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The Decision Problem for a Three-sorted Fragment of Set Theory with Restricted Quantification and Finite Enumerations ⋆

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Abstract

We solve the satisfiability problem for a three-sorted fragment of set theory (denoted $3LQST_0^R$), which admits a restricted form of quantification over individual and set variables and the finite enumeration operator $\{\text{-},\text{-},\ldots,\text{-}\}$ over individual variables, by showing that it enjoys a small model property, i.e., any satisfiable formula ψ of $3LQST_0^R$ has a finite model whose size depends solely on the length of ψ itself. Several set-theoretic constructs are expressible by $3LQST_0^R$-formulae, such as some variants of the power set operator and the unordered Cartesian product. In particular, concerning the latter construct, we show that when finite enumerations are allowed, the resulting formula is exponentially shorter than in their absence.

Keywords: Satisfiability problem, set theory, restricted quantification, finite enumerations.

1 Introduction

Computable set theory studies the decidability problem for specific collections of set-theoretic formulae (also called *syllogistics*). The main results in computable set theory up to 2001 have been collected in [7,13]. We also mention that the most efficient decision procedures for fragments of set theory form the inferential core of the proof verifier ÆtnaNova [17].

In this paper we present a decidability result for the satisfiability problem of the set-theoretic language $3LQST_0^R$ (Three-Level Quantified Syllogistic with finite enumeraTions and Restricted quantifiers), which is a three-sorted quantified syllogistic involving *individual variables*, *set variables*, and *collection variables*, ranging

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over the elements of a given nonempty universe D, the subsets of D, and the collections of subsets of D, respectively. The language of $3LQST_0^R$ admits the predicate symbols $=$ and $\in$ and a restricted form of quantification over individual and set variables. The language $3LQST_0^R$ extends $3LQS^R$ presented in [9] as it admits the finite enumeration operator $\{-,-,\ldots,-\}$ over individual variables. In spite of its simplicity, $3LQST_0^R$ allows one to express several constructs of set theory. Among them, the most comprehensive one is the set former, which in turn allows one to express other set-theoretic operators like some variants of the power set and the unordered Cartesian product. Concerning the latter, we will see that it can be expressed by $3LQST_0^R$-formulae of linear length. On the other hand, if the finite enumeration operator is dropped, exponentially long $3LQS^R$-formulae are required to express it.

Much as for $3LQS^R$, we will show that the fragment $3LQST_0^R$ enjoys a small model property. The proof is carried out by showing how to extract, out of a given model satisfying a $3LQST_0^R$-formula ψ, another model of ψ but of bounded finite cardinality.

The paper is organized as follows. In Section 2 we introduce some related work in computable set theory concerning multi-sorted stratified syllogistics. Then, in Section 3, we first present the syntax and semantics of a more general language, denoted $3LQST_0$, and then provide a decidable semantic restriction to characterize the fragment $3LQST_0^R$ of our interest. Subsequently, in Section 4, we show that several set-theoretic constructs are readily expressible by $3LQST_0^R$-formulae. In Section 5, the machinery needed to prove our main decidability result is provided and, in Section 6, the small model property for $3LQST_0^R$ is sketched, thus solving the satisfiability problem for $3LQST_0^R$. Then, in Section 7 we present two distinct representations of the unordered Cartesian product. The first one, using the finite enumeration operator, is linear in the length of the product, the second one, not involving the finite enumeration operator, is exponentially longer. Finally, in Section 8, we draw our conclusions.

2 Related work

Most of the decidability results established in computable set theory concern one-sorted multi-level syllogistics, namely collections of formulae involving variables of one type only, ranging over the von Neumann universe of sets. On the other hand, few decidability results have been proved for multi-sorted stratified syllogistics, admitting variables of several types. This, despite of the fact that in many fields of computer science and mathematics often one deals with multi-sorted languages.

An efficient decision procedure for the satisfiability of the Two-Level Syllogistic language (2LS), a fragment admitting variables of two sorts (for individuals and for sets of individuals), the basic set-theoretic operators such as $\cup$, $\cap$, $\setminus$, the relators $=$, $\in$, $\subseteq$, and propositional connectives, has been presented in [15]. The three-sorted language 3LSSPU (Three-Level Syllogistic with Singleton, Powerset, and general Union), allowing three types of variables, and the singleton, powerset, and general union operators, in addition to the operators and predicates already in 2LS, has been proved decidable in [4].

More recently, in [9], the three-level quantified syllogistic $3\mathrm{LQS}^{\mathrm{R}}$, involving variables of three sorts has been shown to have a decidable satisfiability problem. The decision algorithm for $3\mathrm{LQS}^{\mathrm{R}}$ was inspired by the procedure presented in [4] to prove the decidability of 3LSSPU. In particular, the notion of relativized interpretation introduced in [9], can be seen as a variant of the notion of assignment of the small model defined in [4]. The language $3\mathrm{LQS}^{\mathrm{R}}$, as well as its extension $3\mathrm{LQST}_0^{\mathrm{R}}$ introduced in this paper, does not allow one to express the construct of general union. The latter construct, on the other hand, is a primitive operator of 3LSSPU.

Later, in [10], the satisfiability problem for $4\mathrm{LQS}^{\mathrm{R}}$, a four-level quantified syllogistic admitting variables of four sorts, has been proved to be decidable. The latter result has been exploited in [8] to prove that the quite expressive description logic $\mathcal{DL}\langle 4\mathrm{LQS}^{\mathrm{R}}\rangle(\mathsf{D})$ has a decidable consistency problem for its knowledge bases.

3 The language $3\mathrm{LQST}_0$ and its fragment $3\mathrm{LQST}_0^{\mathrm{R}}$

We begin by defining the syntax and semantics of a more general three-level quantified language, denoted $3\mathrm{LQST}_0$. Then, in Section 3.1, we characterize $3\mathrm{LQST}_0^{\mathrm{R}}$-formulae by suitably restricting the usage of quantifiers in $3\mathrm{LQST}_0$-formulae.

The three-level quantified language $3\mathrm{LQST}_0$ involves

(i) a collection $\mathcal{V}_0$ of *individual* (or *sort* 0) *variables*, denoted by $x, y, z, \ldots$;

(ii) a collection $\mathcal{V}_1$ of *set* (or *sort* 1) *variables*, denoted by $X, Y, Z, \ldots$;

(iii) a collection $\mathcal{V}_2$ of *collection* (or *sort* 2) *variables*, denoted by $A, B, C, \ldots$.

In addition to variables, $3\mathrm{LQST}_0$ admits also terms of the form $\{x_1, \ldots, x_k\}$ (*finite enumerations*), where $x_1, \ldots, x_k$ are pairwise distinct individual variables with $k \geqslant 1$.

$3\mathrm{LQST}_0$-*quantifier-free atomic formulae* are classified as:

- *level* 0: $x = y$, $x \in X$, $X = \{x_1, \ldots, x_k\}$, $\{x_1, \ldots, x_k\} \in A$, where $x, y, x_1, \ldots, x_k \in \mathcal{V}_0$, $k \geqslant 1$, $X \in \mathcal{V}_1$, and $A \in \mathcal{V}_2$;
- *level* 1: $X = Y$, $X \in A$, where $X, Y \in \mathcal{V}_1$ and $A \in \mathcal{V}_2$.

$3\mathrm{LQST}_0$ *purely universal formulae* are classified as:

- *level* 0: $(\forall z_1) \ldots (\forall z_n)\varphi_0$, with φ_0 a propositional combination of level 0 quantifier-free atoms and $z_1, \ldots, z_n \in \mathcal{V}_0$, where $n \geqslant 1$; [3]
- *level* 1: $(\forall Z_1) \ldots (\forall Z_m)\varphi_1$, with φ_1 a propositional combination of quantifier-free atomic formulae of any level and of purely universal formulae of level 0, and $Z_1, \ldots, Z_m \in \mathcal{V}_1$, where $m \geqslant 1$.

Finally, the *formulae of* $3\mathrm{LQST}_0$ are all the propositional combinations of quantifier-free atomic formulae and of purely universal formulae of levels 0 and 1.

[3] The logical connectives admitted in propositional combinations are the usual ones: negation $\neg$, conjunction $\wedge$, disjunction $\vee$, implication $\rightarrow$, and biimplication $\leftrightarrow$.

