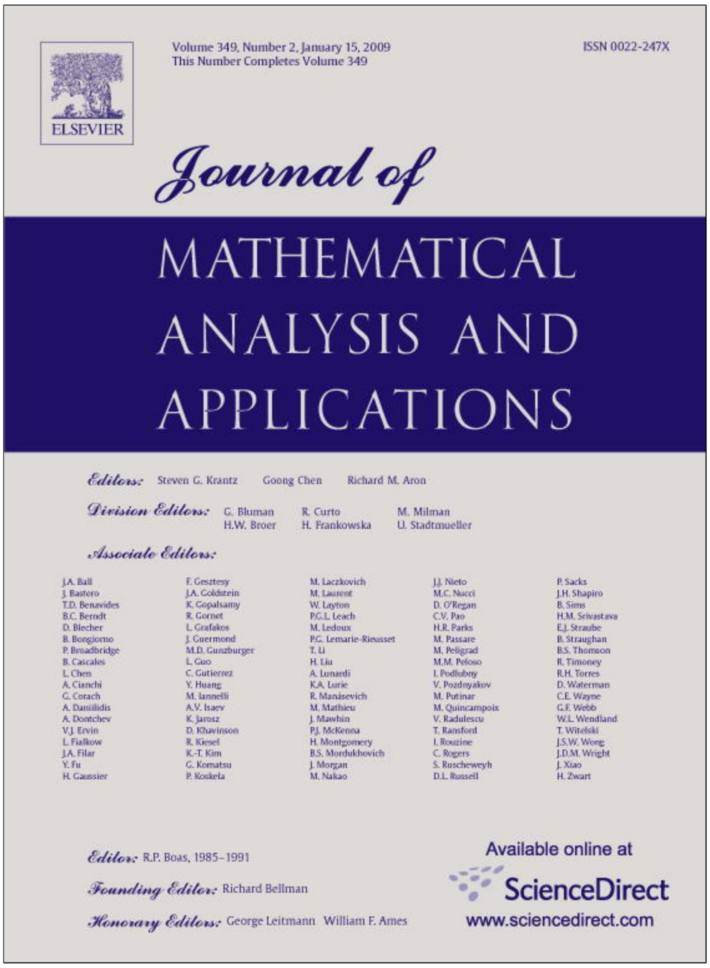
Volume 349, Number 2, January 15, 2009
This Number Completes Volume 349
ISSN 0022-247X
ELSEVIER
Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS
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J. Math. Anal. Appl. 349 (2009) 516–525

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

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Invariants of two- and three-dimensional hyperbolic equations

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ARTICLE INFO

Article history:
Received 11 February 2008
Available online 9 September 2008
Submitted by G. Bluman

Keywords:
Hyperbolic equations
Equivalence transformations
Differential invariants
Point transformations

ABSTRACT

We consider linear hyperbolic equations of the form

$$u_{tt}=\sum_{i=1}^{n}u_{x_ix_i}+\sum_{i=1}^{n}X_i(x_1,\dots,x_n,t)u_{x_i}+T(x_1,\dots,x_n,t)u_t+U(x_1,\dots,x_n,t)u.$$

We derive equivalence transformations which are used to obtain differential invariants for the cases $n=2$ and $n=3$. Motivated by these results, we present the general results for the n-dimensional case. It appears (at least for $n=2$) that this class of hyperbolic equations admits differential invariants of order one, but not of order two. We employ the derived invariants to construct interesting mappings between equivalent equations.

1. Introduction

The theory of invariant quantities for families of equations appeared in the beginning of the theory of partial differential equations. The linear wave equation $u_{xy}=0$ for vibrating strings, was formulated and solved by d'Alembert in 1747. In 1769/1770 Euler [1] and later, in 1773, Laplace [2] derived the invariant quantities

$$h=a_x+ab-c,\qquad k=b_y+ab-c \tag{1}$$

which are known today as the *Laplace invariants*, for the linear hyperbolic equation

$$u_{xy}+a(x,y)u_x+b(x,y)u_y+c(x,y)u=0. \tag{2}$$

An equation of the form (2) can be mapped (by a point transformation) into $u_{xy}=0$ if and only if $h=k=0$. Furthermore an equation of the form (2) can be factorized if and only if $h=0$ or $k=0$. For an interesting and historical review of Laplace invariants one can refer to [3].

