_{\mathbb{P}^1}$, commonly called the *restricted tangent bundle* and the *normal bundle* of the curve C, respectively. It is well known that the classification of rational curves by the splitting type of $f^*\mathcal{T}_{\mathbb{P}^s}$ produces irreducible subvarieties of $\mathcal{H}^{\mathrm{rat}}_{d,s}$; see [Ver83, Ram90]. One can also look at [AR15] for a geometric characterization of rational curves with a given splitting of $f^*\mathcal{T}_{\mathbb{P}^s}$ and at [Iar14] for related results in the commutative algebra language.

Since the early eighties of the past century, a natural question about rational curves in projective spaces has been whether the subschemes of $\mathcal{H}^{\mathrm{rat}}_{d,s}$ characterized by a given splitting of $\mathcal{N}_f$ are irreducible as well. This has been proved to be true for rational curves in $\mathbb{P}^3$, see [EvdV81, EvdV82, GS80]. The irreducibility problem has also been shown to have a positive answer for the general splitting type of $\mathcal{N}_f$, see [Sac80], and more recently other results related to this problem have been obtained in [Ran07] and [Ber14]. However, the general irreducibility problem remained open.

Received 15 Mar 2015, accepted in final form 1 Apr 2016.
2010 Mathematics Subject Classification 14C05, 14H45, 14N05.
Keywords: rational curve, normal bundle, Hilbert scheme.

The first author was partially supported by the national research project "Geometry on Algebraic Varieties" Cofin 2010 of MIUR. The second author was partially supported by the Research Project of the University of Catania FIR 2014 "Aspetti geometrici e algebrici della Weak e Strong Lefschetz Property".

In this paper we show that the irreducibility problem has a negative answer in general, producing the first known example of a reducible Hilbert scheme of rational curves characterized by a given splitting of $\mathcal{N}_f$. In order to achieve this, we develop a new general method to compute the spaces of global sections $H^0\mathcal{N}_f(-k)$ and therefore the splitting type of $\mathcal{N}_f$.

1.1 Notation and summary of results

A rational curve $C \subset \mathbb{P}^s$ is a curve that can be birationally parametrized by a regular map $f\colon \mathbb{P}^1 \to \mathbb{P}^s$. We will always assume that C is non-degenerate, that is, not contained in any hyperplane $H \subset \mathbb{P}^s$, and of degree $d > s$ with $s \geqslant 3$; in particular, we are excluding the well-known case of the rational normal curves. Let $\mathcal{I}_C$ be the ideal sheaf of C in $\mathbb{P}^s$; then the normal sheaf of C is the sheaf $\mathcal{N}_C = \underline{\mathrm{Hom}}_{\mathcal{O}_C}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C)$. Recall also that the tangent sheaf of a noetherian scheme X over $\mathrm{Spec}(\mathbb{C})$ is defined as $\mathcal{T}_X = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega^1_{X/\mathbb{C}}, \mathcal{O}_X)$. Taking the differential of the parametrization map f produces an exact sequence

$$0 \to \mathcal{T}_{\mathbb{P}^1} \xrightarrow{df} f^*\mathcal{T}_{\mathbb{P}^s} \to f^*\mathcal{N}_C\,.$$

When C has ordinary singularities, df is a vector bundle embedding and the sequence

$$0 \to \mathcal{T}_{\mathbb{P}^1} \xrightarrow{df} f^*\mathcal{T}_{\mathbb{P}^s} \to f^*\mathcal{N}_C \to 0$$

is exact and identifies $f^*\mathcal{N}_C$ as the quotient bundle $f^*\mathcal{T}_{\mathbb{P}^s}/df(\mathcal{T}_{\mathbb{P}^1})$. We will write $f^*\mathcal{N}_C = \mathcal{N}_f$ and call this vector bundle the normal bundle to C. Therefore we will assume that C is irreducible and with ordinary singularities when we will be dealing with the normal bundle $\mathcal{N}_f$ associated with a given parametrization $f\colon \mathbb{P}^1 \to C$.

Given a multiset of $s-1$ integers $\overline{c} = c_1, c_2, \dots, c_{s-1}$, ordered in such a way that

$$c_1 \geqslant c_2 \geqslant \cdots \geqslant c_{s-1}\,,$$

we will denote by $\mathcal{H}_{\overline{c}}$ the Hilbert scheme of irreducible degree d rational curves with ordinary singularities $C \subset \mathbb{P}^s$ that can be birationally parametrized by a map $f\colon \mathbb{P}^1 \to \mathbb{P}^s$ such that the normal bundle $\mathcal{N}_f$ splits as $\mathcal{N}_f = \bigoplus_{i=1}^{s-1} \mathcal{O}(c_i + d + 2)$.

Let $U \cong \mathbb{C}^2$ be a 2-dimensional vector space and $\mathbb{P}^1 = \mathbb{P}(U)$ its associated projective line. Let S^dU be the dth symmetric product of U. Let $\nu_d\colon \mathbb{P}(U) \to \mathbb{P}(S^dU)$ be the dth Veronese embedding, and let us consider the rational normal curve $C_d = \nu_d(\mathbb{P}(U))$.

Our main general result is Theorem 4.1. After representing, up to projective transformations, a degree d rational curve as the projection of C_d from a vertex $\mathbb{P}(T) \subset \mathbb{P}(S^dU)$, we prove an identification of the spaces of global sections $H^0\mathcal{T}_f(-d-2-k)$ and $H^0\mathcal{N}_f(-d-2-k)$ with the spaces $\ker D \cap (S^kU \otimes T) \subset S^kU \otimes S^dU$ and $\ker D^2 \cap (S^kU \otimes T) \subset S^kU \otimes S^dU$, respectively, where D is the first-order transvectant operator, that is, $D = \partial_x \otimes \partial_y - \partial_y \otimes \partial_x$, with x, y a basis of U and ∂_x, ∂_y the dual basis, acting by derivation. By means of this result one can relate the splitting types of $\mathcal{T}_f$ and $\mathcal{N}_f$ with the position of the vertex $\mathbb{P}(T)$ with respect to the rational normal curve C_d.

In Section 6 we introduce and discuss our example of a Hilbert scheme $\mathcal{H}_{\overline{c}}$ of rational curves $C \subset \mathbb{P}^8$ of degree $d = 11$ with exactly two irreducible components of dimension 98 whose general points represent smooth rational curves, therefore providing a counterexample to the above-mentioned irreducibility problem.

In Section 7, Theorem 7.3, we give a characterization of smooth rational curves contained in rational normal scrolls in terms of the splitting type of their restricted tangent bundles and compute their normal bundles. The same theorem also shows how to construct these curves as projections of a rational normal curve.

2. Rational curves as projections of a rational normal curve

Given a $\mathbb{C}$-vector space W, we denote by $\mathbb{P}(W)$ the projective space of 1-dimensional subspaces of W. More generally, we denote by $\mathrm{Gr}(e+1, W)$ or $\mathrm{Gr}(e, \mathbb{P}(W))$ the Grassmannian of $(e+1)$-dimensional subspaces of W, or equivalently, of e-dimensional linear subspaces of $\mathbb{P}(W)$. If $T \subseteq W$ is an $(e+1)$-dimensional subspace, we will denote its associated point in $\mathrm{Gr}(e, \mathbb{P}(W))$ by $[T]$ or $[\mathbb{P}(T)]$. Accordingly, if $w \in W$ is a non-zero vector, we will denote its associated point by $[w] \in \mathbb{P}(W)$.

Let $U \cong \mathbb{C}^2$ be a 2-dimensional vector space and $\mathbb{P}^1 = \mathbb{P}(U)$ its associated projective line. Let $S^d U$ be the dth symmetric product of U. Let $\nu_d \colon \mathbb{P}^1 \to \mathbb{P}(S^d U) = \mathbb{P}^d$ the dth Veronese embedding, defined by $\nu_d(p) = [p^d]$. We set $C_d = \nu_d(\mathbb{P}^1)$, which is the rational normal curve given by the set of pure tensors in $S^d U$. For any $b \geqslant 1$, we denote by $\mathrm{Sec}^{b-1} C_d$ the closure of the set of $[\tau] \in \mathbb{P}(S^d U)$ such that $\tau = p_1^d + \cdots + p_b^d$, for $[p_i] \in C_d$ distinct points, that is, the $(b-1)$st secant variety of C_d.

Let $C \subset \mathbb{P}^s = \mathbb{P}(V)$ be a non-degenerate rational curve of degree d. For the next considerations we will not need to assume that C has ordinary singularities. The normalization map $\nu_C \colon \mathbb{P}(U) \to C$ is the restriction of a map $f \colon \mathbb{P}(U) \to \mathbb{P}^s$ such that $f^*\mathcal{O}_{\mathbb{P}^s}(1) = \nu_C^*\mathcal{O}_C(1) = \mathcal{O}_{\mathbb{P}^1}(d)$. The map f is defined by an injection $f^* \colon H^0\mathcal{O}_{\mathbb{P}^s}(1) = V^* \hookrightarrow H^0\mathcal{O}_{\mathbb{P}^1}(d) = S^d U^*$ such that $f^*(V^*)$ spans $\mathcal{O}_{\mathbb{P}^1}(d)$ at any point of $\mathbb{P}^1$. Let us set

$$T = f^*(V^*)^{\perp} \subset S^d U\,, \quad e+1 = \dim T = d - s\,.$$

Then one sees that the map f^* can be identified with the dual of the map $S^d U \to S^d U/T \overset{\cong}{\to} V$. In particular, up to a linear isomorphism, we identify $\mathbb{P}^s$ and $\mathbb{P}(S^d U/T)$, and the map f and the composition $f = \pi_T \circ \nu_d$, where $\pi_T \colon \mathbb{P}(S^d U) \dashrightarrow \mathbb{P}(S^d U/T)$ is the projection of the vertex $\mathbb{P}(T)$. We want to underline the fact that for any $\overline{\psi} \in \mathrm{Aut}(\mathbb{P}^s)$, the curve $C' = \psi(C)$ is obtained by changing $f^* \colon V^* \to S^d U^*$ into $g^* = f^* \circ \psi$, with $\psi \in \mathrm{GL}(V^*)$ a linear automorphism representing $\overline{\psi}$. Hence the space $T = f^*(V^*)^{\perp}$ is not affected by such a transformation. This means that one has a natural bijection between the set of orbits of maps $f \colon \mathbb{P}^1 \to \mathbb{P}^s$ under the left action of $\mathrm{PGL}(s+1)$ and the set of projection vertexes $\mathbb{P}(T)$ obtained as above.

We recall that the condition that $f^*(V^*)$ spans $\mathcal{O}_{\mathbb{P}^1}(d)$ at any point of $\mathbb{P}^1$ is equivalent to $\mathbb{P}(T) \cap C_d = \emptyset$, and the fact that f is birational to the image corresponds to the fact that $\mathbb{P}(T) \cap \mathrm{Sec}^1 C_d$ is finite.

The discussion above shows that the Hilbert scheme $\mathcal{H}_{d,s}^{\mathrm{rat}}$ of rational curves in $\mathbb{P}^s$ is set-theoretically described as the set of images of rational maps $\pi_T \circ \nu_d$ composed with projective transformations of $\mathbb{P}^s$, with the extra condition that the map $\pi_T \circ \nu_d \colon \mathbb{P}^1 \to \mathbb{P}^s$ is birational to the image. More precisely, for $\mathcal{V}$ the open subset of $[T] \in \mathrm{Gr}(e+1, S^d U))$ such that $\mathbb{P}(T) \cap C_d$ is empty and $\mathbb{P}(T) \cap \mathrm{Sec}^1 T$ is finite, we see that there exists a map

$$\mathcal{V} \times \mathrm{PGL}(s+1) \to \mathcal{H}_{d,s}^{\mathrm{rat}}$$

mapping $([T], \phi) \in \mathcal{V} \times \mathrm{PGL}(s+1)$ to the curve $C = \phi(\pi_T(C_d))$, and this map is surjective.

2.1 PGL(2)-action on the space of vertexes $\mathbb{P}(T)$

Let us fix a map $f = \pi_T \circ \nu_d \colon \mathbb{P}^1 \to \mathbb{P}^s$, associated with a vertex $\mathbb{P}(T)$ as in the construction above. Let us consider an automorphism $\phi \in \mathrm{PGL}(2)$. We will denote with the same letter ϕ a fixed representative of the given automorphism as an element of GL(2). One observes that the d-fold symmetric product $S^d\phi$ of the map ϕ acts on S^dU by the action on generators $(S^d\phi)(l^d) = \phi(l)^d$, and one can define the induced action on the Grassmannian $\mathrm{Gr}(e+1, S^dU)$ by $[T] \mapsto [(S^d\phi)(T)]$. Now, let us consider the composition

$$f^\phi = f \circ \phi^{-1} \colon \mathbb{P}^1 \to \mathbb{P}^s .$$

One has the following formula:

$$f^\phi = \pi_{(S^d\phi)(T)} \circ \nu_d . \tag{2.1}$$

Indeed, we know that f is determined by the subspace $T^\perp \subset S^dU^*$; let us write $T^\perp = \langle g_0, \dots, g_s\rangle$. Then f^ϕ is determined by $W = \langle g_0 \circ \phi^{-1}, \dots, g_s \circ \phi^{-1}\rangle$, and by the GL(2)-invariance of the duality pairing $S^dU^* \otimes S^dU \to \mathbb{C}$, one immediately sees that $W = (S^d\phi)(T)^\perp \subset S^dU^*$.