To ease readability, we will write $(\exists z_1)\ldots(\exists z_n)\varphi_0$ and $(\exists Z_1)\ldots(\exists Z_m)\varphi_1$ as shorthands for $\neg(\forall z_1)\ldots(\forall z_n)\neg\varphi_0$ and $\neg(\forall Z_1)\ldots(\forall Z_m)\neg\varphi_1$, respectively.

A $3\mathrm{LQST}_0$-*interpretation* is a pair $\boldsymbol{\mathcal{M}} = (D, M)$, where D is any nonempty collection of objects, called the *domain* or *universe* of $\boldsymbol{\mathcal{M}}$, and M is an assignment over the variables of $3\mathrm{LQST}_0$ such that

- $Mx \in D$, for each individual variable $x \in \mathcal{V}_0$;
- $MX \subseteq D$, for each set variable $X \in \mathcal{V}_1$;
- $MA \subseteq \mathrm{pow}(D)$, for all collection variables $A \in \mathcal{V}_2$;[4]
- $M\{x_1, \ldots, x_k\} =_{\mathrm{Def}} \{Mx_1, \ldots, Mx_k\}$.

Next, let $\boldsymbol{\mathcal{M}} = (D, M)$ be a $3\mathrm{LQST}_0$-interpretation, and let $x_1, \ldots, x_n \in \mathcal{V}_0$, $X_1, \ldots, X_m \in \mathcal{V}_1$, $u_1, \ldots, u_n \in D$, and $U_1, \ldots, U_m \in \mathrm{pow}(D)$.

By $\boldsymbol{\mathcal{M}}[z_1/u_1, \ldots, z_n/u_n, Z_1/U_1, \ldots, Z_m/U_m]$ we denote the $3\mathrm{LQST}_0$-interpretation $\boldsymbol{\mathcal{M}}' = (D, M')$ such that $M'z_i = u_i$ (for $i = 1, ..., n$), $M'Z_j = U_j$ (for $j = 1, ..., m$), and which otherwise coincides with M on the remaining variables. In addition, for any $\mathcal{V}_i' \subseteq \mathcal{V}_i$ (with $i = 0, 1, 2$), we set $M\mathcal{V}_i' =_{\mathrm{Def}} \{M\xi : \xi \in \mathcal{V}_i'\}$.

Throughout the paper we will use the abbreviations:

$$\boldsymbol{\mathcal{M}}^{\boldsymbol{z}} =_{\mathrm{Def}} \boldsymbol{\mathcal{M}}[z_1/u_1, \ldots, z_n/u_n], \qquad \boldsymbol{\mathcal{M}}^{\boldsymbol{Z}} =_{\mathrm{Def}} \boldsymbol{\mathcal{M}}[Z_1/U_1, \ldots, Z_m/U_m],$$

where the variables z_i and Z_j, the individuals u_i, and the subsets U_j are understood from the context.

Let ψ be a $3\mathrm{LQST}_0$-formula and let $\boldsymbol{\mathcal{M}} = (D, M)$ be a $3\mathrm{LQST}_0$-interpretation. The notion of *satisfiability* for ψ with respect to $\boldsymbol{\mathcal{M}}$ (denoted by $\boldsymbol{\mathcal{M}} \models \psi$) is defined recursively over the structure of ψ. The evaluation of quantifier-free atomic formulae is carried out according to the standard meaning of the predicates '$\in$' and '$=$' and of the finite enumeration operator. Purely universal formulae are interpreted as follows:

- $\boldsymbol{\mathcal{M}} \models (\forall z_1)\ldots(\forall z_n)\varphi_0$ iff $\boldsymbol{\mathcal{M}}[z_1/u_1, \ldots, z_n/u_n] \models \varphi_0$, for all $u_1, \ldots, u_n \in D$;
- $\boldsymbol{\mathcal{M}} \models (\forall Z_1)\ldots(\forall Z_m)\varphi_1$ iff $\boldsymbol{\mathcal{M}}[Z_1/U_1, \ldots, Z_m/U_m] \models \varphi_1$, for all $U_1, \ldots, U_n \subseteq D$.

Finally, compound formulae are evaluated according to the standard rules of propositional logic.

Let ψ be a $3\mathrm{LQST}_0$-formula. If $\boldsymbol{\mathcal{M}} \models \psi$ (i.e., $\boldsymbol{\mathcal{M}}$ *satisfies* ψ), then $\boldsymbol{\mathcal{M}}$ is said to be a $3\mathrm{LQST}_0$-*model for* ψ. A $3\mathrm{LQST}_0$-formula is said to be *satisfiable* if it has a $3\mathrm{LQST}_0$-model. A $3\mathrm{LQST}_0$-formula is *valid* if it is satisfied by all $3\mathrm{LQST}_0$-interpretations.

[4] We recall that $\mathrm{pow}(s)$ denotes the power set of s.

3.1 Characterizing the restricted fragment $3\mathrm{LQST}_0^{\mathrm{R}}$

$3\mathrm{LQST}_0^{\mathrm{R}}$ is the collection of all $3\mathrm{LQST}_0$-formulae ψ such that, for *every* purely universal formula $(\forall Z_1)\dots(\forall Z_m)\varphi_1$ of level 1 occurring in ψ and *every* purely universal formula $(\forall z_1)\dots(\forall z_n)\varphi_0$ of level 0 occurring in φ_1, the condition

$$\neg\varphi_0 \rightarrow \bigwedge_{i=1}^{n}\bigwedge_{j=1}^{m} z_i \in Z_j \tag{1}$$

is a valid $3\mathrm{LQST}_0$-formula (in which case we say that the purely universal formula $(\forall z_1)\dots(\forall z_n)\varphi_0$ is *linked to the variables* $Z_1,\dots,Z_m$).

Condition (1) guarantees that, if a given interpretation assigns to $z_1,\dots,z_n$ elements of the domain that make φ_0 false, then all such values must be contained as elements in the intersection of the sets assigned to $Z_1,\dots,Z_m$. This fact has been introduced for technical reasons and it is used in the proof of Lemma 5.7 (which can be found in the extended version of the present paper in [11]) to make sure that satisfiability is preserved in the finite model. Attempts of relaxing such a condition (still maintaining decidability) failed so far.

The following question arises: how can one establish whether a given $3\mathrm{LQST}_0$-formula is a $3\mathrm{LQST}_0^{\mathrm{R}}$-formula? Observe that neither quantification nor collection variables are involved in condition (1). Indeed, it turns out that (1) is a 2LS-formula and therefore its validity can be tested by the decision procedure in [15], as $3\mathrm{LQST}_0$ is a conservative extension of 2LS. As we will see in the next section, in most cases of interest condition (1) is just an instance of the elementary propositional tautology $\neg(\mathbf{p}\rightarrow\mathbf{q})\rightarrow\mathbf{p}$. In such cases, the validity of (1) follows just by inspection.

4 Expressiveness of the language $3\mathrm{LQST}_0^{\mathrm{R}}$

Several constructs of elementary set theory are easily expressible within the language $3\mathrm{LQST}_0^{\mathrm{R}}$. In particular, it is possible to express with $3\mathrm{LQST}_0^{\mathrm{R}}$-formulae a restricted variant of the set former. This, in turn, allows one to express other significant set operators such as binary union, intersection, set difference, set complementation, the powerset operator and some of its variants, and so on. More specifically, a set former of the form $X = \{z : \varphi(z)\}$ can be expressed in $3\mathrm{LQST}_0^{\mathrm{R}}$ by the formula

$$(\forall z)(z \in X \leftrightarrow \varphi(z)) \tag{2}$$

(in which case it is called an *admissible set former of level* 0 *for* $3\mathrm{LQST}_0^{\mathrm{R}}$), provided that after transforming it into prenex normal form, the resulting formula satisfies the syntactic constraints of $3\mathrm{LQST}_0^{\mathrm{R}}$. This, in particular, is always the case whenever $\varphi(z)$ is a quantifier-free formula of $3\mathrm{LQST}_0^{\mathrm{R}}$.