Differential invariants of the Lie groups of continuous transformations can be used in wide fields: classification of invariant differential equations and variational problems arising in the construction of physical theories, solution methods for ordinary and partial differential equations, equivalence problems for geometric structures. First it was noted by S. Lie [4], who proved that every invariant system of differential equations [5], and every variational problem [6], could be directly expressed in terms of differential invariants. Lie also showed [5] how differential invariants play an important role to integrate ordinary differential equations and succeeded in completely classifying all the differential invariants for all possible finite-dimensional Lie groups of point transformations in the case of one independent and one dependent variable. Tresse [7] and Ovsiannikov [8] generalized the Lie's preliminary results on invariant differentiations and existence of finite bases of

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.09.004

differential invariants. The general theory of differential invariants of Lie groups together with algorithms of construction of differential invariants can be found in [8,9].

Recently Ibragimov [10–12] developed a simple method for constructing invariants of families of differential equations. The method is based in the theory of equivalence groups in the infinitesimal form. Basically, the method consists of two steps: Classification of equivalence groups and then use these groups (and extended groups) to derive the desired differential invariants. Ibragimov [13] used his method to solve the Laplace problem. That is, to derive all invariants for the linear hyperbolic equations (2). To achieve this, he constructed a basis for the invariants and then using this basis and invariant differentiation all invariants, of any order, can be derived. The idea of Ibragimov was adopted by a number of authors who derived differential invariants for ordinary differential equations, linear and non-linear partial differential equations [14–21].

In the present work we derive the equivalence transformations for the two-dimensional linear hyperbolic equations

$$u_{tt}=u_{xx}+u_{yy}+X(t,x,y)u_x+Y(t,x,y)u_y+T(t,x,y)u_t+U(t,x,y)u \tag{3}$$

and the three-dimensional linear hyperbolic equations

$$u_{tt}=u_{xx}+u_{yy}+u_{zz}+X(t,x,y,z)u_x+Y(t,x,y,z)u_y+Z(t,x,y,z)u_z+T(t,x,y,z)u_t+U(t,x,y,z)u \tag{4}$$

and, in the spirit of Ibragimov's work, we construct differential invariants with the employment of the derived equivalence transformations. Motivated by the results, we present the general results for the n-dimensional case. It appears (at least for $n=2$) that this class of hyperbolic equations admits differential of order one, but not of order two.

The linear hyperbolic equations have considerable interest in Mathematical Physics and Biology [22–25]. They have a number of applications, for example, in population dynamics, tides and waves, chemical reactors, flame and combustion problems and problems in transonic aerodynamics.

In the next section we determine the equivalence transformations for the class of Eqs. (3). In Section 3, the derived equivalence transformations are employed to obtain differential invariants. In Section 4, we obtain equivalence transformations and differential invariants for (4). Motivated by the results in Sections 2, 3 and 4, we present in Section 5 the corresponding results for the n-dimensional class of hyperbolic equations. In Section 6, we use differential invariants to derive certain mappings that connect equations of the class (3). These mappings can easily be generalized to n-dimensional case. Some final remarks are given in the conclusion. For completeness, we present the corresponding results for the one-dimensional equation in Appendix A [3].

2. Equivalence transformations for (3)

Equivalence transformations play the central part in the theory of invariants. The set of all equivalence transformations of a given family of differential equations forms a group which is called the equivalence group. There exist two methods for calculation of equivalence transformations, the direct which was used first by Lie [26] and the Lie infinitesimal method which was introduced by Ovsiannikov [8]. Although, the direct method involves considerable computational difficulties, it has the benefit of finding the most general equivalence group. For recent applications of the direct method one can refer, for example, to references [27–29]. More detailed description and examples of both methods can be found in [3]. Here we use the infinitesimal method to derive the desired equivalence transformations.