Above, we saw that the space of maps $f \colon \mathbb{P}^1 \to \mathbb{P}^s$ that birationally parametrize a non-degenerate rational curve $C \subset \mathbb{P}^s$ of degree d is identified with $\mathcal{V} \times \mathrm{PGL}(s+1)$, by mapping $([T], \phi)$ to $f = \phi \circ (\pi_T \circ \nu_d)$. We then showed that the right PGL(2)-action on this space of maps can be identified with the left action of PGL(2) on $\mathcal{V} \times \mathrm{PGL}(s+1)$ defined by its left action on $\mathcal{V}$.

2.2 Irreducibility criteria and dimension formulas

To show the irreducibility of a subscheme $\mathcal{H}_P \subseteq \mathcal{H}^{\mathrm{rat}}_{d,s}$ defined by a geometric property P on rational curves $C \subset \mathbb{P}^s$, it will be sufficient to prove the irreducibility of the subvariety $\mathcal{V}_P$ of those $[T] \in \mathrm{Gr}(e+1, S^dU)$ such that the curve $C = \pi_T(C_d)$ satisfies property P. Indeed, in that case $\mathcal{V}_P \times \mathrm{PGL}(s+1) \to \mathcal{H}_P$ is onto, with irreducible domain. To compute $\dim \mathcal{H}_P$ from the map $\pi \colon \mathcal{V}_P \times \mathrm{PGL}(s+1) \to \mathcal{H}_P$, one applies the following result, which is almost obvious and very well known in the special case $\mathcal{H}_P = \mathcal{H}^{\mathrm{rat}}_{d,s}$, but which we will need in the more general form stated here.

Proposition 2.1. *With the notation set above, if $\mathcal{V}_\mathcal{P}$ is irreducible, then $\mathcal{H}_P$ is irreducible of dimension* $\dim \mathcal{H}_P = \dim \mathcal{V}_P + \dim \mathrm{PGL}(s+1) - 3$.

Proof. From the above discussion it follows that the fiber over an arbitrary $[C] \in \mathcal{H}_P$ is

$$\pi^{-1}([C]) = \mathrm{Orb}([T]) \times \mathrm{Stab}(C) ,$$

with $\mathrm{Orb}([T])$ the orbit of $[T]$ under the action of PGL(2) on the Grassmannian $\mathrm{Gr}(e+1, S^dU)$ and $\mathrm{Stab}(C) \subset \mathrm{PGL}(s+1)$ the group of projective transformations preserving C. First, we consider the case when $\dim \mathrm{Orb}([T]) < 3 = \dim \mathrm{PGL}(2)$, that is, when $[T]$ is fixed by some 1-dimensional subgroup of PGL(2). The 1-dimensional subgroups of PGL(2) either fix one point $[x] \in \mathbb{P}^1$ and contain the translations group acting on the basis x, y as $(x, y) \mapsto (x, y + \alpha y)$, with $\alpha \in \mathbb{C}$, or fix two points $[x], [y] \in \mathbb{P}^1$ and contain the group $(x, y) \mapsto (x, \lambda y)$, with $\lambda \in \mathbb{C}^*$. Any subspace $T \subset S^dU$ fixed by a group of the first type must contain the pure tensor $[x^d]$, and hence $[T] \notin \mathcal{V}$. A space fixed by a subgroup of the second type is necessarily monomial; that is, $T = \langle x^{\nu_0} y^{d-\nu_0}, \dots, x^{\nu_e} y^{d-\nu_e}\rangle$. One can see that such a space gives a point $[T] \in \mathcal{V}$, that is, $\mathbb{P}(T) \cap \mathrm{Sec}^1 C_d = \emptyset$ if and only if $d-2 \geqslant \nu_0 \geqslant \cdots \geqslant \nu_e \geqslant 2$, and hence it can exist if $d-3 \geqslant e+1$. In this case one sees $\dim \mathrm{Orb}(T) = \dim \mathrm{PGL}(2) - \dim \mathrm{Stab}(T) = 2$.

Now, we consider the cases when $\dim \mathrm{Stab}(C) > 0$. A classical reference for this class of curves, called the algebraic Klein–Lie curves, or algebraic W-curves, is for example [EC34, libro V, § 24]. In a suitable coordinate system, any 1-dimensional subgroup of $\mathrm{PGL}(s+1)$ whose orbits in $\mathbb{P}^s$ are not lines takes the form $t \mapsto \mathrm{diag}(t^{\mu_0}, \dots, t^{\mu_s})$, with $\mu_i \in \mathbb{Z}$ normalized and ordered such that

$0 = \mu_0 \leqslant \cdots \leqslant \mu_s$. Its orbits $t \mapsto (\alpha_0 t^{\mu_0} : \cdots : \alpha_s t^{\mu_s})$ represent non-degenerate rational curves of degree d if and only if the integers μ_i are distinct, $\alpha_i \neq 0$ for all $i = 0, \ldots, s$ and $\mu_s = d$. Hence there exists only a finite number of possible choices of such integers $\mu_0, \ldots, \mu_s$ for a fixed d, that is, a finite number of non-degenerate degree d Klein–Lie curves in $\mathbb{P}^s$ up to projective equivalence. All of them can be obtained up to projective equivalence as projections $C = \pi_T(C_d)$ in the following way. For any fixed basis $x, y \in U$ consider the vertex $\mathbb{P}(T)$ generated by monomials $x^{\nu_0} y^{d-\nu_0}, \ldots, x^{\nu_e} y^{d-\nu_e}$, with $e + 1 = d - s$ and $\{\nu_0, \ldots, \nu_e\} = \{0, \ldots, d\} \setminus \{\mu_0, \ldots, \mu_s\}$. Then $C = \pi_T(C_d)$ is a curve parametrized as $t \mapsto (t^{\mu_0} : \cdots : t^{\mu_s})$ with respect to the basis $(\bar{x}^{\mu_i} \bar{y}^{d-\mu_i})$ of $S^d U/T$. Hence we have found that non-degenerate rational curves with $\dim \mathrm{Stab}(C) > 0$ come from those vertexes $\mathbb{P}(T)$ with $\dim \mathrm{Orb}([T]) = 2$ that were already analyzed above. In all those cases one has

$$\dim \pi^{-1}([C]) = \dim(\mathrm{Orb}(T) \times \mathrm{Stab}(C)) = 2 + 1 = 3\,.$$

In any other case one has $\dim \mathrm{Orb}(T) = 3$ and $\dim \mathrm{Stab}(C) = 0$. □

2.3 A classification of the projection vertexes $\mathbb{P}(T)$

Let us consider a non-zero subspace $T \subseteq S^d U$, with $d \geqslant 2$. Let us denote by x, y a basis of U and by u, v the dual basis in U^*. Recall that u, v may be identified with ∂_x, ∂_y acting as linear forms on U, and an arbitrary element $\omega \in U^*$ will be written $\omega = \alpha\partial_x + \beta\partial_y$ for suitable $\alpha, \beta \in \mathbb{C}$. We define

$$\partial T = \langle \omega(T) \,|\, \omega \in U^* \rangle\,. \tag{2.2}$$

We remark that if $U = \langle x, y \rangle$, then $\partial T = \partial_x T + \partial_y T$. One observes that in the trivial case $T = S^d U$, we have $\partial T = S^{d-1} U$. One can see that this is the only possible case when $\dim \partial T < \dim T$, either as an easy exercise or as a consequence of Proposition 2.3 below.

We also introduce the space $\partial^{-1} T \subset S^{d+1} U$ defined in the following way:

$$\partial^{-1} T = \bigcap_{\omega \in U^*} \omega^{-1}(T)\,. \tag{2.3}$$

In this case we have $\partial^{-1} T = \partial_x^{-1} T \cap \partial_y^{-1} T$. Of course one has $\partial^{-1} S^d U = S^{d+1} U$.

For $g \in S^{d+b} U$ we introduce the vector space

$$\partial^b(g) = \langle \partial_x^b g, \partial_x^{b-1}\partial_y g, \ldots, \partial_y^b g \rangle \subseteq S^d U\,. \tag{2.4}$$

By convention, we set $\partial^b(g) = 0$ if $b = -1$.

2.4 The numerical type of a subspace $T \subset S^d U$

We will need the following notation and results from the article [AR15].

Definition 2.2. We will say that a proper linear space $\mathbb{P}(S) \subset \mathbb{P}^d$ is C_d-generated if $\mathbb{P}(S)$ is generated by its schematic intersection with C_d. Setting $a + 1 = \dim S$, we will also say in this case that $\mathbb{P}(S)$ is $(a+1)$-secant to C_d. We will say that a vector subspace $S \subseteq S^d U$ is C_d-generated if $\mathbb{P}(S)$ is C_d-generated.

Notation. Given a proper subspace $T \subset S^d U$, we denote by S_T the smallest subspace containing the schematic intersection $\mathbb{P}(T) \cap C_d$ as a subscheme. We set $a = \dim S_T - 1 = \dim \mathbb{P}(S_T)$, with the convention that $\dim \emptyset = -1$.

Proposition 2.3 ([AR15, Theorem 1]). *Let T be a proper subspace of $S^d U$. Let S_T be as defined above. Then $\dim \partial S_T = \dim S_T$. Moreover, if we define $r = \dim \partial T - \dim(T)$, then $r \geqslant 0$ and* *either $r=0$ and in this case one has $T = S_T$ and T is C_d-generated, or $r \geqslant 1$ and there exist forms $f_1, \dots, f_r$, with $f_i \in \mathbb{P}^{d+b_i} \setminus \mathrm{Sec}^{b_i} C_{d+b_i}$ for $i = 1, \dots, r$, with $b_1 \geqslant \dots \geqslant b_r \geqslant 0$, such that T and ∂T are the direct sums*

$$T = S_T \oplus \partial^{b_1}(f_1) \oplus \dots \oplus \partial^{b_r}(f_r)\,,$$
$$\partial T = \partial S \oplus \partial^{b_1+1}(f_1) \oplus \dots \oplus \partial^{b_r+1}(f_r)\,.$$

The $(r+1)$-uple $(a, b_1, \dots, b_r)$ is uniquely determined from T. A space T as above exists if and only if $a \geqslant -1$, $b_i \geqslant 0$ for all $i = 1, \dots, r$ and $a + 1 + \sum(b_i + 2) \leqslant d$.

Definition 2.4. We say that a subspace T as in Proposition 2.3 has *numerical type* $(a, b_1, \dots, b_r)$. If $S_T = 0$, that is, $\mathbb{P}(T) \cap C_d = \emptyset$, then $a = -1$ and we will say T that has type $(b_1, \dots, b_r)$.

Let us also recall the following result from [AR15].

Proposition 2.5 ([AR15, Proposition 5]). *Assume that $T \subseteq S^dU$ has type $(a, b_1, \dots, b_r)$, so that it has a decomposition*

$$T = S_T \oplus \bigoplus_{i=1}^{r} \partial^{b_i}(f_i)$$

satisfying the requirements of Proposition 2.3. Then $\partial^{-1}(S_T) = S_{\partial^{-1}T}$ and $\dim \partial^{-1}(S_T) = \dim S_T = a+1$, and there exists a decomposition

$$\partial^{-1}T = \partial^{-1}S_T \oplus \bigoplus_{i\colon b_i \geqslant 1} \partial^{b_i - 1}(f_i)\,.$$

In particular, $\partial^{-1}T$ has type $(a, b_1 - 1, \dots, b_{r_1} - 1)$ with $r_1 = \max(i\colon b_i \geqslant 1)$.

2.5 The splitting type of the restricted tangent bundle of rational curves

The main result of [AR15] about the splitting type of the restricted tangent bundle $f^*\mathcal{T}_{\mathbb{P}^s}$, that we will write as $\mathcal{T}_f$ for short, of a parametrized rational curve $f\colon \mathbb{P}^1 \to \mathbb{P}^s$ is the following.

Proposition 2.6 ([AR15, Theorem 3]). *Assume that $f\colon \mathbb{P}^1 \to \mathbb{P}^s$ is obtained by projecting the rational normal curve C_d from a vertex $\mathbb{P}(T)$ with T of type $(b_1, \dots, b_r)$. Then $r \leqslant s$ and the splitting type of $\mathcal{T}_f$ is*

$$\mathcal{T}_f = \mathcal{O}_{\mathbb{P}^1}(b_1 + d + 2) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(b_r + d + 2) \oplus \mathcal{O}_{\mathbb{P}^1}^{s-r}(d+1)\,.$$

We also recall the restricted Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to (S^dU/T) \otimes \mathcal{O}_{\mathbb{P}^1}(d) \to \mathcal{T}_f \to 0\,,$$

from which one gets $\deg \mathcal{T}_f = (s+1)d$.

3. Review of some SL(U)-invariant operators

In this section we will review some well-known invariant operators between spaces of tensors on U or U^*, for the convenience of the reader and for later reference. Invariance will mean GL(U)- or SL(U)-invariance.