In Table 1 some examples of formulae expressible by admissible set formers of level 0 for $3\mathrm{LQST}_0^{\mathrm{R}}$ are reported, where $\mathbf{0}$ and $\mathbf{1}$ stand respectively for the empty set and for the domain of the discourse, and $\overline{\cdot}$ is the complementation operator with respect to the domain of the discourse. The formulae in the first column of Table 1

	admissible set formers for $3\mathrm{LQST}_0^{\mathrm{R}}$ *of level* 0
$X = \mathbf{0}$	$X = \{z : z \neq z\}$
$X = \mathbf{1}$	$X = \{z : z = z\}$
$X = \overline{Y}$	$X = \{z : z \notin Y\}$
$X = Y_1 \cup Y_2$	$X = \{z : z \in Y_1 \vee z \in Y_2\}$
$X = Y_1 \cap Y_2$	$X = \{z : z \in Y_1 \wedge z \in Y_2\}$
$X = Y_1 \setminus Y_2$	$X = \{z : z \in Y_1 \wedge z \notin Y_2\}$

Table 1
Some literals expressible by admissible set formers of level 0 for $3\mathrm{LQST}_0^{\mathrm{R}}$.

are the atoms allowed in the fragment 2LS (Two-Level Syllogistic) which has been proved decidable in [15]. Since $X = \{x_1, \ldots, x_k\}$ is a level 0 quantifier-free atomic formula in $3\mathrm{LQST}_0^{\mathrm{R}}$, 2LS with finite enumerations turns out to be expressible by $3\mathrm{LQST}_0^{\mathrm{R}}$-formulae.

In addition to the formulae in Table 1, the following literals

$$Z \subseteq X\,, \quad |Z| \leqslant h\,, \quad |Z| < h+1\,, \quad |Z| \geqslant h+1\,, \quad |Z| = h \tag{3}$$

are also expressible by $3\mathrm{LQST}_0^{\mathrm{R}}$-formulae of level 0, where h stands for a nonnegative integer constant (cf. Table 2). In fact, it turns out that all literals (3) can be expressed by level 0 purely universal $3\mathrm{LQST}_0^{\mathrm{R}}$-formulae which are linked to the variable Z, so that they can freely be used in the matrix $\varphi(Z)$ of a level 1 universal formula of the form $(\forall Z)\varphi(Z)$. Let us consider, for instance, the formula

$$(\forall z_1)\ldots(\forall z_{h+1})\Bigg(\bigwedge_{1\leqslant i\leqslant h+1} z_i \in Z \rightarrow \bigvee_{1\leqslant i<j\leqslant h+1} z_i = z_j\Bigg) \tag{4}$$

which expresses the literal $|Z| \leqslant h$. The linkedness condition for it, relative to the variable Z, is

$$\neg\Bigg(\bigwedge_{1\leqslant i\leqslant h+1} z_i \in Z \rightarrow \bigvee_{1\leqslant i<j\leqslant h+1} z_i = z_j\Bigg) \rightarrow \bigwedge_{1\leqslant i\leqslant h+1} z_i \in Z\,,$$

which is plainly a valid $3\mathrm{LQST}_0^{\mathrm{R}}$-formula since it is an instance of the propositional tautology $\neg(\mathbf{p} \rightarrow \mathbf{q}) \rightarrow \mathbf{p}$, showing that (4) is linked to the variable Z. Similarly, one can show that the remaining formulae in (3) can also be expressed by level 0 purely universal $3\mathrm{LQST}_0^{\mathrm{R}}$-formulae which are linked to the variable Z.

Similar remarks apply also to the set former of the form $A = \{Z : \varphi(Z)\}$. This can be expressed by the $3\mathrm{LQST}_0^{\mathrm{R}}$-formula

$$(\forall Z)(Z \in A \leftrightarrow \varphi(Z)) \tag{5}$$

(in which case it is called an *admissible set former of level* 1 *for* $3\mathrm{LQST}_0^{\mathrm{R}}$) provided that $\varphi(Z)$ does not contain any quantifier over variables of sort 1, and all quantified variables of sort 0 in $\varphi(Z)$ are linked to the variable Z according to condition (1).

	$3\mathrm{LQST}_0^{\mathrm{R}}$-formulae
$Z \subseteq X$	$(\forall z)(z \in Z \rightarrow z \in X)$
$\lvert Z\rvert \leqslant h$	$(\forall z_1)\ldots(\forall z_{h+1})\left(\bigwedge_{1\leqslant i\leqslant h+1} z_i \in Z \rightarrow \bigvee_{1\leqslant i<j\leqslant h+1} z_i = z_j\right)$
$\lvert Z\rvert < h+1$	$\lvert Z\rvert \leqslant h$
$\lvert Z\rvert \geqslant h+1$	$\neg(\lvert Z\rvert < h+1)$
$\lvert Z\rvert \geqslant 0$	$Z = Z$
$\lvert Z\rvert = h$	$\lvert Z\rvert \leqslant h \wedge \lvert Z\rvert \geqslant h$

Table 2
Further formulae expressible by $3\mathrm{LQST}_0^{\mathrm{R}}$-formulae of level 0.

	admissible set formers of level 1 *for* $3\mathrm{LQST}_0^{\mathrm{R}}$
$A = \mathbf{0}$	$X = \{Z : Z \neq Z\}$
$A = \mathbf{1}$	$X = \{Z : Z = Z\}$
$A = \overline{B}$	$A = \{Z : Z \notin B\}$
$A = B_1 \cup B_2$	$A = \{Z : Z \in B_1 \vee Z \in B_2\}$
$A = B_1 \cap B_2$	$A = \{Z : Z \in B_1 \wedge Z \in B_2\}$
$A = B_1 \setminus B_2$	$A = \{Z : Z \in B_1 \wedge Z \notin B_2\}$
$A = \{X_1, \ldots, X_k\}$	$A = \{Z : Z = X_1 \vee \ldots \vee Z = X_k\}$
$A = \mathrm{pow}(X)$	$A = \{Z : Z \subseteq X\}$
$A = \mathrm{pow}_{\leqslant h}(X)$	$A = \{Z : Z \subseteq X \wedge \lvert Z\rvert \leqslant h\}$
$A = \mathrm{pow}_{=h}(X)$	$A = \{Z : Z \subseteq X \wedge \lvert Z\rvert = h\}$
$A = \mathrm{pow}_{\geqslant h}(X)$	$A = \{Z : Z \subseteq X \wedge \lvert Z\rvert \geqslant h\}$
$A = \mathrm{pow}_{<h+1}(X)$	$A = \{Z : Z \subseteq X \wedge \lvert Z\rvert \leqslant h\}$
...	...

Table 3
Some literals expressible by admissible set formers of level 1 for $3\mathrm{LQST}_0^{\mathrm{R}}$.

Some examples of formulae expressible by admissible set formers of level 1 for $3\mathrm{LQST}_0^{\mathrm{R}}$ are reported in Table 3. In this case the symbol $\mathbf{1}$ stands for the powerset of the domain of the discourse. The meaning of the overloaded symbol $\mathbf{1}$ can always be correctly disambiguated from the context. In view of the fact that, as already remarked, the literals (3) can be expressed by level 0 purely universal $3\mathrm{LQST}_0^{\mathrm{R}}$-formulae which are linked to the variable Z, it follows that all set formers in Table 3 are indeed admissible.

Propositional combinations of the following literals

$$\begin{array}{llll} A = \mathbf{0}\,, & A = \mathbf{1}\,, & A = \overline{B}\,, & A = B_1 \cup B_2\,, \\ A = B_1 \cap B_2\,, & A = B_1 \setminus B_2\,, & A = \{X_1, \ldots, X_k\}\,, & A = \mathrm{pow}(X) \end{array} \tag{6}$$

presented in the first column of Table 3 form a proper fragment of 3LSSPU (Three-Level Syllogistic with Singleton, Powerset, and Unionset) whose decision problem has been solved in [4]. We recall that in addition to the formulae in (6), 3LSSPU involves also unionset clauses of the form $X = \bigcup A$ (with X a variable of sort 1 and A a variable of sort 2) which, however, are not expressible by $3\mathrm{LQST}_0^{\mathrm{R}}$-formulae.

Besides the ordinary powerset operator, $3\mathrm{LQST}_0^{\mathrm{R}}$-formulae allow one also to express the variants $\mathrm{pow}_{\leqslant h}(X)$, $\mathrm{pow}_{=h}(X)$, and $\mathrm{pow}_{\geqslant h}(X)$ reported in Table 3, which denote, respectively, the collection of all the subsets of X with at most h distinct elements, with exactly h elements, and with at least h distinct elements. We observe that the satisfiability problem for the propositional combination of literals of the forms $x \in y$, $x = y \cup z$, $x = y \cap z$, $x = y \setminus z$, with at most one literal of the form $y = \mathrm{pow}_{=1}(x)$, has been proved decidable in [6], when set variables are interpreted in the von Neumann hierarchy of sets (cf. [16]).