We call an equivalence transformation of Eqs. (3) an invertible point transformation belonging to the class

$$t'=\alpha(t,x,y,u),\qquad x'=\beta(t,x,y,u),\qquad y'=\gamma(t,x,y,u),\qquad u'=\omega(t,x,y,u)$$

which preserves the order of Eqs. (3) as well as the properties of linearity and homogeneity. In general, the transformed equations have different coefficients X', Y', T' and U'. The functions α,β,γ and ω are such that $\partial(\alpha,\beta,\gamma,\omega)/\partial(t,x,y,u)\neq 0$. If one wants to find the functions α,β,γ and ω, the direct method needs to be employed. It turns out that it is a very difficult task. We therefore use the infinitesimal method.

In order to find continuous group of equivalence transformations for Eqs. (3) by means of the Lie infinitesimal invariance criterion, we search for the equivalent operator in the following form:

$$\Gamma=\xi_1\frac{\partial}{\partial t}+\xi_2\frac{\partial}{\partial x}+\xi_3\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial u}+\mu_1\frac{\partial}{\partial X}+\mu_2\frac{\partial}{\partial Y}+\mu_3\frac{\partial}{\partial T}+\mu_4\frac{\partial}{\partial U} \tag{5}$$

where $\xi_i=\xi_i(t,x,y,u)$ $(i=1,2,3)$, $\eta=\eta(t,x,y,u)$ and $\mu_i=\mu_i(t,x,y,u,X,Y,T,U)$ $(i=1,2,3,4)$. We invoke the determining equation:

$$\Gamma^{(2)}\big(u_{tt}-u_{xx}-u_{yy}-X(t,x,y)u_x-Y(t,x,y)u_y-T(t,x,y)u_t-U(t,x,y)u=0\big)\big|_{\text{Eq. (3)}}=0,$$

where $\Gamma^{(2)}$ is the second extension of Γ. For details of how the operator Γ can be extended, one can refer to [8,11]. The above expression is a multivariable polynomial in variables the derivatives of u, X, Y, T and U. The coefficients of the different powers of these variables must be zero, giving a list of determining equations. These equations enable the equivalence transformations to be derived and ultimately impose restrictions on the functional forms of the infinitesimals $\xi_i(t,x,y,u)$, $\eta(t,x,y,u)$ and $\mu_i(t,x,y,u,T,X,Y,U)$.

We find that the family of Eqs. (3) admits an infinite continuous group $\mathcal{E}$ of equivalence transformations generated by the Lie algebra $L_{\mathcal{E}}$ spanned by the operators

$$\Gamma_1=\frac{\partial}{\partial t},\qquad \Gamma_2=\frac{\partial}{\partial x},\qquad \Gamma_3=\frac{\partial}{\partial y},\qquad \Gamma_4=x\frac{\partial}{\partial t}+t\frac{\partial}{\partial x}+T\frac{\partial}{\partial X}+X\frac{\partial}{\partial T},$$

$$\Gamma_5=y\frac{\partial}{\partial t}+t\frac{\partial}{\partial y}+T\frac{\partial}{\partial Y}+Y\frac{\partial}{\partial T},\qquad \Gamma_6=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}+Y\frac{\partial}{\partial X}-X\frac{\partial}{\partial Y},$$

$$\Gamma_7=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-X\frac{\partial}{\partial X}-Y\frac{\partial}{\partial Y}-T\frac{\partial}{\partial T}-2U\frac{\partial}{\partial U},$$

$$\Gamma_8=\frac{1}{2}\left(t^2+x^2+y^2\right)\frac{\partial}{\partial t}+xt\frac{\partial}{\partial x}+ty\frac{\partial}{\partial y}+(xT-tX)\frac{\partial}{\partial X}+(yT-tY)\frac{\partial}{\partial Y}+(xX+yY-tT+1)\frac{\partial}{\partial T}-2tU\frac{\partial}{\partial U},$$