3.1 The duality pairing

The duality pairing is the natural pairing $S^dU^* \otimes S^dU \to \mathbb{C}$ that identifies either of the two spaces as the dual of the other. It may be defined by considering any element of S^dU^* as a differential operator on S^dU. More precisely, if $x, y \in U$ and $u, v \in U^*$ are dual bases, then one has the formula

$$f(u,v) \in S^dU^*, \quad l = \lambda x + \mu y \in U \Rightarrow f\big(l^d\big) = d!f(\lambda, \mu).$$

3.2 General contractions

The contraction maps

$$S^kU^* \otimes S^bU \to S^{b-k}U,$$

defined for any $0 \leqslant k \leqslant b$, or the analogous maps interchanging U and U^*, can be interpreted in a way similar to that given in 3.1 by letting the tensors in S^kU^* act on S^bU as differential operators. The following formulas are straightforward consequences of the definition of the action of $f \in S^kU^*$ as a differential operator:

$$f(l^b) = \binom{b}{k} f(l^k) l^{b-k}, \tag{3.1}$$

$$f(\eta(g)) = (\eta f)(g), \quad \forall\, f \in S^kU^*,\ \forall\, \eta \in U^*,\ \forall\, g \in S^{b+1}U. \tag{3.2}$$

3.3 The multiplication maps

The multiplication maps are the maps $m\colon S^iU \otimes S^jU \to S^{i+j}U$, or the same with U^* in the place of U, defined on pure generators by $m(l^i \otimes h^j) = l^ih^j$.

3.4 The polarization maps

The polarization maps are maps $p_k\colon S^{d+k}U \to S^kU \otimes S^dU$ proportional to duals of the multiplication maps $m\colon S^kU^* \otimes S^dU^* \to S^{d+k}U^*$, with proportionality factor determined such that $m(p_k(f)) = f$ for any $f \in S^{d+k}U$. For this reason, the polarization maps are always injective. The maps p_k are uniquely defined by

$$p_k(l^{d+k}) = l^k \otimes l^d.$$

One has the following well-known closed formula for p_k in terms of a fixed basis x, y for U:

$$p_k(f) = \frac{(\deg f - k)!}{\deg f!} \sum_{i=0}^{k} \binom{k}{i} x^{k-i}y^i \otimes \partial_x^{k-i}\partial_y^i(f). \tag{3.3}$$

3.5 The multiplication by $\xi = x \otimes y - y \otimes x$

The multiplication by $\xi = x \otimes y - y \otimes x$ is an $\mathrm{SL}(U)$-invariant element of $U \otimes U$, which indeed generates the irreducible subrepresentation of $\mathrm{GL}(U)$ given by $U \wedge U \subset U \otimes U$. The multiplication by ξ acts in the following way:

$$\xi\colon\ S^{i-1}U \otimes S^{j-1}U \to S^iU \otimes S^jU.$$

Observe that for any $k \leqslant d$ one has the direct sum decomposition

$$S^kU \otimes S^dU = p_k\big(S^{d+k}U\big) \oplus \xi p_{k-1}\big(S^{d+k-2}U\big) \oplus \cdots \oplus \xi^k p_0\big(S^{d-2k}U\big). \tag{3.4}$$

Here we set $S^iU = 0$ if $i < 0$. This decomposition is equal to the *Pieri decomposition* of $S^kU \otimes S^dU$ as a $\mathrm{GL}(U)$-representation, for which we refer to [FH91]. Note that grouping the terms in (3.4) in a suitable way, one obtains

$$S^kU \otimes S^dU = p_k\big(S^{d+k}U\big) \oplus \xi\big(S^{k-1}U \otimes S^{d-1}U\big)\,, \tag{3.5}$$

$$S^kU \otimes S^dU = p_k\big(S^{d+k}U\big) \oplus \xi p_{k-1}\big(S^{d+k-2}U\big) \oplus \xi^2\big(S^{k-2}U \otimes S^{d-2}U\big)\,. \tag{3.6}$$

3.6 The operator $D = D_{x,y} = \partial_x \otimes \partial_y - \partial_y \otimes \partial_x$

The operator $D = D_{x,y} = \partial_x \otimes \partial_y - \partial_y \otimes \partial_x$ is classically know as the *first-order transvectant*; see, for example, [Olv99, Definition 5.2]. If $(x', y') = (x, y)A$ is a new basis for U, then the operator D transforms as $D_{x',y'} = (\det A)^{-1}D_{x,y}$; see [Olv99, formula (5.3)]. In particular, D is invariant with respect to the $\mathrm{SL}(U)$-representation on $U^* \otimes U^*$. In this article we will consider the following actions of D as a differential operator:

$$D\colon\ S^kU \otimes S^dU \to S^{k-1}U \otimes S^{d-1}U\,.$$

The operator D satisfies the following property.

Lemma 3.1. *For any $\tau \in S^{k-1}U \otimes S^{d-1}U$ one has*

$$D(\xi\tau) = (d + k)\tau + \xi D(\tau)\,.$$

Moreover, one has $D(p_k(f)) = 0$ for any $f \in S^{d+k}U$.

We omit the proof, that can be achieved by a direct computation, reducing oneself to the case $\tau = x^{k-1} \otimes y^{d-1}$ by linearity and $\mathrm{SL}(2)$-invariance. One consequence of the lemma above is the following.

Corollary 3.2. *For any $d, k \geqslant 1$ or $d, k \geqslant 2$, respectively, the following sequences are exact:*

$$0 \to p_k(S^{d+k}U) \to S^kU \otimes S^dU \xrightarrow{D} S^{k-1}U \otimes S^{d-1}U \to 0\,,$$

$$0 \to p_k(S^{d+k}U) \oplus \xi p_{k-1}(S^{d+k-2}U) \to S^kU \otimes S^dU \xrightarrow{D^2} S^{k-2}U \otimes S^{d-2}U \to 0\,.$$

Proof. We start with the first sequence. The fact that the sequence is a complex is the second statement of Lemma 3.1. By the first statement of Lemma 3.1 and by (3.4) and (3.5), the operator D maps the subspace $\xi(S^{k-1}U \otimes S^{d-1}U)$ of the space $S^kU \otimes S^dU = p_k(S^{d+k}U) \oplus \xi(S^{k-1}U \otimes S^{d-1}U)$ onto $S^{k-1}U \otimes S^{d-1}U$. The exactness in the middle also follows from the decomposition (3.5). The proof of the exactness of the second sequence is very similar. One first shows

$$D^2\big(p_k\big(S^{d+k}U\big) \oplus \xi p_{k-1}\big(S^{d+k-2}U\big)\big) = 0$$

by applying Lemma 3.1 twice. Then the exactness follows from (3.4) and (3.6) in a way similar to that for the first sequence. □

In a different vein, one can use the operator D^2 to produce the invariant map

$$S^kU \otimes S^bU^* \xrightarrow{D^2} S^{k-2}U \otimes S^{b+2}U^*\,. \tag{3.7}$$

In this map, the tensor $D^2 = \partial_x^2 \otimes \partial_y^2 - 2\partial_x\partial_y \otimes \partial_x\partial_y + \partial_y^2 \otimes \partial_x^2$ acts by contraction on the S^kU-components and by multiplication on the S^bU^*-component. Later, we will need the following result.

Proposition 3.3. *The map (3.7) has maximal rank for any $b \geqslant 0$ and $k \geqslant 2$.*

Proof. We use the identification $\phi\colon U^* \to U$ that maps $\alpha\partial_x + \beta\partial_y$ to $-\beta x + \alpha y$. Note that ϕ is SL(2)-invariant, as $\phi \wedge \phi$ maps $\partial_x \wedge \partial_y$ to $y \wedge (-x) = x \wedge y$. Then for any $i, j \geqslant 0$, the map $1 \otimes S^j(\phi)\colon S^iU \otimes S^jU^* \to S^iU \otimes S^jU$ is a isomorphism. We can rewrite the map (3.7) in terms of these identifications as follows:

$$S^kU \otimes S^bU \xrightarrow{\delta^2} S^{k-2}U \otimes S^{b+2}U\,,$$

with $\delta^2 = \partial_x^2 \otimes x^2 + 2\partial_x\partial_y \otimes xy + \partial_y^2 \otimes y^2 = (\partial_x \otimes x + \partial_y \otimes y)^2$, acting as before by contraction on S^kU and by multiplication on S^bU. Now the fact that δ^2 has maximal rank is a consequence of the following more general result. □

Lemma 3.4. *For any $(n+1)$-dimensional $\mathbb{C}$-vector space $V = \langle x_0, \dots, x_n\rangle$ and any $k \geqslant a$ and $b \geqslant 0$, for $\delta = (\partial_{x_0} \otimes x_0 + \dots + \partial_{x_n} \otimes x_n)$, the map*

$$S^kV \otimes S^bV \xrightarrow{\delta^a} S^{k-a}V \otimes S^{b+a}V \tag{3.8}$$

has maximal rank.

The result above is already known; for example, one can see that it is a consequence of [Re12, Theorem 2]. However, we find it more convenient to give a new proof here, since we did not find any clear reference for the statement above in the existing literature.

Sketch of proof. We use the invariance of δ and the Pieri decompositions of $S^kV \otimes S^bV$ and $S^{k-a}V \otimes S^{b+a}V$ as SL(V)-modules. As is well known,

$$S^kV \otimes S^bV = \bigoplus_{i=0}^{\min(k,b)} S_{(k+b-i,i)}V\,, \tag{3.9}$$

where $S_{(k+b-i,i)}V$ is the SL(V)-irreducible tensor space resulting by applying to V the Schur functor associated with the Young diagram with two rows of lengths $k+b-i$ and i, respectively. One has the similar decomposition

$$S^{k-a}V \otimes S^{b+a}V = \bigoplus_{i=0}^{\min(k-a,a+b)} S_{(k+b-i,i)}V\,. \tag{3.10}$$

Note that if $b \leqslant k-a$, then all the summands $S_{(k+b-i,i)}V$ appearing in (3.9) also appear in (3.10) and, on the other hand, if $b \geqslant k-a$, then all the summands in (3.10) appear in (3.9). Then the proof is complete if one shows that for any summand appearing in both the formulas above, the composition

$$S_{(k+b-i,i)}V \hookrightarrow S^kV \otimes S^bV \xrightarrow{\delta^a} S^{k-a}V \otimes S^{b+a}V \twoheadrightarrow S_{(k+b-i,i)}V$$

is non-zero and hence an isomorphism. It is well known that the first invariant inclusion identifies $S_{(k+b-i,i)}V$ as the subspace of $S^kV \otimes S^bV$ generated by tensors of the form $\xi_1 \cdots \xi_i f$, where the ξ_j are tensors of the form $x_h \otimes x_k - x_k \otimes x_h$ and $f \in p_{k-i}(S^{k+b-2i}V) \subset S^{k-i}V \otimes S^{b-i}V$. Then one observes the fundamental fact that $\delta(x_h \otimes x_k - x_k \otimes x_h) = 0$. Since δ is a derivation on the commutative ring $S^\bullet V \otimes S^\bullet V$, one deduces that δ commutes with $x_h \otimes x_k - x_k \otimes x_h$ and hence $\delta^a(\xi_1 \cdots \xi_i f) = \xi_1 \cdots \xi_i \delta^a(f)$. Then one concludes by the observation that $f = p_{k-i}(g)$ and one can easily check that $\delta^a(f) = \delta^a(p_{k-i}(g)) = p_{k-i-a}(g)$, up to some non-zero rational factor. Hence the map δ^a is non-zero when restricted to $S_{(k+b-i,i)}V$. □

3.7 The invariant embeddings $\psi_k\colon U \otimes S^{d+k-1}U \to S^kU \otimes S^dU$

We define the invariant embeddings ψ_k as the compositions

$$U \otimes S^{d+k-1}U \xrightarrow{1\otimes p_k} U \otimes S^kU \otimes S^{d-1}U \xrightarrow{\widetilde{m}} S^kU \otimes S^dU,$$

where $\widetilde{m}$ is the multiplication of the first and the third tensor components of $U \otimes S^kU \otimes S^{d-1}U$. The maps ψ_k are obviously $\mathrm{SL}(U)$-invariant. We will show that the maps ψ_k are invariant embeddings for any $k \geqslant 1$.

Proposition 3.5. *For any $d \geqslant 2$ and $k \geqslant 1$, the map ψ_k is injective and*

$$\psi_k\big(U \otimes S^{d+k-1}U\big) = \ker\left(D^2\colon S^kU \otimes S^dU \to S^{k-2}U \otimes S^{d-2}U\right),$$

where the map above is set to be the zero map in the case $k = 1$.

Proof. We use the decomposition $U \otimes S^{d+k-1}U = p_1(S^{d+k}U) \oplus \xi S^{d+k-2}U$, which is a particular case of (3.5). Since the two summands are irreducible representations of $\mathrm{SL}(U)$ and the map ψ_k is $\mathrm{SL}(U)$-invariant, to show the injectivity of ψ_k it will be sufficient to show that ψ_k is non-zero on the summands $p_1(S^{d+k}U)$ and $\xi S^{d+k-2}U$. We will achieve that by computing ψ_k on some special elements of these summands.