A useful variant of the powerset is the pow* operator, introduced in the solution to the satisfiability problem for the extension of MLS with the powerset and singleton operators (cf. [3,12]). We recall that given sets $X_1, \ldots, X_k$, $\mathrm{pow}^*(X_1, \ldots, X_k)$ denotes the collection of all the subsets of $\bigcup_{i=1}^{k} X_i$ which have nonempty intersection with each set X_i, for $i = 1, \ldots, k$. In symbols,

$$\begin{aligned} \mathrm{pow}^*(X_1, \ldots, X_k) &=_{\mathrm{Def}} \left\{ Z : Z \subseteq \textstyle\bigcup_{i=1}^{k} X_i \wedge \bigwedge_{i=1}^{k} Z \cap X_i \neq \emptyset \right\} \\ &= \quad \left\{ Z : Z \subseteq \textstyle\bigcup_{i=1}^{k} X_i \wedge \bigwedge_{i=1}^{k} \neg (Z \subseteq \overline{X}_i) \right\} . \end{aligned}$$

From the latter expression, it follows that the literal $A = \mathrm{pow}^*(X_1, \ldots, X_k)$ can be readily expressed by a $3\mathrm{LQST}_0^{\mathrm{R}}$-formula.

5 Relativized interpretations

Small models of satisfiable $3\mathrm{LQST}_0^{\mathrm{R}}$-formulae will be expressed in terms of *relativized interpretations* with respect to a suitable (small) domain.

Definition 5.1 (Relativized interpretation) Let $\boldsymbol{\mathcal{M}} = (D, M)$ be a $3\mathrm{LQST}_0$-interpretation and let $D^* \subseteq D$, $d^* \in D^*$, and $\mathcal{V}'_1 \subseteq \mathcal{V}_1$. The *relativized interpretation* $\mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, d^*, \mathcal{V}'_1)$ of $\boldsymbol{\mathcal{M}}$ with respect to D^*, d^*, and $\mathcal{V}'_1$ is the interpretation $\boldsymbol{\mathcal{M}}^* = (D^*, M^*)$ such that

$$\begin{aligned} &M^*x = \begin{cases} Mx\,, & \text{if } Mx \in D^* \\ d^*\,, & \text{otherwise} \end{cases} \\ &M^*X = MX \cap D^* \\ &M^*A = \big(MA \cap \mathrm{pow}(D^*) \setminus M^*\mathcal{V}'_1\big) \cup \{M^*X : X \in \mathcal{V}'_1,\ MX \in MA\}\,. \end{aligned}$$

For ease of notation, sometimes we will omit the reference to the element $d^* \in D^*$ and write simply $\mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, \mathcal{V}'_1)$ in place of $\mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, d^*, \mathcal{V}'_1)$. □

Our goal is to show that any satisfiable $3\mathrm{LQST}_0^{\mathrm{R}}$-formula ψ is satisfied by a small model of the form $\mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, \mathcal{V}'_1)$, where $\boldsymbol{\mathcal{M}} = (D, M)$ is a model of ψ, D^* is a subset of D of bounded finite size, and $\mathcal{V}'_1 \subseteq \mathcal{V}_1$ is a suitable collection of set variables of bounded size.

Example 5.2 Consider the formula

$$\begin{aligned} \psi \equiv (\forall Z)(Z \in A \leftrightarrow (\exists x_1)(\exists x_2)(x_1 \in X_1 \wedge x_2 \in X_2 \wedge \{x_1, x_2\} = Z)) \\ \wedge (\forall z)(z \in X_1 \rightarrow z \notin X_2). \end{aligned}$$

ψ is satisfied by the $3\mathrm{LQST}_0$-interpretation $\boldsymbol{\mathcal{M}} = (D, M)$ such that $D = \{0, 1, \ldots\}$ is the set of natural numbers, $MX_1 = \{0, 2, 4, \ldots\}$ is the set of even natural numbers, $MX_2 = \{1, 3, 5, \ldots\}$ is the set of odd natural numbers, and $MA = \{\{0, 1\}, \{2, 1\}, \{0, 3\}, \{2, 3\}, \ldots\}$ is the unordered Cartesian product of MX_1 and MX_2.

Let $D^* = \{0, 1, 2, 3, 4, 5\}$, d^* any element of D^*, and $\mathcal{V}'_1 = \{X_1, X_2\}$. Then, according to Definition 5.1, $\boldsymbol{\mathcal{M}}^* = \mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, d^*, \mathcal{V}'_1)$ interprets the variables X_1, X_2, and A as follows:

- $M^*X_1 = \{0, 2, 4\}$,
- $M^*X_2 = \{1, 3, 5\}$, and
- $M^*A = \{\{0, 1\}, \{0, 3\}, \{0, 5\}, \{2, 1\}, \{2, 3\}, \{2, 5\}, \{4, 1\}, \{4, 3\}, \{4, 5\}\}$.

It is easy to check that $\boldsymbol{\mathcal{M}}^* \models \psi$ as well.

We start by stating a slightly stronger result for propositional combinations of quantifier-free atomic $3\mathrm{LQST}_0^{\mathrm{R}}$-formulae of levels 0 and 1.

Lemma 5.3 *Let $\boldsymbol{\mathcal{M}} = (D, M)$ and $\boldsymbol{\mathcal{M}}^* = \mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, d^*, \mathcal{V}'_1)$ be, respectively, a $3\mathrm{LQST}_0$-interpretation and the relativized interpretation of $\boldsymbol{\mathcal{M}}$ with respect to $D^* \subseteq D$, $d^* \in D^*$, and $\mathcal{V}'_1 \subseteq \mathcal{V}_1$. Furthermore, let K be a fixed positive number, ψ_0 a level 0 quantifier-free atomic formula of the form $x = y$ or $x \in X$, with $x, y \in \mathcal{V}_0$ and $X \in \mathcal{V}_1$, ψ'_0 a level 0 quantifier-free atomic formula of the form $X = \{x_1, \ldots, x_k\}$ or $\{x_1, \ldots, x_k\} \in A$, with $x_1, \ldots, x_k \in \mathcal{V}_0$, $X \in \mathcal{V}_1$, $A \in \mathcal{V}_2$, $k \leqslant K$, and let ψ_1 be a level 1 quantifier-free atomic formula of the form $X = Y$ or $X \in A$, with $X, Y \in \mathcal{V}'_1$, and $A \in \mathcal{V}_2$. Then we have:*

- *(a) if $Mx \in D^*$, for every $x \in \mathcal{V}_0$ in ψ_0, then $\boldsymbol{\mathcal{M}} \models \psi_0$ iff $\boldsymbol{\mathcal{M}}^* \models \psi_0$;*
- *(b) if (b1) $Mx \in D^*$, for every $x \in \mathcal{V}_0$ in ψ'_0, (b2) $M^*X = MX$, if $|MX| \leqslant K$, and $|M^*X| > K$ otherwise, for every $X \in \mathcal{V}'_1$, and (b3) $M^*X = MX$, for $X \in \mathcal{V}_1 \setminus \mathcal{V}'_1$ occurring in ψ'_0, then $\boldsymbol{\mathcal{M}} \models \psi'_0$ iff $\boldsymbol{\mathcal{M}}^* \models \psi'_0$;*
- *(c) if (c1) $M^*X = MX$, if $|MX| \leqslant K$, and $|M^*X| > K$ otherwise, for $X \in \mathcal{V}'_1$, and (c2) $(MX \,\Delta\, MY) \cap D^* \neq \emptyset$,[5] for all $X, Y \in \mathcal{V}'_1$ such that $MX \neq MY$, then $\boldsymbol{\mathcal{M}} \models \psi_1$ iff $\boldsymbol{\mathcal{M}}^* \models \psi_1$.* ■

The interested reader can find the proof of the preceding lemma in [11]. By propositional logic, Lemma 5.3 implies at once the following result.

Corollary 5.4 *Let $\boldsymbol{\mathcal{M}} = (D, M)$ and $\boldsymbol{\mathcal{M}}^* = \mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, d^*, \mathcal{V}'_1)$ be, respectively, a $3\mathrm{LQST}_0$-interpretation and the relativized interpretation of $\boldsymbol{\mathcal{M}}$ with respect to $D^* \subseteq D$, $d^* \in D^*$, and $\mathcal{V}'_1 \subseteq \mathcal{V}_1$. Furthermore, let $K \geqslant 1$ and let ψ be a propositional combination of quantifier-free atomic formulae of the types*

$$x = y\,,\quad x \in X\,,\quad X = \{x_1, \ldots, x_k\}\,,\quad \{x_1, \ldots, x_k\} \in A\,,\quad X = Y\,,\quad X \in A$$

such that

[5] We recall that Δ denotes the symmetric difference operator defined by $s \,\Delta\, t = (s \setminus t) \cup (t \setminus s)$.