$$\Gamma_9=xt\frac{\partial}{\partial t}+\frac{1}{2}\left(t^2+x^2-y^2\right)\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}-(xX+yY-tT+1)\frac{\partial}{\partial X}+(yX-xY)\frac{\partial}{\partial Y}+(tX-xT)\frac{\partial}{\partial T}-2xU\frac{\partial}{\partial U},$$

$$\Gamma_{10}=ty\frac{\partial}{\partial t}+xy\frac{\partial}{\partial x}+\frac{1}{2}\left(t^2-x^2+y^2\right)\frac{\partial}{\partial y}+(xY-yX)\frac{\partial}{\partial X}-(xX+yY-tT+1)\frac{\partial}{\partial Y}+(tY-yT)\frac{\partial}{\partial T}-2yU\frac{\partial}{\partial U},$$

$$\Gamma_\alpha=\alpha u\frac{\partial}{\partial u}-2\alpha_x\frac{\partial}{\partial X}-2\alpha_y\frac{\partial}{\partial Y}+2\alpha_t\frac{\partial}{\partial T}+(\alpha_{tt}-\alpha_{xx}-\alpha_{yy}-\alpha_x X-\alpha_y Y-\alpha_t T)\frac{\partial}{\partial U},$$

where $\alpha=\alpha(t,x,y)$ is an arbitrary function. In order to construct the one-parameter continuous Lie group which corresponds to each of the above infinitesimal generators, one needs to employ Lie's first fundamental theorem. We consider these equivalence transformations in the next section to derive invariants for the class of Eqs. (3).

3. Invariants for (3)

We call a function

$$J(t,x,y,u,X,Y,T,U,X_i,Y_i,T_i,U_i,\ldots),\quad i=t,x,y,$$

an *invariant* of the family of hyperbolic equations (3) if it is differential invariant under the equivalence group $\Gamma_1,\ldots,\Gamma_{10}$ and Γ_α. The function J is called *semi-invariant* if it is invariant only under the generator Γ_α. The order of the invariant is equal to the order of the highest derivative that appears in the form of J. If no derivatives appear, we say that we have invariants of zero order.

Note 1. The Laplace invariants h and k are semi-invariants, while the quantity $\frac{h}{k}$ is an invariant of first order for the linear hyperbolic equations (2).

Any system of equations $E_i(t,x,y,u,X,Y,T,U,X_i,Y_i,T_i,U_i,\ldots)=0$ that satisfies the condition

$$\Gamma_k^{(s)}(E_i)|_{E_1=0,E_2=0,\ldots}=0,\quad i=1,2,\ldots,$$

is called an *invariant system*. If

$$\Gamma_k^{(s)}(E_j)|_{E_j=0}=0,\quad j=1,2,\ldots,$$

then $E_j=0$ is called an *invariant equation*.

Note 2. Equations $h=0$ and $k=0$, where h and k are the Laplace invariants, are invariant equations for the linear hyperbolic equations (2).

Here we consider the problem of finding differential invariants of the class of Eqs. (3). First, we seek for differential invariants of zero order, i.e. invariants of the form

$$J=J(t,x,y,u,X,Y,T,U).$$

We apply the invariant test

$$\Gamma_i(J)=0,\quad i=1,2,\ldots,10,\alpha.$$

It is straightforward to show that $J=constant$. Hence, the family of Eqs. (3) does not admit differential invariants of zero order.