For $l \otimes l^{d+k-1} \in p_1(S^{d+k}U)$ we see that

$$\begin{aligned}\psi_k\big(l \otimes l^{d+k-1}\big) &= \widetilde{m}\big((1 \otimes p_k)\big(l \otimes l^{d+k-1}\big)\big)\\ &= \widetilde{m}\big(l \otimes l^k \otimes l^{d-1}\big)\\ &= l^k \otimes l^d \in p_k\big(S^{d+k}U\big) \subset S^kU \otimes S^dU\,.\end{aligned}$$

Now, let us consider the element $\xi x^{d+k-2} = x \otimes x^{d+k-2}y - y \otimes x^{d+k-1} \in \xi S^{d+k-2}U$. We compute separately $\psi_k(x \otimes x^{d+k-2}y)$ and $\psi_k(y \otimes x^{d+k-1})$. One finds easily

$$\psi_k\big(y \otimes x^{d+k-1}\big) = x^k \otimes x^{d-1}y\,.$$

From formula (3.3) one has

$$\begin{aligned}p_k(x^{d+k-2}y) &= \frac{(d-1)!}{(d+k-1)!}\big(x^k \otimes \partial_x^k(x^{d+k-2}y) + kx^{k-1}y \otimes \partial_x^{k-1}\partial_y\big(x^{d+k-2}y\big)\big)\\ &= \frac{(d-1)!}{(d+k-1)!}\left(\frac{(d+k-2)!}{(d-2)!}x^k \otimes x^{d-2}y + kx^{k-1}y \otimes \frac{(d+k-2)!}{(d-1)!}x^{d-1}\right)\\ &= \frac{1}{d+k-1}\big((d-1)x^k \otimes x^{d-2}y + kx^{k-1}y \otimes x^{d-1}\big)\,.\end{aligned}$$

Hence one obtains

$$\psi_k\big(x \otimes x^{d+k-2}y\big) = \frac{1}{d+k-1}\big((d-1)x^k \otimes x^{d-1}y + kx^{k-1}y \otimes x^d\big)\,.$$

Then we find

$$\begin{aligned}\psi_k\big(\xi x^{d+k-1}\big) &= \psi_k\big(x\otimes x^{d+k-2}y\big)-\psi_k\big(y\otimes x^{d+k-1}\big)\\ &= \frac{1}{d+k-1}\big((d-1)x^k\otimes x^{d-1}y+kx^{k-1}y\otimes x^d\big)\\ &\quad -\frac{1}{d+k-1}\big((d+k-1)x^k\otimes x^{d-1}y\big)\\ &= \frac{k}{d+k-1}\big(x^{k-1}y\otimes x^d-x^k\otimes x^{d-1}y\big)\\ &= -\frac{k}{d+k-1}\xi\big(x^{k-1}\otimes x^{d-1}\big)\in \xi p_{k-1}\big(S^{d+k-2}U\big).\end{aligned}$$

The calculations made above show that ψ_k restricts to a non-zero $\mathrm{SL}(U)$-invariant map on $p_1(S^{d+k}U)$ and $\xi S^{d+k-2}U$. In particular, by the $\mathrm{SL}(U)$-irreducibility of these spaces, one gets

$$\begin{aligned}\psi_k\big(p_1\big(S^{d+k}U\big)\big) &= p_k\big(S^{d+k}U\big)\,,\\ \psi_k\big(\xi S^{d+k-2}U\big) &= \xi p_{k-1}\big(S^{d+k-2}U\big)\,,\end{aligned}$$

proving the global injectivity of ψ_k. Moreover, applying Corollary 3.2, one has

$$\psi_k\big(U\otimes S^{d+k-1}U\big)=p_k\big(S^{d+k}U\big)\oplus\xi p_{k-1}\big(S^{d+k-2}U\big)=\ker D^2\,. \qquad \square$$

4. A new setup for computing the cohomology of $\mathcal{N}_f$

From now on we will assume that $f\colon \mathbb{P}^1\to\mathbb{P}^s$ parametrizes a rational curve with ordinary singularities and that $f=\pi_T\circ\nu_d$, so the parametrized curve arises as projection of the rational normal curve C_d from a vertex $\mathbb{P}(T)$. Let us recall the operator

$$D^2\colon\; S^kU\otimes S^dU\to S^{k-2}U\otimes S^{d-2}U$$

discussed in Section 3. We state the main theorem of this article, whose proof will be given at the end of this section.

Theorem 4.1. *For any $k\geqslant 1$ one has*

$$\begin{aligned}h^0\mathcal{T}_f(-d-2-k) &= \dim\big(\ker D\cap\big(S^kU\otimes T\big)\big)\,,\\ h^0\mathcal{N}_f(-d-2-k) &= \dim\big(\ker D^2\cap\big(S^kU\otimes T\big)\big)\,.\end{aligned}$$

4.1 Euler sequence and its consequences

Let $C\subset\mathbb{P}^s$ be a degree d rational curve with ordinary singularities. As in the notation above we assume that there is a parametrization map $f\colon\mathbb{P}^1\to\mathbb{P}^s$ obtained by projecting the rational normal curve C_d from a vertex $\mathbb{P}(T)\subset\mathbb{P}(S^dU)$. Since $f=\pi_T\circ\nu_d$, we have $\mathbb{P}^s=\mathbb{P}(S^dU/T)$. Note also that the natural inclusion $(S^dU/T)^*\subset S^dU^*$ identifies $(S^dU/T)^*$ and $T^\perp$. Hence we can set

$$\dim T=e+1\,,\quad \dim T^\perp=s+1=d-e\,.$$

We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & \mathcal{O}_{\mathbb{P}^1} & \xrightarrow{=} & \mathcal{O}_{\mathbb{P}^1} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & U \otimes \mathcal{O}_{\mathbb{P}^1}(1) & \xrightarrow{J(f)} & (T^{\perp})^* \otimes \mathcal{O}_{\mathbb{P}^1}(d) & \longrightarrow & \mathcal{N}_f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathcal{T}_{\mathbb{P}^1} & \xrightarrow{df} & \mathcal{T}_f & \longrightarrow & \mathcal{N}_f & \longrightarrow & 0\,.
\end{array}$$

Indeed, if the map $f\colon \mathbb{P}^1 \to \mathbb{P}((T^{\perp})^*) = \mathbb{P}^s$ is given in coordinates by

$$f(u:v) = (g_0(u,v) : \cdots : g_s(u,u))\,,$$

with $g_i(u,v) \in S^d U^*$, then the map $J(f)\colon U \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to (T^{\perp})^* \otimes \mathcal{O}_{\mathbb{P}^1}(d)$ in the diagram above is given fiberwise by the differentials $df|_{(u,v)}\colon T_{(u,v)}(C\mathbb{P}^1) \to T_{f(u,v)}(C\mathbb{P}^s)$ of the map $f\colon C\mathbb{P}^1 \to C\mathbb{P}^s$ between the associated affine cones. Hence it has associated matrix

$$J(f) = \begin{pmatrix} \partial_u g_0(u,v) & \partial_v g_0(u,v) \\ \vdots & \vdots \\ \partial_u g_s(u,v) & \partial_v g_s(u,v) \end{pmatrix}.$$

Let us consider the exact sequence

$$0 \to U \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to \left(T^{\perp}\right)^* \otimes \mathcal{O}_{\mathbb{P}^1}(d) \to \mathcal{N}_f \to 0\,. \tag{4.1}$$

From this sequence we get

$$\deg \mathcal{N}_f(-d-1) = -(d-e) + 2d = d+e\,.$$

Writing, as in the introduction,

$$\mathcal{N}_f = \mathcal{O}_{\mathbb{P}^1}(c_1 + d + 2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(c_{s-1} + d + 2) \tag{4.2}$$

with $c_1 \geqslant \cdots \geqslant c_{s-1}$, we see that

$$\sum_{i=1}^{s-1} (c_i + 1) = d + e\,, \quad \sum_{i=1}^{s-1} c_i = 2(e+1)\,. \tag{4.3}$$

Taking the cohomology exact sequence from (4.1) we obtain, for any $k \geqslant d+1$,

$$H^0\mathcal{N}_f(-k) \hookrightarrow U \otimes H^1\mathcal{O}_{\mathbb{P}^1}(1-k) \to (T^{\perp})^* \otimes H^1\mathcal{O}_{\mathbb{P}^1}(d-k) \twoheadrightarrow H^1\mathcal{N}_f(-k)\,. \tag{4.4}$$

If $k = d+1$, one obtains $H^0\mathcal{N}_f(-d-1) \cong U \otimes H^1\mathcal{O}_{\mathbb{P}^1}(-d)$. Let us now consider the cases $k \geqslant d+2$. We have $T^{\perp} = \langle g_0, \ldots, g_s \rangle$, and we denote by $g_0^*, \ldots, g_s^*$ the dual basis of the g_i in $(T^{\perp})^* = S^d U/T$. Recall that if we write $U^* = \langle u, v \rangle$, with u, v the dual basis of $x, y \in U$, then the first non-zero map in (4.1) is defined by $x \otimes l \mapsto \sum_i g_i^* \otimes l \partial_u g_i$ and $y \otimes l' \mapsto \sum_i g_i^* \otimes l' \partial_v g_i$, for any local sections l, l' of $\mathcal{O}_{\mathbb{P}^1}(1)$.

As is well known, by Serre duality one can identify the spaces $H^1\mathcal{O}_{\mathbb{P}^1}(1-k)$ and $H^1\mathcal{O}_{\mathbb{P}^1}(d-k)$ appearing in the exact sequence (4.4) with $(H^0\mathcal{O}_{\mathbb{P}^1}(k-3))^* = S^{k-3}U$ and $(H^0\mathcal{O}_{\mathbb{P}^1}(k-d-2))^* = S^{k-d-2}U$, respectively. Moreover, it is well known that any sheaf map $\mathcal{O}_{\mathbb{P}^1}(1-k) \xrightarrow{\sigma} \mathcal{O}_{\mathbb{P}^1}(d-k)$ associated with a global section $\sigma \in H^0\mathcal{O}_{\mathbb{P}^1}(d-1) = S^{d-1}U^*$ induces a map $H^1\mathcal{O}_{\mathbb{P}^1}(1-k) \xrightarrow{\sigma} H^1\mathcal{O}_{\mathbb{P}^1}(d-k)$ between the cohomology spaces that, under the identifications above, can be written as the linear map $S^{k-3}U \xrightarrow{\sigma} S^{k-d-2}U$ defined by letting σ act as a differential operator on $S^{k-3}U$. In our case the sheaf map $U\otimes\mathcal{O}_{\mathbb{P}^1}(1-k)\to(T^\perp)^*\otimes\mathcal{O}_{\mathbb{P}^1}(d-k)$ arising from (4.1), after the identifications $U\cong\mathbb{C}^2$ and $T^\perp\cong\mathbb{C}^{s+1}$ by means of the mentioned bases x, y and $g_0,\dots,g_s$ can be seen as a sheaf map $\mathcal{O}^2_{\mathbb{P}^1}(1-k)\to\mathcal{O}^{s+1}_{\mathbb{P}^1}(d-k)$ whose components have the form $\mathcal{O}_{\mathbb{P}^1}(1-k)\overset{\partial_u g_i}{\to}\mathcal{O}_{\mathbb{P}^1}(d-k)$ and $\mathcal{O}_{\mathbb{P}^1}(1-k)\overset{\partial_v g_i}{\to}\mathcal{O}_{\mathbb{P}^1}(d-k)$. The induced maps on the H^1 cohomology spaces are therefore $\partial_u g_i\colon S^{k-3}U\to S^{k-2-d}U$ and $\partial_v g_i\colon S^{k-3}U\to S^{k-2-d}U$, acting as differential operators of order $d-1$.

From the discussion above it follows that one can compute $H^0\mathcal{N}_f(-k)$ as the kernel of the linear map

$$U\otimes S^{k-3}U\to\left(T^\perp\right)^*\otimes S^{k-d-2}U \tag{4.5}$$

defined by $x\otimes f\mapsto\sum_i g_i^*\otimes(\partial_u g_i)(f)$ and $y\otimes f'\mapsto\sum_i g_i^*\otimes(\partial_v g_i)(f')$, where $\partial_u g_i,\partial_v g_i\colon S^{k-3}U\to S^{k-2-d}U$ act as differential operators of order $d-1$. Let us compute the kernel $H^0\mathcal{N}_f(-k)$ of the linear map (4.5).

The space $H^0\mathcal{N}_f(-k)$, seen as a subspace of $U\otimes S^{k-3}U$, is the space of tensors $x\otimes f_0+y\otimes f_1\in U\otimes S^{k-3}U$ such that $(\partial_u g_i)(f_0)+(\partial_v g_i)(f_1)=0\in S^{k-d-2}U$ for all $i=0,\dots,s$. This is equivalent to imposing that $f_0(P\partial_u g)+f_1(P\partial_v g)=0$ for any $g\in T^\perp$ and any $P\in S^{k-d-2}U^*$. This is equivalent to saying that

$$P(f_0)(\partial_u g)+P(f_1)(\partial_v g)=0 \tag{4.6}$$

for any $P\in S^{k-d-2}U^*$ and any $g\in T^\perp$. By applying the version of formula (3.2) with the roles of U and U^* interchanged and recalling that the elements $x,y\in U$ act as ∂_u,∂_v on $\mathbb{C}[u,v]$, respectively, one sees that for any $\phi\in S^{d-1}U$ and any $g\in S^dU^*$ one has $\phi(\partial_u g)=(x\phi)(g)$ and similarly $\phi(\partial_v g)=(y\phi)(g)$. Hence we can rewrite (4.6) in the following form:

$$(xP(f_1)+yP(f_2))(g)=0\,,\quad\forall\, g\in T^\perp\,,\quad\forall\, P\in S^{k-d-2}U^*\,,$$

which means

$$xP(f_1)+yP(f_2)\in T\,,\quad\forall\, P\in S^{k-d-2}U^*\,. \tag{4.7}$$

Notation. The calculations made above hold for any $k\geqslant d+2$. We find it convenient, from now on, to redefine k to be what was first $k-d-2$. Accordingly, we set, for any $k\geqslant 0$,

$$T_k=\left\{x\otimes f_0+y\otimes f_1\in U\otimes S^{d+k-1}U\,|\,xP(f_0)+yP(f_1)\in T,\,\forall\, P\in S^kU^*\right\}.$$

Hence we can summarize the discussion above in the following result.