- *$Mx \in D^*$, for every level 0 variable x in ψ;*
- *$k \leqslant K$;*
- *$X \in \mathcal{V}_1'$, for every variable X of level 1 in quantifier-free atomic formulae of level 1 (namely of the form $X = Y$ or $X \in A$) occurring in ψ;*
- *$M^*X = MX$, if $|MX| \leqslant K$, and $|M^*X| > K$, otherwise, for every $X \in \mathcal{V}_1'$;*
- *$(MX \,\Delta\, MY) \cap D^* \neq \emptyset$, for all $X, Y \in \mathcal{V}_1'$ such that $MX \neq MY$;*

Then $\boldsymbol{\mathcal{M}} \models \psi$ if and only if $\boldsymbol{\mathcal{M}}^* \models \psi$. ■

The preceding corollary yields at once a small model property for the collection $3\mathrm{LST}_0$ of propositional combinations of quantifier-free atomic formulae of the types

$$x = y\,,\ \ x \in X\,,\ \ X = \{x_1, \ldots, x_k\}\,,\ \ \{x_1, \ldots, x_k\} \in A\,,\ \ X = Y\,,\ \ X \in A\,.$$

Indeed, let ψ be a satisfiable $3\mathrm{LST}_0$-formula and $\boldsymbol{\mathcal{M}} = (D, M)$ a model for it. Also, let K_ψ be the maximal length of any finite enumeration $\{x_1, \ldots, x_k\}$ occurring in ψ, and let $\mathcal{V}_0^\psi$ and $\mathcal{V}_1^\psi$ be the collections of variables of levels 0 and 1, respectively, occurring in ψ.

We construct a small model for ψ as follows. Let D_1 be a subset of D of cardinality not larger than $(K_\psi + 1) \cdot |\mathcal{V}_1^\psi|$ and such that $|J \cap D_1| \geqslant \min(K_\psi + 1, |J|)$, for every $J \in M\mathcal{V}_1^\psi$. For each pair of variables $X, Y \in \mathcal{V}_1^\psi$ such that $MX \neq MY$, select an element $d_{XY} \in MX \,\Delta\, MY$. Then we put

$$D^* \ =_{\mathrm{Def}} M\mathcal{V}_0^\psi \cup \big(\{d_{XY} : X, Y \in \mathcal{V}_1^\psi\,,\ MX \neq MY\} \cup D_1\big)$$

and select an arbitrary element d^* in D^*. Then, from Corollary 5.4 it follows that the relativized interpretation $\boldsymbol{\mathcal{M}}^* = \mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, \mathcal{V}_1^\psi)$ is a *small* model for ψ, as $|D^*| \leqslant |\mathcal{V}_0^\psi| + (K_\psi + 1) \cdot |\mathcal{V}_1^\psi| + |\mathcal{V}_1^\psi|^2$. In fact, by suitably choosing the elements d_{XY} in $MX \,\Delta\, MY$, we can enforce the bound $|D^*| < |\mathcal{V}_0^\psi| + (K_\psi + 2) \cdot |\mathcal{V}_1^\psi|$ (see [5]). Summing up, the following result holds:

Lemma 5.5 (Small model property for $3\mathrm{LST}_0$-formulae) *Let ψ be a $3\mathrm{LST}_0$-formula, i.e., a propositional combination of quantifier-free atomic formulae of the following types*

$$x = y\,,\ \ x \in X\,,\ \ X = \{x_1, \ldots, x_k\}\,,\ \ \{x_1, \ldots, x_k\} \in A\,,\ \ X = Y\,,\ \ X \in A\,.$$

Also, let K_ψ be the maximal length of any finite enumeration $\{x_1, \ldots, x_k\}$ occurring in ψ, and let $\mathcal{V}_0^\psi$ and $\mathcal{V}_1^\psi$ be the collections of variables of sort 0 and of sort 1 occurring in ψ, respectively. Then ψ is satisfiable if and only if it is satisfied by a $3\mathrm{LQST}_0$-interpretation $\boldsymbol{\mathcal{M}} = (D, M)$ such that $|D| < |\mathcal{V}_0^\psi| + (K_\psi + 2) \cdot |\mathcal{V}_1^\psi|$. ■

Since the $3\mathrm{LQST}_0$-interpretations over a bounded domain are finitely many and they can be effectively generated, the decidability of the satisfiability problem for $3\mathrm{LST}_0$-formulae follows.

To state the main results for quantified formulae, namely that the relativized interpretation $\boldsymbol{\mathcal{M}}^* = \mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, d^*, \mathcal{V}'_1)$ of a model $\boldsymbol{\mathcal{M}} = (D, M)$ for a purely universal $3\mathrm{LQST}_0^\mathrm{R}$-formula ψ of level 0 or 1 also satisfies ψ, under suitable conditions on D^* and $\mathcal{V}'_1 \subseteq \mathcal{V}_1$ (see Lemmas 5.6 and 5.7 below), it is convenient to introduce the following abbreviations:

$$\begin{aligned}
\boldsymbol{\mathcal{M}}^{\boldsymbol{z},*} &=_{\mathrm{Def}} \mathrm{Rel}(\boldsymbol{\mathcal{M}}^{\boldsymbol{z}}, D^*, d^*, \mathcal{V}'_1)\\
\boldsymbol{\mathcal{M}}^{*,\boldsymbol{z}} &=_{\mathrm{Def}} \boldsymbol{\mathcal{M}}^*[z_1/u_1, \ldots, z_n/u_n]\\
\boldsymbol{\mathcal{M}}^{\boldsymbol{Z},*} &=_{\mathrm{Def}} \mathrm{Rel}(\boldsymbol{\mathcal{M}}^{\boldsymbol{Z}}, D^*, d^*, \mathcal{V}'_1)\\
\boldsymbol{\mathcal{M}}^{*,\boldsymbol{Z}} &=_{\mathrm{Def}} \boldsymbol{\mathcal{M}}^*[Z_1/U_1, \ldots, Z_m/U_m]\,.
\end{aligned}$$

Lemma 5.6 *Let $\boldsymbol{\mathcal{M}} = (D, M)$ be a $3\mathrm{LQST}_0$-interpretation, K a fixed positive number, $D^* \subseteq D$, $d^* \in D^*$, $\mathcal{V}'_1 \subseteq \mathcal{V}_1$, and let $\boldsymbol{\mathcal{M}}^* = \mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, d^*, \mathcal{V}'_1)$ be such that $M^*X = MX$, if $|MX| \leqslant K$, and $|M^*X| > K$ otherwise, for every $X \in \mathcal{V}'_1$. Furthermore, let $(\forall z_1) \ldots (\forall z_n)\varphi_0$ be a purely universal $3\mathrm{LQST}_0^\mathrm{R}$-formula of level 0 such that*

(i) $Mx \in D^*$, for every $x \in \mathcal{V}_0$ occurring free in it;

(ii) each enumeration term $\{x_1, \ldots, x_k\}$ in ψ has size at most K (i.e., $k \leqslant K$);

(iii) $M^*X = MX$, for every variable X in ψ such that $X \in \mathcal{V}_1 \setminus \mathcal{V}'_1$.