Next we consider the problem of existence of differential invariants of the form

$$J(t,x,y,u,X,Y,T,U,X_i,Y_i,T_i,U_i),\quad i=t,x,y.$$

We need to derive the once-extended generators of Γ_j, $j = 1, 2, \ldots, 10, \alpha$, using the following formulas:

$$\Gamma_j^{(1)} = \Gamma_j + \sigma_i^k \frac{\partial}{\partial f_i^k}, \quad i = 1, 2, 3,\ k = 1, 2, 3, 4,\ j = 1, 2, \ldots, 10, \alpha.$$

Here we use the local notation $f^1 = X$, $f^2 = Y$, $f^3 = T$, $f^4 = U$, $f_1^k = f_t^k$, $f_2^k = f_x^k$, $f_3^k = f_y^k$ and

$$\sigma_i^k = \widetilde{D}_i(\mu_k) - f_1^k \widetilde{D}_i(\xi_1) - f_2^k \widetilde{D}_i(\xi_2) - f_3^k \widetilde{D}_i(\xi_3), \quad i = 1, 2, 3,\ k = 1, 2, 3, 4,$$

where $\widetilde{D}_j$ $(j = 1, 2, 3)$ denote the total derivatives with respect to t, x and y, respectively.

We first calculate semi-invariants of first order by considering the invariant criterion

$$\Gamma_\alpha^{(1)}(J) = 0. \tag{6}$$

Eq. (6) is a polynomial in the derivatives of $\alpha(t, x, y)$. Using the fact that $\alpha(t, x, y)$ is arbitrary, we set the coefficients of the derivatives of it equal to zero. This leads to a system of linear first order partial differential equations. First we note that $\Gamma_i^{(1)} = \Gamma_i$, $i = 1, 2, 3$, and therefore, $\Gamma_i^{(1)}(J) = 0$ implies $J_t = J_x = J_y = 0$. Furthermore the coefficients of α, α_{xxy}, α_{xxt}, α_{xyy} in (6) give $J_u = J_{U_y} = J_{U_t} = J_{U_x} = 0$. Hence,

$$J = J(X, Y, T, U, X_t, X_x, X_y, Y_t, Y_x, Y_t, T_t, T_x, T_y).$$

Now coefficients of α_x, α_y, α_t, α_{xx}, α_{xy}, α_{xt}, α_{yy}, α_{yt} and α_{tt} in (6) give

$$\begin{aligned}
&2J_X + XJ_U = 0,\\
&2J_Y + YJ_U = 0,\\
&2J_T - TJ_U = 0,\\
&2J_{X_x} + J_U = 0,\\
&J_{X_y} + J_{Y_x} = 0,\\
&J_{X_t} - J_{T_x} = 0,\\
&2J_{Y_y} + J_U = 0,\\
&J_{Y_t} - J_{T_y} = 0,\\
&2J_{T_t} + J_U = 0.
\end{aligned}$$

Solving this system we obtain four independent integrals which form the set of semi-invariants of first order for the class of Eqs. (3):

$$J_1 = Y_x - X_y, \qquad J_2 = X_t + T_x, \qquad J_3 = Y_t + T_y, \qquad J_4 = X^2 + Y^2 - T^2 + 2X_x + 2Y_y + 2T_t - 4U. \tag{7}$$

Now, we apply the full equivalence group to derive the desired invariants, which are expected to be certain functions of the semi-invariants. That is, in addition to (6), we apply the invariance criterion

$$\Gamma_j^{(1)}(J) = 0, \quad j = 4, 5, \ldots, 10,$$

we obtain a list of seven equations which are tabulated in Appendix B. Using the semi-invariants (7), this latter system simplifies to

$$\begin{aligned}
&E_4: \quad J_1 J_{J_3} + J_3 J_{J_1} = 0,\\
&E_5: \quad J_1 J_{J_2} + J_2 J_{J_1} = 0,\\
&E_6: \quad J_3 J_{J_2} - J_2 J_{J_3} = 0,\\
&E_7: \quad J_1 J_{J_1} + J_2 J_{J_2} + J_3 J_{J_3} + J_4 J_{J_4} = 0,\\
&E_8: \quad 2tE_7 + xE_4 - yE_5 = 0,\\
&E_9: \quad tE_4 + 2xE_7 + yE_6 = 0,\\
&E_{10}: \quad tE_5 + xE_6 - 2yE_7 = 0.
\end{aligned}$$