Proposition 4.2. *Under the notation above, we have the following relation for any $k\geqslant 0$:*

$$H^0\mathcal{N}_f(-d-2-k)=T_k\,. \tag{4.8}$$

The following proposition collects some facts that will be needed later, as well as some first applications of the result above.

Proposition 4.3. *Assume that $\mathcal{N}_f$ has a splitting of the form (4.2). Then the following hold:*

(i) *One has $h^0\mathcal{N}_f(-d-k-2)=\sum_{i\colon c_i\geqslant k}(c_i-k+1)$ for any $k\in\mathbb{Z}$.*

(ii) *Setting $f(-k)=h^0\mathcal{N}_f(-d-k-2)$ for any $k\in\mathbb{Z}$, one has*

$$\#\{i\,|\,c_i=k\}=\Delta^2f(-k)=f(-k)-2f(-k-1)+f(-k-2)\,.$$

(iii) $\sum_{i=1}^{s-1}(c_i+1)=d+e=d+\dim\mathbb{P}(T)$.

(iv) $\sum_{i=1}^{s-1}c_i=2(e+1)=2\dim T$.

(v) $c_{s-1} \geqslant 0$.

Proof. Items (i) and (ii) are easy and well known. The relations (iii) and (iv) coincide with formulas (4.3) and therefore have already been proven.

From Proposition 4.2 we have the identification

$$H^0\mathcal{N}_f(-d-2) = \left\{x \otimes f_1 + y \otimes f_2 \in U \otimes S^{d-1}U \,|\, xf_1 + yf_2 \in T\right\},$$

and therefore we see that

$$H^0N_f(-d-2) \cong m^{-1}(T) \subset U \otimes S^{d-1}U\,, \tag{4.9}$$

where m is the multiplication map $m\colon U \otimes S^{d-1}U \to S^dU$. Now, the kernel of m is given by the tensors of the form $x \otimes yh - y \otimes xh$, with arbitrary $h \in S^{d-2}U$. Then one has

$$h^0\mathcal{N}_f(-d-2) = \dim m^{-1}(T) = d - 1 + \dim T = d + e\,. \tag{4.10}$$

On the other hand, by (4.3) we know

$$d + e = h^0\mathcal{N}_f(-d-2) = \sum_{i\colon c_i \geqslant 0}(c_i + 1) \geqslant \sum_{i=1}^{s-1}(c_i + 1) = d + e\,.$$

This implies $c_1 \geqslant \cdots \geqslant c_{s-1} \geqslant -1$. We will also need to know the value of $h^0\mathcal{N}_f(-d-1)$. This is obtained from the exact sequence (4.1), from which it easily follows that $H^0\mathcal{N}_f(-d-1) \cong U \otimes H^1\mathcal{O}_{\mathbb{P}^1}(-d)$ and hence $h^0\mathcal{N}_f(-d-1) = 2(d-1)$. Now, applying fact (ii) for $k = -1$ and using relations (iii) and (iv) and the above calculation of $f(1) = h^0\mathcal{N}_f(-d-1)$, we see that $\#\{i \,|\, c_i = -1\} = 2(d-1) - 2(d+e) + 2(e+1) = 0$, which completes the proof of relation (v). □

4.2 Completion of the proof of Theorem 4.1

Proof of Theorem 4.1. We start with the part of the statement about $\mathcal{T}_f$. At the beginning of [AR15, Section 6.2, p. 1334] we showed the equality

$$h^0\mathcal{T}_f(-d-2-k) = \dim \partial^{-k}T\,.$$

Moreover, from Corollary 3.2 we know $p_k(S^{d+k}U) = \ker D \subset S^kU \otimes S^dU$. Then one finds

$$\begin{aligned}\ker D \cap \left(S^kU \otimes T\right) &= p_k\left(S^{k+d}U\right) \cap \left(S^kU \otimes T\right)\\ &= p_k\left(\left\{f \in S^{d+k}U \,|\, \partial_x^{k-i}\partial_y^i(f) \in T,\, \forall\, i = 0,\ldots,k\right\}\right)\\ &\cong \partial^{-k}T.\end{aligned}$$

Hence we find the equality $h^0\mathcal{T}_f(-d-2-k) = \dim(\ker D \cap (S^kU \otimes U))$.

Now, we prove the statement about $\mathcal{N}_f$. By Proposition 4.2 we know

$$H^0\mathcal{N}_f(-d-2-k) = T_k$$

with $T_k \subseteq U \otimes S^{d+k-1}U$ the subspace consisting of those elements $x \otimes f_0 + y \otimes f_1$ such that $xP(f_0) + yP(f_1) \in T$ for any $P \in S^kU^*$. This is equivalent to the condition

$$x\partial_x^{k-i}\partial_y^i(f_0) + y\partial_x^{k-i}\partial_y^i(f_1) \in T\,, \quad \forall\, i = 0,\ldots,k\,.$$

Recall that by formula (3.3) one has

$$\begin{aligned}\psi_k(x \otimes f_0 + y \otimes f_1) &= \widetilde{m}(x \otimes p_k(f_0) + y \otimes p_k(f_1))\\ &= \text{const} \cdot \sum_{i=1}^{k}\binom{k}{i}x^{k-i}y^i \otimes \left(x\partial_x^{k-i}\partial_y^i(f_0) + y\partial_x^{k-i}\partial_y^i(f_1)\right).\end{aligned}$$

Therefore, by the definition of T_k, we have

$$x \otimes f_0 + y \otimes f_1 \in T_k \iff x\partial_x^{k-i}\partial_y^i(f_0) + y\partial_x^{k-i}\partial_y^i(f_1) \in T \quad \forall\, i = 0, \dots, k\,,$$
$$\iff \psi_k(x \otimes f_0 + y \otimes f_1) \in S^k U \otimes T\,.$$

On the other hand, by Proposition 3.5, one has $\psi_k(x \otimes f_0 + y \otimes f_1) \in \mathrm{Im}(\psi_k) = \ker D^2$ and ψ_k is injective for $k \geqslant 1$. Hence for any $k \geqslant 1$ one has

$$H^0\mathcal{N}_f(-d-2-k) \cong T_k \overset{\psi_k}{\cong} \ker D^2 \cap \left(S^k U \otimes T\right). \qquad \square$$

5. Some general consequences of Theorem 4.1

5.1 The dimension $h^0\mathcal{N}_f(-d-2-k)$ for $k = 0, 1, 2$

Proposition 5.1. *The spaces $H^0\mathcal{N}_f(-d-2-k)$ have the following dimensions for $k = 0, 1, 2$:*

$$k = 0: \quad h^0\mathcal{N}_f(-d-2) = d - 1 + \dim T\,,$$
$$k = 1: \quad h^0\mathcal{N}_f(-d-3) = 2\dim T\,,$$
$$k = 2: \quad h^0\mathcal{N}_f(-d-4) = 3\dim T - \dim \partial^2 T\,.$$

Proof. The case $k = 0$ is the formula (4.10) and has already been discussed.

The case $k = 1$ is a consequence of the degree of $\mathcal{N}_f$ and was already established by the formulas (4.3), but it also follows from the fact that $D^2 = 0$ on the space $U \otimes S^d U$ and therefore, by Theorem 4.1, one has $H^0\mathcal{N}_f(-d-3) \cong U \otimes T$.

Finally, for $k = 2$, by Theorem 4.1 we have to compute

$$\dim\left(\left(S^2U \otimes T\right) \cap \ker D^2\right) = \dim \ker D^2\big|_{S^2U\otimes T}\,.$$

Note that $\dim(S^2U \otimes T) = 3\dim T$, and hence the claim on $h^0\mathcal{N}_f(-d-4)$ follows if we show that $D^2(S^2U \otimes T) = \partial^2 T$. We know

$$D^2\left(\left(ax^2 + bxy + cy^2\right) \otimes \tau\right) = 2a\tau_{xx} - 2b\tau_{xy} + 2c\tau_{yy}\,.$$

By choosing $\tau \in T$ and a, b, c appropriately, one sees that $\tau_{xx}, \tau_{xy}, \tau_{yy} \in D^2(S^2U \otimes T)$ and since these elements generate $\partial^2 T$, one obtains $\partial^2 T \subseteq D^2(S^2U \otimes T)$. The converse inclusion is obvious. $\square$

Corollary 5.2. *The number of summands equal to $\mathcal{O}_{\mathbb{P}^1}(d+2)$ in the splitting type (4.2) of $\mathcal{N}_f$ is equal to $d - 1 - \dim \partial^2 T$.*

Proof. This follows immediately from Proposition 4.3(v) applied to $k = 0$ and the dimensions computed in Proposition 5.1. $\square$

5.2 Some general results on $h^0\mathcal{N}_f(-d-2-k)$ with $k \geqslant 3$

The computation of kernels and images of the maps

$$D^2\colon\ S^kU \otimes T \to S^{k-2}U \otimes S^{d-2}U$$

for $k \geqslant 3$ may be not easy for an arbitrary T. Sometimes one can reduce this computation to the case of subspaces of smaller dimension. This is possible by means of the following easy lemma.

Lemma 5.3. *Assume that for a given decomposition $T = T_1 \oplus T_2$ one also has $\partial^2 T = \partial^2 T_1 \oplus \partial^2 T_2$. Then for any $k \geqslant 2$ the map $D^2\colon S^kU \otimes T \to S^{k-2}U \otimes S^{d-2}U$ is the direct sum of its restrictions to $S^kU \otimes T_i$ for $i = 1, 2$. In particular its rank is the sum of the ranks of the two restrictions.*

Proof. This is immediate, since the image of $\mathrm{res}(D^2)\colon S^kU\otimes T_i\to S^{k-2}U\otimes S^{d-2}U$ is contained in $S^{k-2}U\otimes\partial^2T_i$ for $i=1,2$. □

From Lemma 5.3 one deduces the following result.

Proposition 5.4. *Assume $T=\partial^{b_1}(f_1)\oplus\cdots\oplus\partial^{b_r}(f_r)$, of type $(b_1,\dots,b_r)$, and that ∂T has type $(b_1+1,\dots,b_r+1)$. Let us denote by*

$$D_i^2\colon\ S^kU\otimes\partial^{b_i}(f_i)\to S^{k-2}U\otimes\partial^{b_i+2}(f_i)$$

the restriction of D^2 for any $i=1,\dots,r$. Then the maps D_i^2 have maximal rank for any $i=1,\dots,r$, and the rank of $D^2\colon S^kU\otimes T\to S^{k-2}U\otimes S^{d-2}U$ is the sum of their ranks.

Proof. In view of Lemma 5.3 we only need to show that D_i^2 has maximal rank for any $i=1,\dots,r$. Note that by Proposition 2.3 the assumption that the type of ∂T is $(b_1+1,\dots,b_r+1)$ in particular implies $\dim\partial^{b_i+2}(f_i)=b_i+3$ for all i, hence one has an isomorphism $S^{b_i+2}U^*\to\partial^{b_i+2}(f_i)$ defined by $\Omega\mapsto\Omega(f_i)$ for any $\Omega\in S^{b_i+2}U^*$. Recall also that since T has type $(b_1,\dots,b_r)$, one knows $\dim\partial^{b_i}(f_i)=b_i+1$, and hence one has an isomorphism $S^{b_i}U^*\to\partial^{b_i}(f_i)$ defined in the same way as above. Under these isomorphisms, the maps D_i^2 are identified with the map (3.7) with $b=b_i$ and hence, by Proposition 3.3, they have maximal rank. □

As an application of the result above, we compute the normal bundles of rational curves obtained from vertexes T of the most special type, that is, $T=\partial^e(g)$ with $g\in\mathbb{P}(S^{d+e})\setminus\mathrm{Sec}^e\,C_{d+e}$.