Then $\boldsymbol{\mathcal{M}} \models (\forall z_1) \ldots (\forall z_n)\varphi_0 \implies \boldsymbol{\mathcal{M}}^* \models (\forall z_1) \ldots (\forall z_n)\varphi_0\,.$ ■

Lemma 5.7 *Let $\boldsymbol{\mathcal{M}} = (D, M)$ be a $3\mathrm{LQST}_0$-interpretation, $D^* \subseteq D$, $d^* \in D^*$, $\mathcal{V}'_1 \subseteq \mathcal{V}_1$, $\boldsymbol{\mathcal{M}}^* = \mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, d^*, \mathcal{V}'_1)$, $K \geqslant 1$, and let $(\forall Z_1) \ldots (\forall Z_m)\varphi_1$ be a purely universal $3\mathrm{LQST}_0^\mathrm{R}$-formula of level 1 such that*

(i) *$Z_1, \ldots, Z_m \notin \mathcal{V}'_1$;*

(ii) *$X \in \mathcal{V}'_1$, for every variable $X \in \mathcal{V}_1$ occurring free in $(\forall Z_1) \ldots (\forall Z_m)\varphi_1$;*

(iii) *$Mx \in D^*$, for every variable $x \in \mathcal{V}_0$ occurring free in $(\forall Z_1) \ldots (\forall Z_m)\varphi_1$;*

(iv) *$M^*X = MX$, if $|MX| \leqslant K$, and $|M^*X| > K$ otherwise, for every $X \in \mathcal{V}'_1$;*

(v) *$(MX \,\Delta\, MY) \cap D^* \neq \emptyset$, for all $X, Y \in \mathcal{V}'_1$ such that $MX \neq MY$;*

(vi) *each enumeration term $\{x_1, \ldots, x_k\}$ in φ_1 has size at most K;*

(vii) *for every purely universal formula $(\forall z_1) \ldots (\forall z_n)\varphi_0$ of level 0 occurring in φ_1 and variables $X_1, \ldots, X_m \in \mathcal{V}'_1$ such that $\boldsymbol{\mathcal{M}} \not\models ((\forall z_1) \ldots (\forall z_n)\varphi_0)^{Z_1,\ldots,Z_m}_{X_1,\ldots,X_m}$, there are $u_1, \ldots, u_n \in D^*$ such that $\boldsymbol{\mathcal{M}}[z_1/u_1, \ldots, z_n/u_n] \not\models (\varphi_0)^{Z_1,\ldots,Z_m}_{X_1,\ldots,X_m}$;*[6]

Then $\boldsymbol{\mathcal{M}} \models (\forall Z_1) \ldots (\forall Z_m)\varphi_1 \implies \boldsymbol{\mathcal{M}}^* \models (\forall Z_1) \ldots (\forall Z_m)\varphi_1\,.$

Proofs of Lemmas 5.6 and 5.7 can be found in [11].

---

[6] Given a formula ψ and variables $X_1, \ldots, X_m, Z_1, \ldots, Z_m$, by $\psi^{Z_1,\ldots,Z_m}_{X_1,\ldots,X_m}$ we mean the formula obtained by simultaneously substituting each occurrence of Z_i in ψ with X_i for every $i \in \{1, \ldots, m\}$.

6 The satisfiability problem for $3\mathrm{LQST}_0^{\mathrm{R}}$-formulae

We will solve the satisfiability problem for $3\mathrm{LQST}_0^{\mathrm{R}}$ as follows:

(a) firstly, we will reduce effectively the satisfiability problem for $3\mathrm{LQST}_0^{\mathrm{R}}$-formulae to the same problem for *normalized* $3\mathrm{LQST}_0^{\mathrm{R}}$-*conjunctions* (these will be defined precisely below);

(b) secondly, we will prove that the collection of normalized $3\mathrm{LQST}_0^{\mathrm{R}}$-conjunctions enjoys a small model property.

From (a) and (b), the solvability of the satisfiability problem for $3\mathrm{LQST}_0^{\mathrm{R}}$ will follow immediately.

6.1 Normalized $3\mathrm{LQST}_0^{\mathrm{R}}$-conjunctions

Let ψ be a formula of $3\mathrm{LQST}_0^{\mathrm{R}}$ and let ψ_{DNF} be a disjunctive normal form of ψ. We observe that the disjuncts of ψ_{DNF} are conjunctions of $3\mathrm{LQST}_0^{\mathrm{R}}$-literals, namely quantifier-free atomic formulae of levels 0 and 1, or their negations, and of purely universal formulae of levels 0 and 1, or their negations, satisfying the linkedness condition (1).

By a suitable renaming of variables, we can assume that no bound variable can occur in more than one quantifier in the same disjunct of ψ_{DNF} and that no variable can have both bound and free occurrences in the same disjunct.

Without disrupting satisfiability, we replace negative literals of the form $\neg(\forall z_1)\dots(\forall z_n)\varphi_0$ and $\neg(\forall Z_1)\dots(\forall Z_m)\varphi_1$ occurring in ψ_{DNF} by their negated matrices $\neg\varphi_0$ and $\neg\varphi_1$, respectively, since for any given $3\mathrm{LQST}_0$-interpretation $\boldsymbol{\mathcal{M}} = (D, M)$ one has $\boldsymbol{\mathcal{M}} \models \neg(\forall z_1)\dots(\forall z_n)\varphi_0$ if and only if $\boldsymbol{\mathcal{M}}[z_1/u_1,\dots,z_n/u_n] \models \neg\varphi_0$, for some $u_1,\dots,u_n \in D$, and, likewise, $\boldsymbol{\mathcal{M}} \models \neg(\forall Z_1)\dots(\forall Z_m)\varphi_1$ if and only if $\boldsymbol{\mathcal{M}}[Z_1/U_1,\dots,Z_m/U_m] \models \neg\varphi_1$, for some $U_1,\dots,U_m \in \mathrm{pow}(D)$. Then, if needed, we bring back the resulting formula into disjunctive normal form, eliminate as above the residual negative literals of the form $\neg(\forall z_1)\dots(\forall z_n)\varphi_0$ which might have been introduced by the previous elimination of negative literals of the form $\neg(\forall Z_1)\dots(\forall Z_m)\varphi_1$ from ψ_{DNF}, and transform again the resulting formula into disjunctive normal form. Let ψ'_{DNF} be the formula so obtained. Observe that all the above steps preserve satisfiability, so that our initial formula ψ is satisfiable if and only if so is ψ'_{DNF}. In addition, the formula ψ'_{DNF} is satisfiable if and only if so is at least one of its disjuncts.

It is an easy matter to check that each disjunct of ψ'_{DNF} is a conjunction of $3\mathrm{LQST}_0^{\mathrm{R}}$-literals of the following types (I,II,III):

$$
\begin{array}{cccc}
x = y\,, & x \in X\,, & X = \{x_1,\dots,x_k\}\,, & \{x_1,\dots,x_k\} \in A\,,\\
\neg(x = y)\,, & \neg(x \in X)\,, & \neg(X = \{x_1,\dots,x_k\})\,, & \neg(\{x_1,\dots,x_k\} \in A)\,,\\
X = Y\,, & X \in A\,, & \neg(X = Y)\,, & \neg(X \in A)\,,
\end{array}
\qquad \text{(I)}
$$

where $x, y, x_1, \ldots, x_k \in \mathcal{V}_0$, $X, Y \in \mathcal{V}_1$, and $A \in \mathcal{V}_2$;

$$(\forall z_1) \ldots (\forall z_n)\varphi_0\,, \tag{II}$$

where $n \geqslant 1$ and φ_0 is a propositional combination of quantifier-free level 0 atoms; and

$$(\forall Z_1) \ldots (\forall Z_m)\varphi_1\,, \tag{III}$$

where $m \geqslant 1$ and φ_1 is a propositional combination of quantifier-free atomic formulae of any level and of purely universal formulae of level 0, where the propositional components in φ_1 of type $(\forall z_1) \ldots (\forall z_n)\varphi_0$ are linked to the bound variables $Z_1, \ldots, Z_m$.

We call such formulae *normalized* $3\mathrm{LQST}_0^{\mathrm{R}}$*-conjunctions*.

The above discussion can then be summarized in the following lemma.

Lemma 6.1 *The satisfiability problem for* $3\mathrm{LQST}_0^{\mathrm{R}}$*-formulae can be effectively reduced to the satisfiability problem for* $3\mathrm{LQST}_0^{\mathrm{R}}$*-conjunctions.* ■

6.2 A small model property for normalized $3\mathrm{LQST}_0^{\mathrm{R}}$-conjunctions

Let ψ be a normalized $3\mathrm{LQST}_0^{\mathrm{R}}$-conjunction and assume that $\boldsymbol{\mathcal{M}} = (D, M)$ is a model for ψ. We show how to construct, out of $\boldsymbol{\mathcal{M}}$, a finite *small* $3\mathrm{LQST}_0$-interpretation $\boldsymbol{\mathcal{M}}^* = (D^*, M^*)$ which is a model of ψ. We proceed as follows. First we outline a procedure to build a nonempty finite universe $D^* \subseteq D$ whose size depends solely on ψ and can be computed *a priori*. Then, following Definition 5.1, we construct a relativized $3\mathrm{LQST}_0$-interpretation $\boldsymbol{\mathcal{M}}^* = (D^*, M^*)$ with respect to a suitable collection $\mathcal{V}_1'$ of variables, and show that it satisfies ψ.