Solving the system that contains equations E_4, E_5 and E_7, we obtain

$$J = \frac{J_2^2 + J_3^2 - J_1^2}{J_4^2}.$$

This solution also satisfies the remaining equations, E_6, E_8, E_9 and E_{10}. Hence we have derived the differential invariant of first order

$$J = \frac{(X_t + T_x)^2 + (Y_t + T_y)^2 - (Y_x - X_y)^2}{(X^2 + Y^2 - T^2 + 2X_x + 2Y_y + 2T_t - 4U)^2}. \tag{8}$$

This result re-confirms that the family of Eqs. (3) does not admit differential invariants of zero order. Furthermore we obtain the invariant system

$$X_t + T_x = 0, \qquad Y_t + T_y = 0, \qquad Y_x - X_y = 0 \tag{9}$$

and the invariant equation

$$X^2 + Y^2 - T^2 + 2X_x + 2Y_y + 2T_t - 4U = 0. \tag{10}$$

That is,

$$\Gamma_j^{(1)}(X_t + T_x)|_{(9)} = 0, \qquad \Gamma_j^{(1)}(Y_t + T_y)|_{(9)} = 0, \qquad \Gamma_j^{(1)}(Y_x - X_y)|_{(9)} = 0$$

and

$$\Gamma_j^{(1)}\big(X^2 + Y^2 - T^2 + 2X_x + 2Y_y + 2T_t - 4U\big)\big|_{(10)} = 0.$$

We point out that the calculation of invariant equations or/and systems is executed simultaneously with the calculation of the differential invariants. Here, the above invariant criteria only satisfied by the system (9) and by Eq. (10).

Now, in order to derive differential invariants of second order we need to consider the invariant criterion

$$\Gamma_i^{(2)}(J) = 0, \quad i = 1, 2, \ldots, 10, \alpha,$$

where $\Gamma_i^{(2)}$ is the second order extension of Γ_i. Without presenting any calculations we state that we only re-obtained the differential invariant (8). That is, there do not exist differential invariants of second order. The absence of differential invariants of second order can be shown via calculations of ranks of sub-matrices of the matrix whose entries are coefficients of the operators.

4. Equivalence transformations and invariants for (4)

We employ the same procedure used in the previous sections, to derive equivalence transformations and then differential invariants for the class (4).

We use Lie infinitesimal method for calculating the equivalence transformations of the class of Eqs. (4). We find that Eqs. (4) admit an infinite continuous group $\mathcal{E}$ of equivalence transformations generated by the Lie algebra $L_{\mathcal{E}}$ spanned by the operators:

$$\Gamma_1 = \frac{\partial}{\partial t}, \qquad \Gamma_2 = \frac{\partial}{\partial x}, \qquad \Gamma_3 = \frac{\partial}{\partial y}, \qquad \Gamma_4 = \frac{\partial}{\partial z},$$
$$\Gamma_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y} - T\frac{\partial}{\partial T} - Z\frac{\partial}{\partial Z} - 2U\frac{\partial}{\partial U},$$
$$\Gamma_6 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} + T\frac{\partial}{\partial X} + X\frac{\partial}{\partial T}, \qquad \Gamma_7 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y} + T\frac{\partial}{\partial Y} + Y\frac{\partial}{\partial T},$$
$$\Gamma_8 = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z} + Z\frac{\partial}{\partial T} + T\frac{\partial}{\partial Z}, \qquad \Gamma_9 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - Y\frac{\partial}{\partial X} + X\frac{\partial}{\partial Y},$$
$$\Gamma_{10} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + Z\frac{\partial}{\partial Y} - Y\frac{\partial}{\partial Z}, \qquad \Gamma_{11} = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + Z\frac{\partial}{\partial X} - X\frac{\partial}{\partial Z},$$
$$\Gamma_{12} = \frac{1}{2}\big(t^2 + x^2 + y^2 + z^2\big)\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} + tz\frac{\partial}{\partial z} + (xT - tX)\frac{\partial}{\partial X} + (yT - tY)\frac{\partial}{\partial Y}$$
$$+ (zT - tZ)\frac{\partial}{\partial Z} + (xX + yY - tT + zZ + 2)\frac{\partial}{\partial T} - 2tU\frac{\partial}{\partial U},$$
$$\Gamma_{13} = tx\frac{\partial}{\partial t} + \frac{1}{2}\big(t^2 + x^2 - y^2 - z^2\big)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z} - (xX + yY - tT + zZ + 2)\frac{\partial}{\partial X} + (yX - xY)\frac{\partial}{\partial Y}$$
$$+ (zX - xZ)\frac{\partial}{\partial Z} + (tX - xT)\frac{\partial}{\partial T} - 2xU\frac{\partial}{\partial U},$$
$$\Gamma_{14} = ty\frac{\partial}{\partial t} + xy\frac{\partial}{\partial x} + \frac{1}{2}\big(t^2 - x^2 + y^2 - z^2\big)\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z} + (xY - yX)\frac{\partial}{\partial X} - (xX + yY - tT + zZ + 2)\frac{\partial}{\partial Y}$$

$$
\begin{aligned}
&\quad + (zY - yZ)\frac{\partial}{\partial Z} + (tY - yT)\frac{\partial}{\partial T} - 2yU\frac{\partial}{\partial U},\\
&\Gamma_{15} = tz\frac{\partial}{\partial t} + xz\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} + \frac{1}{2}\big(t^2 - x^2 - y^2 + z^2\big)\frac{\partial}{\partial z} + (xZ - zX)\frac{\partial}{\partial X} + (yZ - zY)\frac{\partial}{\partial Y} + (tZ - zT)\frac{\partial}{\partial T}\\
&\quad - (xX + yY - tT + zZ + 2)\frac{\partial}{\partial Z} - 2zU\frac{\partial}{\partial U},\\
&\Gamma_\alpha = \alpha u\frac{\partial}{\partial u} - 2\alpha_x\frac{\partial}{\partial X} - 2\alpha_y\frac{\partial}{\partial Y} - 2\alpha_z\frac{\partial}{\partial Z} + 2\alpha_t\frac{\partial}{\partial T} + (\alpha_{tt} - \alpha_{xx} - \alpha_{yy} - \alpha_{zz} - \alpha_x X - \alpha_y Y - \alpha_t T - \alpha_z Z)\frac{\partial}{\partial U},
\end{aligned}
$$

where $\alpha = \alpha(x, t, y, z)$.

The invariant criterion $\Gamma_\alpha^{(1)}(J) = 0$ leads to seven semi-invariants:

$$
\begin{aligned}
&J_1 = Y_x - X_y, \qquad J_2 = X_t + T_x, \qquad J_3 = Y_t + T_y,\\
&J_4 = Z_x - X_z, \qquad J_5 = T_z + Z_t, \qquad J_6 = Z_y - Y_z,\\
&J_7 = X^2 + Y^2 + Z^2 - T^2 + 2X_x + 2Y_y + 2Z_z + 2T_t - 4U.
\end{aligned}
$$

Now using the complete equivalence group we find that the family of Eqs. (4) admits two functionally independent differential invariants of first order:

$$
\begin{aligned}
&J = \frac{(T_x + X_t)^2 + (T_y + Y_t)^2 + (T_z + Z_t)^2 - (Y_x - X_y)^2 - (Z_x - X_z)^2 - (Z_y - Y_z)^2}{(X^2 + Y^2 + Z^2 - T^2 + 2X_x + 2Y_y + 2Z_z + 2T_t - 4U)^2},\\
&I = \frac{(T_x + X_t)(Y_z - Z_y) - (T_y + Y_t)(X_z - Z_x) + (T_z + Z_t)(X_y - Y_x)}{(X^2 + Y^2 + Z^2 - T^2 + 2X_x + 2Y_y + 2Z_z + 2T_t - 4U)^2}.
\end{aligned}
$$