Proposition 5.5. *If the curve $C\subset\mathbb{P}^s$ is obtained from a vertex T of numerical type (e), that is, $T=\partial^e(g)$ with $g\in\mathbb{P}(S^{d+e})\setminus\mathrm{Sec}^e\,C_{d+e}$, then*

$$\mathcal{N}_f=\mathcal{O}^2_{\mathbb{P}^1}(d+e+3)\oplus\mathcal{O}^{d-e-4}_{\mathbb{P}^1}(d+2)\,.$$

Proof. One can apply Proposition 5.4 and find

$$h^0\mathcal{N}_f(-d-2-k)=\max(0,(k+1)(e+1)-(k-1)(e+3))=\max(0,2e+4-2k)\,.$$

Setting $f(-k)=h^0\mathcal{N}_f(-d-2-k)$ for $k\geqslant 0$, as in Proposition 4.3, we see that the sequence $f(-k)$ is

$$d+e,2e+2,2e,\dots,2,0,\dots\,.$$

Its second difference is

$$d-e-4,0,\dots,0,2,0,\dots\,,$$

where the last 2 appears at the place $k=e+1$. Hence, by Proposition 4.3, one has $(c_1,\dots,c_{s-1})=(e+1,e+1,0,\dots,0)$, with $s-1=d-e-2$. By formula (4.2), we obtain the stated splitting type of $\mathcal{N}_f$. □

6. Example of a reducible Hilbert scheme of rational curves with fixed normal bundle: $\mathcal{H}_{\overline{c}}$ with $\overline{c}=(2,2,1,1,0,0,0)$

This section is dedicated to the construction of the first known example, to our knowledge, of a reducible Hilbert scheme of rational curves with a given splitting type of the normal bundle.

As in the introduction, we will denote by $\mathcal{H}_{\overline{c}}$ the Hilbert scheme of degree d irreducible, non-degenerate rational curves in $\mathbb{P}^s$, with ordinary singularities and with normal bundle with splitting type $\bigoplus\mathcal{O}_{\mathbb{P}^1}(c_i+d+2)$. We will consider the case $\overline{c}=(2,2,1,1,0,0,0)$; therefore we have $s-1=7$. Moreover, from $\sum(c_i+1)=13=d+e$ and $\sum c_i=6=2(e+1)$ we get $e=2$

and $d = 11$; that is, we are dealing with rational curves of degree 11 in $\mathbb{P}^8$. More precisely, we are dealing with parametrized curves of degree 11 in $\mathbb{P}^8$ with splitting type of the normal bundle given by

$$\mathcal{N}_f = \mathcal{O}^2_{\mathbb{P}^1}(15) \oplus \mathcal{O}^2_{\mathbb{P}^1}(14) \oplus \mathcal{O}^3_{\mathbb{P}^1}(13)\,.$$

These curves are obtained, up to a projective transformation in $\mathbb{P}^8$, as projections of the rational curve $C_{11} = \nu_{11}(\mathbb{P}^1) \subseteq \mathbb{P}(S^{11}U)$ from a 2-dimensional vertex $\mathbb{P}(T)$, so that

$$\dim T = e + 1 = 3\,.$$

We recall that the knowledge of the $(s-1)$-uple $(c_1, \dots, c_{s-1})$ is equivalent to the knowledge of the dimensions of the spaces $H^0\mathcal{N}_f(-d-2-k) = T_k$. In our case these dimensions are the following:

$$\begin{aligned}
&\dim T_0 = \sum_{i\colon c_i \geqslant 0} (c_i + 1) = 13\,, \quad \dim T_1 = \sum_{i\colon c_i \geqslant 1} c_i = 6\,,\\
&\dim T_2 = \sum_{i\colon c_i \geqslant 2} (c_i - 1) = 2\,, \quad \dim T_3 = \sum_{i\colon c_i \geqslant 3} (c_i - 2) = 0\,,\\
&\dim T_k = 0\,, \quad \forall\, k \geqslant 3\,.
\end{aligned}$$

We also recall that $T_k \cong \ker(D^2\colon S^kU \otimes T \to S^{k-2}U \otimes \partial^2 T)$ for all $k \geqslant 1$. Since the vertex $\mathbb{P}(T)$ must not intersect C_{11}, we have only three possibilities for the numerical type of T, namely the type (2), the type $(1, 0)$ and the type $(0, 0, 0)$. We can immediately rule out the type (2) by the following argument. By Proposition 5.1 one has

$$\dim \partial^2 T = \dim S^2U \otimes T - \dim T_2 = 7\,. \tag{6.1}$$

If T is of type (2), then $T = \partial^2(f)$ for some polynomial $f \in S^{13}U$ and hence $\partial^2 T = \partial^4(g)$, which has dimension at most 5. Therefore we are left with the possibilities that T has type $(1, 0)$ or $(0, 0, 0)$.

6.1 Curves from spaces T of type $(1, 0)$

We will show that from a general vertex T of type $(1, 0)$ we always obtain a curve with splitting of the normal bundle corresponding to $\bar{c} = (2, 2, 1, 1, 0, 0, 0)$. Recall that such a vertex has the form

$$T = \partial(f) \oplus \langle g \rangle\,,$$

with sufficiently general $f \in \mathbb{P}(S^{12}U)$ and $g \in \mathbb{P}(S^{11}U)$, where the latter is determined by T up to an element of $\partial(f)$. Hence the dimension of the space of such T is given by $\dim \mathbb{P}(S^{12}U) + \dim \mathbb{P}(S^{11}U/\partial(f)) = 12 + 9 = 21$. The same conclusion can be reached by means of the dimension formula provided by [AR15, Theorem 2].

Now, we know that a general $T \subset S^{11}U$ of type $(1, 0)$ has ∂T of type $(2, 1)$. This may be shown starting from a particular T, for example $T = \langle x^3y^8, x^4y^7, x^7y^4\rangle = \partial(x^4y^8) \oplus (x^7y^4)$, from which we get the direct sum decompositions $\partial T = \partial^2(x^4y^8) \oplus \partial(x^7y^4)$ and $\partial^2 T = \partial^3(x^4y^8) \oplus \partial^2(x^7y^4)$. Then one can extend the result to a general T of type $(1, 0)$ by lower semicontinuity of $\dim \partial^2 T$. Hence for a general T of type $(1, 0)$ we find $\dim \partial^2 T = \dim \partial T + 2 = 7$, as required by (6.1). In particular, one obtains $\partial^2 T = \partial^3(f) \oplus \partial^2(g)$ and for any $k \geqslant 2$ the map $D^2\colon S^kU \otimes T \to$

$S^{k-2}U \otimes \partial^2 T$ can be written as the direct sum of the maps

$$D^2\colon S^kU \otimes \partial(f) \to S^{k-2}U \otimes \partial^3(f)\,,$$
$$D^2\colon S^kU \otimes (g) \to S^{k-2}U \otimes \partial^2(g)\,.$$

By construction one has $\dim \partial(f) = 2$, $\dim \partial^3(f) = 4$, $\dim(g) = 1$ and $\dim \partial^2(g) = 3$, hence one has the identifications $S^iU^* \cong \partial^i(f)$ for $i = 1, 3$ and $S^jU^* \cong \partial^j(g)$ for $j = 0, 2$. By means of these identifications the maps above become

$$D^2\colon S^kU \otimes U^* \to S^{k-2}U \otimes S^3U^*\,,$$
$$D^2\colon S^kU \otimes S^0U^* \to S^{k-2}U \otimes S^2U^*\,,$$

where D^2 now operates as in Proposition 3.3. Hence the maps have maximal rank. For $k = 3$ the map $D^2\colon S^3U \otimes \partial(f) \to U \otimes \partial^3(f)$ has domain of dimension 8 and codomain of dimension 8, hence is an isomorphism. The map $D^2\colon S^3U \otimes (g) \to U \otimes \partial^2(g)$ has domain of dimension 4 and codomain of dimension 6; hence it is injective. In conclusion, we obtain $T_3 = 0$, and hence also $T_k = 0$ for all $k \geqslant 3$. So we get the dimensions of the spaces T_k that correspond to $\bar{c} = (2, 2, 1, 1, 0, 0, 0)$. By Proposition 2.1 we have obtained an irreducible subscheme of $\mathcal{H}_{\bar{c}}$ of dimension $21 + \dim \mathrm{PGL}(9) - \dim \mathrm{PGL}(2) = 98$.

We observe that the general curve in the subscheme of $\mathcal{H}_{\bar{c}}$ just defined is a smooth rational curve. Indeed, this is equivalent to showing that a general $\mathbb{P}(T)$ with T of type $(1, 0)$ as above does not intersect $\mathrm{Sec}^1 C_{11}$. Let us fix $[g] \in \mathbb{P}^{11} \setminus \mathrm{Sec}^1 C_{11}$; then the dimension of the cone over $\mathrm{Sec}^1 C_{11}$ with vertex $[g]$, defined as the join $J = J([g], \mathrm{Sec}^1 C_{11})$, is $\dim J = 4$. Let us define

$$J' = \bigl\{[f'] \in \mathbb{P}\bigl(S^{12}U\bigr) \mid \exists\, \omega \in U^*\colon [\omega(f')] \in J\bigr\}\,.$$

Then one finds $\dim J' \leqslant 6$; indeed, $J' = \bigcup_{q \in J, \omega \in \mathbb{P}(U^*)} \mathbb{P}(\omega^{-1}(q))$. Therefore there exists an $[f] \in \mathbb{P}^{12} \setminus J'$. Then one can conclude that for a general $T = \partial(f) \oplus \langle g \rangle$ one has

$$\mathbb{P}(T) \cap \mathrm{Sec}^1 C_d = \emptyset\,.$$

6.2 Curves from spaces T of type $(0, 0, 0)$

Unlike the previous case of T of type $(1, 0)$, it will not be true that a general $T \subseteq S^{11}U$ of type $(0, 0, 0)$ can produce a rational curve in $\mathcal{H}_{\bar{c}}$. Instead, we will show that the space of all T of type $(0, 0, 0)$ whose general element produces curves in $\mathcal{H}_{\bar{c}}$ is a proper irreducible subvariety of the space of all T of type $(0, 0, 0)$.

Now, we have $\dim \partial T = \dim T + 3 = 6$. Recall that to obtain a curve in $\mathcal{H}_{\bar{c}}$ one must have $\dim \partial^2 T = 7$. Hence, under the notations of Proposition 2.3, the space ∂T has type (a, b_1) with $\dim \partial T = a + 1 + b_1 + 1 = 6$, that is, $(a, b_1) = (a, 4 - a)$.

Case $a = -1$. One has $a = -1$ if and only if $\mathbb{P}(\partial T)$ does not intersect $C_{10} \subset \mathbb{P}(S^{10}U)$, so we see that ∂T has type $(b_1) = (5)$, that is, $\partial T = \partial^5(g)$ for some $[g] \notin \mathrm{Sec}^5 C_{15} \subset \mathbb{P}^{15}$ and hence $\partial^2 T = \partial^6(g)$ has dimension 7, as required.

We compute the dimension of the variety of the spaces T under consideration. We observe that for a fixed general $[g] \in \mathbb{P}(S^{15}U)$, any sufficiently general $T \subseteq \partial^{-1}T = \partial^4(g)$ will have type $(0, 0, 0)$ and $\partial T = \partial^5(g)$. One can first show the claim for a special pair g, T, for example $g = x^8y^7$ and $T = \langle x^4y^7, x^6y^5, x^8y^3\rangle$. Then the result holds for general g, T by semicontinuity, more precisely by the upper semicontinuity of $\dim \partial^{-1}T$, which is equal to 0 if and only if T has type $(0, 0, 0)$, by Proposition 2.5. Hence we can find spaces T meeting our requirements in a dense open subset of $\mathrm{Gr}(3, \partial^4(g))$, whose dimension is $\dim \mathrm{Gr}(3, \partial^4(g)) = 6$. Moreover, since a general $T \subset \partial^4(g)$ constructed as above has $\partial T = \partial^5(g)$, the space $\langle g \rangle = \partial^{-5}(\partial T)$ is uniquely determined by T. Hence the final count of parameters for spaces T as above is the following:

$$\dim \mathbb{P}\big(S^{15}U\big) + \dim \mathrm{Gr}(3,5) = 15 + 6 = 21\,.$$

Case $a \geqslant 0$. By Proposition 2.3 a general T of type $(a, 4-a)$ has the form

$$\partial T = \big\langle p_0^{10}, \ldots, p_a^{10} \big\rangle \oplus \partial^{4-a}(g)$$

for a suitable $[g] \notin \mathrm{Sec}^{4-a}\, C_{14-a} \subset \mathbb{P}(S^{14-a}U)$. Note that the C_{10}-generated part of ∂T is uniquely determined by ∂T and hence by T; that is, the points $p_0, \ldots, p_a$ are uniquely determined. On the other hand, g is determined only modulo $W = \langle p_0^{14-a}, \ldots, p_a^{14-a} \rangle$. We have

$$T \subseteq \partial^{-1}\partial T = \big\langle p_0^{10}, \ldots, p_a^{10} \big\rangle \oplus \partial^{3-a}(g)\,,$$

which is again a space of dimension 5, uniquely determined by T. However, we now have $[g] \in \mathbb{P}(S^{14-a}U/W)$, which gives us $13-2a$ parameters. Hence a dimension count similar to the one above provides us with a number of parameters equal to $13-2a+a+1+\dim \mathrm{Gr}(3,5) = 20-a$. So in the case $a \geqslant 0$ we find a variety of vertexes $\mathbb{P}(T)$ of smaller dimension than in the case $a = -1$. Since we are looking for components of $\mathcal{H}_{\overline{c}}$ of maximal dimension, we will be satisfied if we get one such component from the case $a = -1$.