6.2.1 Construction of the universe D^*

Let $\mathcal{V}_0^{\psi}$, $\mathcal{V}_1^{\psi}$, and $\mathcal{V}_2^{\psi}$ be the collections of the variables of sort 0, 1, and 2 occurring in ψ, respectively, and let K_ψ be smallest integer such that $k \leqslant K_\psi$, for every finite enumeration term $\{x_1, \ldots, x_k\}$ occurring in ψ. We construct the domain D^* by means of the procedure below.

Let $\psi_1, \ldots, \psi_h$ be the conjuncts of ψ of the form (III). To each such conjunct $\psi_i \equiv (\forall Z_{i1}) \ldots (\forall Z_{im_i})\varphi_i$, we associate the collection $\varphi_{i1}, \ldots, \varphi_{i\ell_i}$ of the purely universal atomic formulae of level 0 occurring in its matrix φ_i and call the variables $Z_{i1}, \ldots, Z_{im_i}$ the *arguments of* $\varphi_{i1}, \ldots, \varphi_{i\ell_i}$. Then we put

$$\Phi_\psi \;=_{\mathrm{Def}} \{\varphi_{ij} : 1 \leqslant i \leqslant h \text{ and } 1 \leqslant j \leqslant \ell_i\}.$$

By applying the procedure *Distinguish* described in [5] to the collection $M\mathcal{V}_1^{\psi}$, it is possible to construct a set D_0 such that

- $MX \cap D_0 \neq MY \cap D_0$, for all $X, Y \in \mathcal{V}_1^{\psi}$ such that $MX \neq MY$, and
- $|D_0| \leqslant |\mathcal{V}_1^{\psi}| - 1$.

Next, we construct a set D_1 such that $|J \cap D_1| \geqslant \min(K_\psi + 1, |J|)$, for every $J \in M\mathcal{V}_1^\psi$. Plainly, we can assume that $|D_1| \leqslant (K_\psi + 1) \cdot |\mathcal{V}_1^\psi|$.

Then, after initializing D^* to the set $M\mathcal{V}_0^\psi \cup (D_0 \cup D_1)$, we insert in D^* elements $u_1, \ldots, u_n \in D$ such that $\boldsymbol{\mathcal{M}}[z_1/u_1, \ldots, z_n/u_n] \not\models (\varphi_0)^{Z_1, \ldots, Z_m}_{X_{i_1}, \ldots, X_{i_m}}$, for each $\varphi \in \Phi_\psi$ of the form $(\forall z_1) \ldots (\forall z_n)\varphi_0$ having $Z_1, \ldots, Z_m$ as arguments and for each ordered m-tuple $(X_{i_1}, \ldots, X_{i_m})$ of variables in $\mathcal{V}_1^\psi$ such that $\boldsymbol{\mathcal{M}} \not\models \varphi^{Z_1, \ldots, Z_m}_{X_{i_1}, \ldots, X_{i_m}}$.

The above construction yields easily that

$$|D^*| \leqslant |\mathcal{V}_0^\psi| + (l_\psi + 2) \cdot |\mathcal{V}_1^\psi| - 1 + N_\psi \cdot |\mathcal{V}_1^\psi|^{L_\psi} \cdot |\Phi_\psi|\,, \tag{7}$$

where L_ψ and N_ψ are, respectively, the maximal number of quantifiers in any purely universal formula of level 1 in Φ_ψ and the maximal number of quantifiers in purely universal formulae of level 0 occurring in any purely universal formula of level 1 in Φ_ψ. Thus, in general, the size of the domain D^* is exponential in the size of the input formula ψ.

6.2.2 Correctness of the relativization

Let $\boldsymbol{\mathcal{M}}^* =_{\mathrm{Def}} \mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, d^*, \mathcal{V}_1^\psi)$. The next theorem, whose proof can be found in [11], states that if $\boldsymbol{\mathcal{M}} \models \psi$, then $\boldsymbol{\mathcal{M}}^* \models \psi$.

Theorem 6.2 *Let $\boldsymbol{\mathcal{M}}$ be a* $3\mathrm{LQST}_0$*-interpretation satisfying a normalized* $3\mathrm{LQST}_0^{\mathrm{R}}$*-conjunction ψ. Further, let $\boldsymbol{\mathcal{M}}^* = \mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, d^*, \mathcal{V}_1^\psi)$ be the* $3\mathrm{LQST}_0$*-interpretation defined according to Definition 5.1, where D^* is constructed as above and $\mathcal{V}_1^\psi$ is the collection of variables of level* 1 *occurring in ψ. Then $\boldsymbol{\mathcal{M}}^* \models \psi$.* ∎

The above reduction and relativization steps yield easily the following result:

Corollary 6.3 *The fragment* $3\mathrm{LQST}_0^{\mathrm{R}}$ *enjoys a small model property (and therefore its satisfiability problem is solvable).* ∎

Much as in [10], it is possible to define a class of subtheories $(3\mathrm{LQST}_0^{\mathrm{R}})^h$ of $3\mathrm{LQST}_0^{\mathrm{R}}$, for $h \geqslant 2$, having an NP-complete satisfiability problem. In addition to certain syntactic constraints (see [10]), all quantifier prefixes in $(3\mathrm{LQST}_0^{\mathrm{R}})^h$-formulae have length bounded by the constant h. It turns out that such subtheories are quite expressive: in fact, several set-theoretic constructs considered in Section 4 (such as, for instance, some variants of the powerset operator) can be expressed in them. Moreover, it can be shown that the modal logic S5 can be represented in $(3\mathrm{LQST}_0^{\mathrm{R}})^3$.

7 The unordered Cartesian product

Given sets $X_1, \ldots, X_n$, the unordered Cartesian product $X_1 \otimes \ldots \otimes X_n$ is the set

$$X_1 \otimes \ldots \otimes X_n =_{\mathrm{Def}} \Big\{ \{x_1, \ldots, x_n\} : x_1 \in X_1, \ldots, x_n \in X_n \Big\}.$$

Then, the literal

$$A = X_1 \otimes \ldots \otimes X_n \,, \tag{8}$$

where A is a variable of level 2 and $X_1, \ldots, X_n$ are variables of level 1, can be expressed by the $3\mathrm{LQST}_0^{\mathrm{R}}$-formula

$$(\forall Z)\Big(Z \in A \longleftrightarrow (\exists x_1)\ldots(\exists x_n)\Big(\bigwedge\nolimits_{i=1}^{n} x_i \in X_i \wedge \{x_1, \ldots, x_n\} = Z\Big)\Big)\,. \tag{9}$$

One may wonder if it is possible to express the Cartesian product (8) without making use of the finite enumeration operator (hence, by a $3\mathrm{LQS}^{\mathrm{R}}$-formula). Since the atom $\{x_1, \ldots, x_n\} = Z$ can be expressed by the $3\mathrm{LQS}^{\mathrm{R}}$-formula

$$(\forall z)(z \in Z \leftrightarrow \bigvee\nolimits_{i=1}^{n} z = x_i)\,, \tag{10}$$

a straightforward attempt consists in replacing the occurrence of $\{x_1, \ldots, x_n\} = Z$ in (9) with (10). The resulting formula:

$$(\forall Z)\Big(Z \in A \longleftrightarrow (\exists x_1)\ldots(\exists x_n)(\forall z)\Big(\bigwedge_{i=1}^{n} x_i \in X_i \wedge (z \in Z \leftrightarrow \bigvee_{i=1}^{n} z = x_i)\Big)\Big)\,,$$

however, is not in $3\mathrm{LQS}^{\mathrm{R}}$ because the formula

$$(\exists x_1)\ldots(\exists x_n)(\forall z)\Big(\bigwedge_{i=1}^{n} x_i \in X_i \wedge (z \in Z \leftrightarrow \bigvee_{i=1}^{n} z = x_i)\Big)$$

is not a purely universal formula of level 0, and the variables $x_1, \ldots, x_n$ are not linked to Z. As we will see below, in the general case we need $3\mathrm{LQS}^{\mathrm{R}}$-formulae of an exponential length in n, thus showing that the fragment $3\mathrm{LQST}_0^{\mathrm{R}}$ is strictly more expressive than $3\mathrm{LQS}^{\mathrm{R}}$.