So we have reduced ourselves to showing that a general T of type $(0,0,0)$ with ∂T of type (5) produces a curve in $\mathcal{H}_{\overline{c}}$. Note that from the known data $d = 11$, $\dim T = 3$ and $\dim \partial^2 T = 7$ we already have $\dim T_0 = d + \dim T = 13$, $\dim T_1 = 2\dim T = 6$ and $\dim T_2 = 3 \dim T - \dim \partial^2 T = 2$. From the characterization of $\overline{c} = (2,2,1,1,0,0,0)$ in terms of the dimensions of the spaces T_k, we will get a curve in $\mathcal{H}_{\overline{c}}$ from the vertex T if and only if $\dim T_3 = 0$. By semicontinuity, if we show this for a special T of type $(0,0,0)$ and ∂T of type (5), then the same will hold for such T in general. We take the same example as above.

$$g = x^8y^7, \quad T = \big\langle x^8y^3, x^6y^5, x^4y^7 \big\rangle\,.$$

Notation. To simplify calculations, we denote by $[h]$ any fixed non-zero rational multiple of the polynomial h. Similarly, $[h] + [g]$ will denote a fixed linear combination of h and g with non-zero rational coefficients.

We compute T_3 as the kernel of $D^2\colon S^3U \otimes T \to U \otimes \partial^2 T$. In particular, we will get $T_3 = 0$ if we show that the image of that map has dimension 12. Recalling that $D^2 = \partial_x^2 \otimes \partial_y^2 - 2\partial_x\partial_y \otimes \partial_x\partial_y + \partial_y^2 \otimes \partial_x^2$, we see the following:

$$\begin{aligned}
&D^2\big(\big\langle x^3, x^2y, xy^2, y^3 \big\rangle \otimes \big\langle x^8y^3 \big\rangle\big)\\
&\quad = \big\langle \big[x \otimes x^8y\big], \big[y \otimes x^8y\big] + \big[x \otimes x^7y^2\big], \big[y \otimes x^7y^2\big] + \big[x \otimes x^6y^3\big], \big[y \otimes x^6y^3\big] \big\rangle\,,\\
&D^2\big(\big\langle x^3, x^2y, xy^2, y^3 \big\rangle \otimes \big\langle x^6y^5 \big\rangle\big)\\
&\quad = \big\langle \big[x \otimes x^6y^3\big], \big[y \otimes x^6y^3\big] + \big[x \otimes x^5y^4\big], \big[y \otimes x^5y^4\big] + \big[x \otimes x^4y^5\big], \big[y \otimes x^4y^5\big] \big\rangle\,,\\
&D^2\big(\big\langle x^3, x^2y, xy^2, y^3 \big\rangle \otimes \big\langle x^4y^7 \big\rangle\big)\\
&\quad = \big\langle \big[x \otimes x^4y^5\big], \big[y \otimes x^4y^5\big] + \big[x \otimes x^3y^6\big], \big[y \otimes x^3y^6\big] + \big[x \otimes x^2y^7\big], \big[y \otimes x^2y^7\big] \big\rangle\,.
\end{aligned}$$

The space $D^2(S^3 \otimes T)$ is generated by the 12 elements shown on the right-hand sides of the equalities above. After taking suitable linear combinations of them, they are reduced to the following set of generators:

$$\begin{gathered}[x\otimes x^8y]\,,\quad [y\otimes x^8y]+[x\otimes x^7y^2]\,,\quad [y\otimes x^7y^2]\,,\quad [y\otimes x^6y^3]\,,\\ [x\otimes x^6y^3]\,,\quad [x\otimes x^5y^4]\,,\quad [y\otimes x^5y^4]\,,\quad [y\otimes x^4y^5]\,,\\ [x\otimes x^4y^5]\,,\quad [x\otimes x^3y^6]\,,\quad [y\otimes x^3y^6]+[x\otimes x^2y^7]\,,\quad [y\otimes x^2y^7]\,.\end{gathered}$$

After this simplification, one can easily see that the 12 generators are linearly independent. This completes the proof that $T_3 = 0$.

Finally, we observe that in the given example of $T = \langle x^8y^3, x^6y^5, x^4y^7\rangle$ one has

$$T^\perp = \langle u^{11}, u^{10}v, u^9v^2, u^7v^4, u^5v^6, u^3v^8, u^2v^9, uv^{10}, v^{11}\rangle,$$

and since the elements of given basis of $T^\perp$ serve also as the components of a parametrization map $f = \pi_T \circ \nu_d \colon \mathbb{P}^1 \to \mathbb{P}^s$, one easily sees that the parametrized curve is smooth. Hence the general curve in the same component of $\mathcal{H}_{\bar{c}}$ is also smooth.

Conclusion. We have found that for $\bar{c} = (2, 2, 1, 1, 0, 0, 0)$ the Hilbert scheme $\mathcal{H}_{\bar{c}}$ is the union of two irreducible components, each of dimension equal to $21 + \dim \mathrm{PGL}(9) - \dim \mathrm{PGL}(2) = 98$, by Proposition 2.1. One component has general point representing a smooth rational curve constructed from a general vertex T of type $(1, 0)$ with ∂T of type $(2, 1)$. The other component has general point representing smooth rational curves constructed from a general vertex T of type $(0, 0, 0)$ with ∂T of type (5). We also observe that, by Proposition 2.6, the restricted tangent bundles are the following (setting $d = 11$):

$$\begin{aligned} f^*\mathcal{T}_{\mathbb{P}^s} &= \mathcal{O}_{\mathbb{P}^1}(d+3) \oplus \mathcal{O}_{\mathbb{P}^1}(d+2) \oplus \mathcal{O}^6_{\mathbb{P}^1}(d+1) \quad \text{for } T \text{ of type } (1,0)\,,\\ f^*\mathcal{T}_{\mathbb{P}^s} &= \mathcal{O}^3_{\mathbb{P}^1}(d+2) \oplus \mathcal{O}^5_{\mathbb{P}^1}(d+1) \quad \text{for } T \text{ of type } (0,0,0)\,. \end{aligned}$$

On the other hand, for any $[C] \in \mathcal{H}_{\bar{c}}$ one has

$$\mathcal{N}_f = \mathcal{O}^2_{\mathbb{P}^1}(d+4) \oplus \mathcal{O}^2_{\mathbb{P}^1}(d+3) \oplus \mathcal{O}^3_{\mathbb{P}^1}(d+2)\,.$$

Remark 6.1. One may note that the decomposition type given above has the form $\mathcal{N}_f = \mathcal{F} \oplus \mathcal{O}^3_{\mathbb{P}^1}(d+2)$ with $\mathcal{F} = \mathcal{O}^2_{\mathbb{P}^1}(d+4) \oplus \mathcal{O}^2_{\mathbb{P}^1}(d+3)$ of *almost balanced type*, and hence $\mathcal{N}_f$ has the most general possible type among the vector bundles on $\mathbb{P}^1$ of the same rank and degree and with summand $\mathcal{O}^3_{\mathbb{P}^1}(d+2)$. Therefore the same counterexample discussed in this section also gives the following.

Example 6.2. The variety parametrizing the rational curves of degree $d = 11$ in $\mathbb{P}^8$ with normal bundle $\mathcal{N}_f$ with three summands of degree $d + 2 = 13$ is reducible.

This is actually a counterexample to [Ber14, Theorem 4.8]. It seems that in the preparatory results leading to Theorem 4.8, especially Lemma 4.3, the author has overlooked his own more detailed treatment of the same results given in his Ph.D. thesis [Ber11], where more restrictive hypotheses are given. In [Ber11], Theorem 4.8 of [Ber14] is stated as Theorem 3.4.16, which in turn is deduced from Theorems 3.3.9 and 3.4.10. Our counterexample corresponds to the case $n = 11$, $d = 8$, $k = 3$, $r = 2$ and $\rho^{n,k}_r = 3$ in the author's notation, and it is not covered by Theorems 3.3.9 and 3.4.10 of [Ber11].

7. Smooth rational curves in rational normal scrolls

In this section we will characterize smooth rational curves contained in rational normal scroll surfaces in terms of the splitting type of their restricted tangent bundles $\mathcal{T}_f$, and we will also compute the splitting type of their normal bundles $\mathcal{N}_f$. Our main result can be viewed as a generalization of [EvdV81, Propositions 5 and 6], where the authors characterized smooth rational curves contained in a smooth quadric in $\mathbb{P}^3$ by their restricted tangent bundles and computed their normal bundles. The general purpose of this section is to illustrate the idea that especially the splitting type of $\mathcal{T}_f$ may have a deep impact on the extrinsic geometry of the curve $C \subset \mathbb{P}^s$.

NOTATION. Following the notation of [Har77, II, Section 7], we denote by $\mathbf{P}(\mathcal{E})$ the projective bundle associated with a vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ of rank $t \geqslant 1$. Recall that an epimorphism of vector bundles $\mathbb{C}^{s+1} \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{E}$ defines a regular map $g\colon \mathbf{P}(\mathcal{E}) \to \mathbb{P}^s$ such that, for H the pullback of an hyperplane of $\mathbb{P}^s$, one has $\deg H^{t-1} = \deg \mathcal{E} = \deg \wedge^t \mathcal{E}$. If the map $g\colon \mathbf{P}(\mathcal{E}) \to \mathbb{P}^s$ is birational to the image, then, setting $S = \operatorname{Im}(g)$, one finds $\deg S = \deg \mathcal{E}$.

Let $C \subset \mathbb{P}^s$ be a smooth non-degenerate rational curve of degree d, biregularly parametrized by a map $f\colon \mathbb{P}^1 \to \mathbb{P}^s$ which, as discussed in preceding sections, we can assume of the form $f = \pi_T \circ \nu_d$ up to a projective transformation of $\mathbb{P}^s$. As usual we will set $\dim T = e+1$ and $s = d-e-1$. Throughout this section we will assume $s \geqslant 3$ and $d \geqslant s+1$, that is, $T \neq 0$. We first study a sufficient condition for C to be smooth.

LEMMA 7.1. *Let $T = \partial^e(g)$ be a vertex of type (e). Then the curve $C = \pi_T(C_d)$ is smooth if and only if $g \in \mathbb{P}(S^{d+e}U) \setminus \operatorname{Sec}^{e+1} C_{d+e}$.*

Proof. Our strategy of proof will be to show that when T has type (e), the curve C is smooth if and only if ∂T has type $(e+1)$. Indeed, by Proposition 2.3 one sees that $\partial T = \partial^{e+1}(g)$ being of type $(e+1)$ is equivalent to $[g] \notin \operatorname{Sec}^{e+1} C_{d+e}$. Note that the point $[g] \in \mathbb{P}(S^{d+e}U)$ such that $\partial T = \partial^{e+1}(g)$ has type $(e+1)$ is unique, since one sees that $\langle g \rangle = \partial^{-e-1}(\partial T)$ by iteratively applying Proposition 2.5.

The condition that C is smooth is given by $\mathbb{P}(T) \cap \operatorname{Sec}^1 C_d = \emptyset$. Observe that T being of type (e) in particular implies $\mathbb{P}(T) \cap C_d = \emptyset$ and $\dim \mathbb{P}(\partial T) = \dim \mathbb{P}(T) + 1$. Hence the space $\mathbb{P}(\partial T)$, which a priori is the join $\mathbb{P}(\langle \omega(T) \,|\, [\omega] \in \mathbb{P}(U^*) \rangle)$, in this case is also the union $\mathbb{P}(\partial T) = \bigcup_{\omega \in U^*} \mathbb{P}(\omega(T))$. Then one has $\mathbb{P}(\partial T) \cap C_{d-1} \neq \emptyset$ if and only if there exist $\omega \in U^*$ and $l \in U$ such that $[l^{d-1}] \in \mathbb{P}(\omega(T))$. Setting $\langle m \rangle = \omega^\perp$, this is equivalent to saying that in $\mathbb{P}(T)$ there exists an element of the form $[\lambda l^d + \mu m^d]$ if $[m] \neq [l]$ and an element of the form $[l^{d-1}n]$ if $[m] = [l]$. This is equivalent to the condition $\mathbb{P}(T) \cap \operatorname{Sec}^1 C_d \neq \emptyset$, that is, to C not being smooth.

Therefore, we have shown that C is smooth if and only if $\mathbb{P}(\partial T) \cap C_{d-1} = \emptyset$, that is, $S_{\partial T} = 0$, with the notation of Proposition 2.3. Moreover, for $T = \partial^e(g)$, one has $\partial T = \partial^{e+1}(g)$ and $\partial^2 T = \partial^{e+2}(g)$, hence $\dim \partial^2 T - \dim \partial T \leqslant 1$. Then, by Proposition 2.3 applied to the space ∂T, we see that C is smooth if and only if ∂T has type $(e+1)$. □

Remark 7.2. Note that the open set $\mathbb{P}(S^{d+e}U) \setminus \operatorname{Sec}^{e+1} C_{d+e}$ is non-empty and of dimension $d+e = 2d-s-1$ if and only if $\dim \operatorname{Sec}^{e+1} C_{d+e} = 2e+3 \leqslant d+e-1$, which is true, as we are assuming $s = d-e-1 \geqslant 3$.

Now, we can state and prove the main result of this section.

THEOREM 7.3. *Let us assume that C is a non-degenerate irreducible smooth rational curve of degree $d \geqslant s+1$ with parametrization map $f = \pi_T \circ \nu_d\colon \mathbb{P}^1 \to C \subset \mathbb{P}^s$. Then the following conditions are equivalent:*

(i) *The vertex T is of type (e), that is, $T = \partial^e(g)$ with $[g] \in \mathbb{P}(S^{d+e}U) \setminus \operatorname{Sec}^{e+1} C_{d+e}$.*
(ii) *$\mathcal{T}_f = \mathcal{O}_{\mathbb{P}^1}(d+2+e) \oplus \mathcal{O}_{\mathbb{P}^1}^{s-1}(d+1)$.*

(iii) *The curve C is contained in a smooth rational normal scroll $S \cong \mathbf{P}(\mathcal{E}) \subset \mathbb{P}^s$, with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta)$, where $\alpha, \beta > 0$ and $\alpha + \beta = s - 1$.*

Moreover, under any of the conditions above, the following also hold:

(1) *The rational normal scroll containing C is uniquely determined by C.*

(2) *The normal bundle $\mathcal{N}_f$ has splitting type $\mathcal{N}_f \cong \mathcal{O}^2_{\mathbb{P}^1}(d+e+3) \oplus \mathcal{O}^{s-3}_{\mathbb{P}^1}(d+2)$.*

Proof. (i) $\iff$ (ii). By Proposition 2.6 one sees that T has type (e), that is, $T = \partial^e(g)$ with $[g] \notin \operatorname{Sec}^e C_{d+e}$ if and only if $\mathcal{T}_f = \mathcal{O}_{\mathbb{P}^1}(d+2+e) \oplus \mathcal{O}^{s-1}_{\mathbb{P}^1}(d+1)$. Since we are assuming C smooth, by Lemma 7.1 one actually has $[g] \notin \operatorname{Sec}^{e+1} C_{d+e}$.

(ii)$\Rightarrow$(iii). We set $V = T^\perp$ and recall the restricted Euler sequence appearing in the second column of the diagram of Section 4.1:

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to V^* \otimes \mathcal{O}_{\mathbb{P}^1}(d) \to \mathcal{T}_f \to 0\,.$$

From this sequence and the existence of the sub-line bundle $\mathcal{O}_{\mathbb{P}^1}(d+2+e) \to \mathcal{T}_f$, we deduce a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(-d) & \longrightarrow & \mathcal{E}^* & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(e+2) & \longrightarrow & 0 \\
 & & \downarrow{\cong} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(-d) & \longrightarrow & V^* \otimes \mathcal{O}_{\mathbb{P}^1} & \longrightarrow & \mathcal{T}_f(-d) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}^{s-1}_{\mathbb{P}^1}(1) & \xrightarrow{\cong} & \mathcal{O}^{s-1}_{\mathbb{P}^1}(1)\,, & &
\end{array}$$

where $\mathcal{E}^*$ is defined as the preimage of $\mathcal{O}_{\mathbb{P}^1}(e+2)$ in $V^* \otimes \mathcal{O}_{\mathbb{P}^1}$. Dually, we get a exact sequence $0 \to \mathcal{O}^{s-1}_{\mathbb{P}^1}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{E} \to 0$. It immediately follows that $\mathcal{E}$ has splitting type $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta)$ with $\alpha, \beta \geqslant 0$ and $\alpha + \beta = s - 1$. Moreover, the sheaf map $V \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(d)$ that is naturally associated with f is the composition of the sheaf epimorphisms $V \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(d)$. Let us set $Y = \mathbf{P}(\mathcal{E})$. Then the sheaf epimorphism $V \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{E}$ provides a map $Y \to \mathbb{P}^s$ whose image S is a ruled surface of minimal degree $s-1$, and the existence of the factorization $V \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(d)$ shows that the curve C is contained in S as the image of a section $\widetilde{C}$ of the $\mathbb{P}^1$-bundle $Y \to \mathbb{P}^1$. We only have to show that $\alpha, \beta > 0$. Indeed, if for example $\alpha = 0$ and $\beta = s-1$, then S is a cone over a rational normal curve in $\mathbb{P}^{s-1}$; more precisely, the map $Y \to S$ contracts the unique curve C_0 of Y with $C_0^2 = 1 - s$ to the vertex of the cone S. In this case the section $\widetilde{C} \subset Y$ has divisor class $\widetilde{C} \equiv C_0 + dF$, with F a fiber of $Y \to \mathbb{P}^1$, and intersection number $\widetilde{C} \cdot C_0 = d + 1 - s \geqslant 2$ for $d \geqslant s+1$. Hence C cannot be smooth for $d \geqslant s+1$. This argument excludes the case of the cone; therefore $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta)$, with $\alpha + \beta = s-1$ and $\alpha, \beta > 0$. In this case one also sees that the map $Y \to \mathbb{P}^1$ is an embedding, that is, $Y \cong S$, so S is a smooth rational normal scroll.

(iii)$\Rightarrow$(ii). Assume $C \subset S \subset \mathbb{P}^s$, with S a smooth rational normal scroll. In particular, S is isomorphic to a rational ruled surface $\mathbf{P}(\mathcal{E})$, embedded in $\mathbb{P}^s$ by means of a surjection of vector bundles $V \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{E}$. The fact that $\deg S = s-1$ is equivalent to $\deg \mathcal{E} = s-1$. The fact that $C \subset S \cong \mathbf{P}(\mathcal{E})$ is a section of the projection map $\mathbf{P}(\mathcal{E}) \to \mathbb{P}^1$ implies the existence of a sheaf epimorphism $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(d)$ such that the epimorphism $V \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(d)$ associated with the embedding $C \subset \mathbb{P}^s$ factors as $V \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(d)$. Setting $\mathcal{L} = \ker(\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(d))$, we see that $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(s-1-d) = \mathcal{O}_{\mathbb{P}^1}(-e-2)$. Now, we can dualize all the sheaf morphisms that we have introduced so far, obtaining a diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(-d) & \longrightarrow & \mathcal{E}^* & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(e+2) & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(-d) & \longrightarrow & V^* \otimes \mathcal{O}_{\mathbb{P}^1} & \longrightarrow & \mathcal{T}_f(-d) & \longrightarrow & 0\,. \end{array} \tag{7.1}$$

That is, we have obtained a sheaf embedding $\mathcal{O}_{\mathbb{P}^1}(d+e+2) \to \mathcal{T}_f$. Since $\deg \mathcal{T}_f = (s+1)d = (s-1)(d+1)+d+e+2$ and the degree of any summand $\mathcal{O}_{\mathbb{P}^1}(\delta)$ in a splitting of $\mathcal{T}_f$ is at least $d+1$, we can conclude that $\mathcal{T}_f$ has the form stated in condition (ii).

Proof of statement (1). After fixing homogeneous coordinates on $\mathbb{P}^s$, the last row of the diagram (7.1) is uniquely determined by the parametrization map $f\colon \mathbb{P}^1 \to \mathbb{P}^s$, since this map defines uniquely the sheaf embedding $\mathcal{O}_{\mathbb{P}^1}(-d) \to V^* \otimes \mathcal{O}_{\mathbb{P}^1}$. Hence it is determined by C up to the action of $\mathrm{PGL}(2) = \mathrm{Aut}(\mathbb{P}^1)$. Moreover, there exists only one sheaf embedding $\mathcal{O}_{\mathbb{P}^1}(e+2) \to \mathcal{T}_f(-d)$ for the given splitting type $\mathcal{T}_f = \mathcal{O}_{\mathbb{P}^1}(d+e+2) \oplus \mathcal{O}_{\mathbb{P}^1}^{s-1}(d+1)$. Hence the sheaf embedding $\mathcal{E}^* \to V^* \otimes \mathcal{O}_{\mathbb{P}^1}$ in the diagram (7.1) is also uniquely determined by C up to the action of $\mathrm{PGL}(2)$ on $\mathbb{P}^1$. This means that the parametrization map $\mathbb{P}(\mathcal{E}) \to S \subset \mathbb{P}^s$ is uniquely determined by C, up to the (equivariant) action of $\mathrm{PGL}(2)$ on $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$. Hence S is uniquely determined by C.

Proof of statement (2). The stated formula for the splitting type of $\mathcal{N}_f$ is an immediate consequence of Proposition 5.5. □

Remark 7.4. There is a classical connection between the property of a non-degenerate irreducible curve C of sufficiently high degree of being contained in a rational normal scroll and the number of independent quadric hypersurfaces containing C. Indeed, one has the following result, essentially due to Castelnuovo.

Proposition 7.5. *A non-degenerate and irreducible curve $C \subset \mathbb{P}^s$ of degree $d \geqslant 2s+1$ has $h^0\mathcal{I}_C(2) \leqslant (s-1)(s-2)/2$. If in addition C is smooth and rational, the equality holds if and only if C is contained in a smooth rational normal scroll of dimension* 2*.*

Sketch of proof. Let $\Gamma = C \cap H$ be a general hyperplane section of C, which is in general linear position. Then, from the exact sequence

$$0 \to \mathcal{I}_C(1) \to \mathcal{I}_C(2) \to \mathcal{I}_{\Gamma,H}(2) \to 0\,,$$

one finds $h^0\mathcal{I}_C(2) \leqslant h^0\mathcal{I}_{\Gamma,H}(2)$. By a classical argument of Castelnuovo, any $2s-1$ points of Γ impose independent conditions on the quadrics of $H \cong \mathbb{P}^{s-1}$, hence $h^0\mathcal{I}_{\Gamma,H}(2) \leqslant h^0\mathcal{O}_H(2) - 2s + 1 = s(s+1)/2 - 2s + 1 = (s-1)(s-2)/2$, proving the stated inequality.

If the equality holds, then Γ imposes exactly $2s-1$ conditions on the quadrics of $H \cong \mathbb{P}^{s-1}$, and since $\deg H \geqslant 2s+1 = 2(s-1)+3$, one can apply Castelnuovo's lemma as in [GH78, Chapter 4, p. 531], and conclude that Γ is contained in a unique rational normal curve of $\mathbb{P}^{s-1}$. Hence, by the arguments in the proof of the lemma in [GH78, Chapter 4, pp. 531–532], either the curve C is contained in a rational normal scroll or $s = 5$ and C is contained in a Veronese surface in $\mathbb{P}^5$. When C is a smooth rational curve, we can exclude that S is the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$, because any non-degenerate smooth curve $C \subset S$ would come from a smooth curve of degree at least 3 of $\mathbb{P}^2$, hence cannot be rational. Therefore we are left with the case of S a rational normal scroll. As in the proof of the implication (ii)⇒(iii) of Theorem 7.3, it is easy to see that S is smooth. The converse follows from the fact that a rational normal scroll $S \subset \mathbb{P}^s$ is contained in $(s-1)(s-2)/2$ independent quadrics. □

We conclude this section with a discussion of the relevance of the smoothness assumption in Theorem 7.3. Indeed, one can see that the implication (iii)⇒(ii) of Theorem 7.3 is false if one does not assume C to be smooth. To this purpose, one can find counterexamples already in $\mathbb{P}^3$. This fact was not explicitly observed in [EvdV81], where the case $s = 3$ of Theorem 7.3 was proved. Here it is such an example.

Example 7.6. Let us consider $g\colon \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$g(u,v) = \left(u^2 : v^2; u^3 : v^3\right)$$

and compose it with the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ so as to obtain $f\colon \mathbb{P}^1 \to \mathbb{P}^3$ defined by

$$f(u,v) = \left(u^5 : u^2v^3 : v^2u^3 : v^5\right).$$

This is a parametrization of a rational curve C (with two cusps) of degree 5 contained in the quadric $Q \subset \mathbb{P}^3$ of equation $x_0x_3 - x_1x_2 = 0$, which is a very simple rational normal scroll. Therefore C satisfies condition (iii) of Theorem 7.3. Note that C is a curve of divisor class $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, so C is not a section of any of the two $\mathbb{P}^1$-bundle structures $Q \to \mathbb{P}^1$. We have, by construction,

$$T^\perp = \left\langle u^5, u^2v^3, v^2u^3, v^5\right\rangle.$$

One immediately sees that $T = \langle x^4y, xy^4\rangle$ and therefore $\partial T = \langle x^4, x^3y, xy^3, y^4\rangle$, so that $\dim \partial T = \dim T + 2$. Hence, from Proposition 2.3 and Definition 2.4 one sees that T has numerical type $(0,0)$, and by Proposition 2.6 one finds

$$\mathcal{T}_f = \mathcal{O}^2_{\mathbb{P}^1}(7) \oplus \mathcal{O}_{\mathbb{P}^1}(6). \tag{7.2}$$

This contradicts condition (ii) of Theorem 7.3. Observe that the curve C has no ordinary singularities, but it can be deformed to a rational curve $C' \subset Q$ of divisor class $(2,3)$ with two nodes. Since the vertex T relative to C has numerical type $(0,0)$ and this is the *general* numerical type for subspaces $T \subset S^5U$ of dimension 2, the vertex T' relative to C' will have type $(0,0)$ as well. Hence the restricted tangent sheaf to C' has splitting type as in formula (7.2), providing a counterexample to condition (ii) of Theorem 7.3 by means of a curve with ordinary singularities.

Acknowledgements

We thank G. Ottaviani and F. Russo for many stimulating and helpful discussions during the development of this work.

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