When the sets $X_1, \ldots, X_n$ are pairwise disjoint or, on the opposite side, when they all coincide, we can express the literal (8) by a simple $3\mathrm{LQS}^{\mathrm{R}}$-formula. For instance, if the sets $X_1, \ldots, X_n$ are pairwise disjoint, then $Z \in X_1 \otimes \ldots \otimes X_n$ if and only if

(i) $|Z| = n$, and

(ii) there exist $x_1 \in X_1, \ldots, x_n \in X_n$ such that $x_1 \in Z, \ldots, x_n \in Z$.

The above conditions can be used to express the literal (8) by the following $3\mathrm{LQS}^{\mathrm{R}}$-formula

$$(\forall Z)\Big(Z \in A \longleftrightarrow \Big(|Z| = n \wedge (\exists x_1)\ldots(\exists x_n)\Big(\bigwedge_{i=1}^{n}(x_i \in X_i \wedge x_i \in Z)\Big)\Big)\Big)\,,$$

as is easy to check, where

- $|Z| = n \quad \equiv_{\mathrm{Def}} |Z| \leqslant n \wedge |Z| \geqslant n$
- $|Z| \leqslant n \quad \equiv_{\mathrm{Def}} (\forall x_1)\dots(\forall x_{n+1})\Big(\bigwedge_{i=1}^{n+1} x_i \in Z \rightarrow \bigvee_{1\leqslant i<j\leqslant n+1} x_i = x_j \Big)$
- $|Z| \geqslant n \quad \equiv_{\mathrm{Def}} \neg(|Z| \leqslant n-1)$

(notice that $|Z| \leqslant n$ is linked to the variable Z).

When $X_1 = \ldots = X_n$, then $Z \in X_1 \otimes \ldots \otimes X_n$ if and only if $|Z| \leqslant n$ and $Z \subseteq X_1$. Thus, in this particular case, the literal (8) can be expressed by the $3\mathrm{LQS}^{\mathrm{R}}$-formula

$$(\forall Z)\Big(Z \in A \longleftrightarrow \Big(|Z| \leqslant n \wedge (\forall x)(x \in Z \rightarrow x \in X_1)\Big)\Big)\,.$$

However, if we make no assumption on the sets $X_1, \ldots, X_n$, in order to characterize the sets Z belonging to $X_1 \otimes \ldots \otimes X_n$ by a $3\mathrm{LQS}^{\mathrm{R}}$-formula, we have to consider separately the cases in which $|Z| = n$, $|Z| = n-1$, etc., listing explicitly, for each of them, all the allowed membership configurations of the members of Z. For instance, if $n = 2$, we have $Z \in X_1 \otimes X_2$ if and only if

- $|Z| = 2$ and there exist distinct $x_1 \in X_1$ and $x_2 \in X_2$ such that $x_1, x_2 \in Z$; or
- $|Z| = 1$ and the intersection $X_1 \cap X_2 \cap Z$ is nonempty.

Thus the following $3\mathrm{LQS}^{\mathrm{R}}$-formula expresses the literal $A = X_1 \otimes X_2$:

$$\begin{aligned}(\forall Z)\Big(Z \in A \longleftrightarrow \Big(\Big(|Z| = 2 \wedge (\exists x_1)(\exists x_2)\Big(x_1 \neq x_2 \wedge \bigwedge_{i=1}^{2}(x_i \in X_i \wedge x_i \in Z)\Big)\Big)\\ \vee \Big(|Z| = 1 \wedge (\exists x_1)(x_1 \in X_1 \wedge x_1 \in X_2 \wedge x_1 \in Z)\Big)\Big)\Big)\end{aligned}$$

Likewise, in the case $n = 3$, we have $Z \in X_1 \otimes X_2 \otimes X_3$ if and only if

- $|Z| = 3$ and there exist pairwise distinct $x_1 \in X_1$, $x_2 \in X_2$, and $x_3 \in X_3$ such that $x_1, x_2, x_3 \in Z$; or
- $|Z| = 2$ and there exist distinct x_1 and x_2 such that either
  - · $x_1 \in X_1 \cap X_2$ and $x_2 \in X_3$, or
  - · $x_1 \in X_1 \cap X_3$ and $x_2 \in X_2$, or
  - · $x_1 \in X_2 \cap X_3$ and $x_2 \in X_1$,

  and such that $x_1, x_2 \in Z$; or
- $|Z| = 1$ and the intersection $X_1 \cap X_2 \cap X_3 \cap Z$ is nonempty.

More in general, we have the following lemma, proved in [11].

Lemma 7.1 *Let $X_1, \ldots, X_n$ be given sets. Then $Z \in X_1 \otimes \ldots \otimes X_n$ if and only there exists a partition P of the set $\{1, \ldots, n\}$ and a bijection $\sigma : Z \rightarrow P$ such that*

$$\text{if } i \in \sigma(x), \text{ then } x \in X_i, \text{ for } x \in Z \text{ and } i \in \{1, \ldots, n\}. \tag{11}$$

Let $\mathfrak{P}_n$ be the collection of all partitions of the set $\{1, \ldots, n\}$. For any partition $P \in \mathfrak{P}_n$, we will assume that the blocks $b_1(P), \ldots, b_{|P|}(P)$ of P are ordered by a total order $\prec$ in such a way that

$$b_i(P) \prec b_j(P) \quad \text{if and only if} \quad \min b_i(P) < \min b_j(P)\,.$$

Then, based on Lemma 7.1, the literal $A = X_1 \otimes \ldots \otimes X_n$ can be expressed by the following $3\mathrm{LQS}^{\mathrm{R}}$-formula

$$(\forall Z)\Big(Z \in A \leftrightarrow \bigwedge_{P \in \mathfrak{P}_n} \Big(|Z| = |P| \wedge (\exists z_1) \ldots (\exists z_{|P|})\Big(\bigwedge_{1 \leqslant i < j \leqslant |P|} z_i \neq z_j \\ \wedge \bigwedge_{i=1}^{|P|} \Big(z_i \in Z \wedge \bigwedge_{j \in b_i(P)} z_i \in X_j\Big)\Big)\Big)\Big)\,. \quad (12)$$

The following bounds on the length ℓ_n of the formula (12) hold:

$$\ell_n = \Omega(nB_n)\,, \qquad \ell_n = \mathcal{O}(n^2 B_n)\,, \quad (13)$$

where $B_n = |\mathfrak{P}_n|$ is the nth Bell's number. Using the bounds $\left(\frac{n}{e \ln n}\right)^n < B_n < \left(\frac{0.792n}{\ln(n+1)}\right)^n$ by Berend and Tassa (cf. [1]), the bounds (13) yield

$$\ell_n = \Omega\left(n\left(\frac{n}{e \ln n}\right)^n\right)\,, \qquad \ell_n = \mathcal{O}\left(n^2\left(\frac{0.792n}{\ln(n+1)}\right)^n\right)\,.$$

Thus, the representation (12) of the unordered Cartesian product of n sets has exponential length in n.

8 Conclusions and future work

We have presented a three-sorted stratified set-theoretic fragment, $3\mathrm{LQST}_0^{\mathrm{R}}$, and have shown that it has a decidable satisfiability problem. The fragment $3\mathrm{LQST}_0^{\mathrm{R}}$ turns out to be quite expressive as it allows to represent efficiently several set-theoretic constructs, such as variants of the powerset operator and the unordered Cartesian product.

Much as in [10], it is possible to single out a family $\{(3\mathrm{LQST}_0^{\mathrm{R}})^h\}_{h \geqslant 2}$ of subfragments of $3\mathrm{LQST}_0^{\mathrm{R}}$ (characterized by imposing suitable syntactic constraints) having an NP-complete satisfiability problem. It is not hard to see that the modal logic S5 can be formalized in $(3\mathrm{LQST}_0^{\mathrm{R}})^3$.

We intend to study the possibility of formalizing further non-classical logics into suitable extensions of the $3\mathrm{LQST}_0^{\mathrm{R}}$ fragment also in consideration of the fact that techniques to translate modal formulae in set-theoretic terms have already been proposed in [2], in the context of hyperset theory, and in [14] in the ambit of weak set theories not involving the axiom of extensionality and the axiom of foundation.

We also plan to extend the fragment $3\mathrm{LQST}_0^{\mathrm{R}}$ so as to express the set-theoretic construct of general union, thus subsuming the theory 3LSSPU.

Finally, another direction of future research concerns the investigation of the satisfiability problem for stratified set-theoretic fragments involving n levels, for any $n > 4$.

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