We note that, as expected, these two invariants are specific functions of the semi-invariants. In addition, the calculation of the two invariants produces an invariant system with six equations:

$$
X_t + T_x = 0, \qquad T_z + Z_t = 0, \qquad Y_t + T_y = 0, \qquad Y_x - X_y = 0, \qquad Z_x - X_z = 0, \qquad Z_y - Y_z = 0 \tag{11}
$$

and the invariant equation

$$
X^2 + Y^2 + Z^2 - T^2 + 2X_x + 2Y_y + 2Z_z + 2T_t - 4U = 0. \tag{12}
$$

In other words, system (11) is the only one which satisfies the invariant criterion

$$
\begin{aligned}
&\Gamma_j^{(1)}(X_t + T_x)|_{(11)} = 0, \qquad \Gamma_j^{(1)}(T_z + Z_t)|_{(11)} = 0, \qquad \Gamma_j^{(1)}(Y_t + T_y)|_{(11)} = 0,\\
&\Gamma_j^{(1)}(Y_x - X_y)|_{(11)} = 0, \qquad \Gamma_j^{(1)}(Z_x - X_z)|_{(11)} = 0, \qquad \Gamma_j^{(1)}(Z_y - Y_z)|_{(11)} = 0
\end{aligned}
$$

and Eq. (12) is the only one which satisfies the invariant criterion

$$
\Gamma_j^{(1)}\big(X^2 + Y^2 + Z^2 - T^2 + 2X_x + 2Y_y + 2Z_z + 2T_t - 4U\big)\big|_{(12)} = 0,
$$

where $\Gamma_j^{(1)}$ is the first order extension of generators Γ_j admitted by the family of Eqs. (4).

We note that the results are similar to the two-dimensional equations (3), with the exception that the three-dimensional equations (4) admit two differential invariants.

5. On *n*-dimensional hyperbolic equations

Motivated by the results of the previous sections, we can generalize them to n dimensions. We consider the linear hyperbolic equation

$$
u_{tt} = \sum_{i=1}^{n} u_{x_i x_i} + \sum_{i=1}^{n} X_i(x_1, x_2, \ldots, x_n, t)u_{x_i} + T(x_1, x_2, \ldots, x_n, t)u_t + U(x_1, x_2, \ldots, x_n, t)u, \quad n \geqslant 2. \tag{13}
$$

We tabulate the results in the following two theorems.

Theorem 1. *Eqs.* (13) *admit an infinite continuous group $\mathcal{E}$ of equivalence transformations generated by the Lie algebra $L_{\mathcal{E}}$ spanned by the operators*

$$\Gamma_{1_i} = \frac{\partial}{\partial x_i}, \quad i = 1, 2, \ldots, n, \qquad \Gamma_{1_{n+1}} = \frac{\partial}{\partial t},$$

$$\Gamma_2 = t\frac{\partial}{\partial t} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} X_i \frac{\partial}{\partial X_i} - T\frac{\partial}{\partial T} - 2U\frac{\partial}{\partial U},$$

$$\Gamma_{3_{ij}} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} + X_i \frac{\partial}{\partial X_j} - X_j \frac{\partial}{\partial X_i}, \quad i = 1, 2, \ldots, n-1, \ j = i+1, \ldots n,$$

$$\Gamma_{4_i} = x_i \frac{\partial}{\partial t} + t\frac{\partial}{\partial x_i} + T\frac{\partial}{\partial X_i} + X_i \frac{\partial}{\partial T}, \quad i = 1, 2, \ldots, n,$$

$$\Gamma_{5_i} = x_i t\frac{\partial}{\partial t} + \frac{1}{2}\left(t^2 + x_i^2 - \sum_{j=1,\, j\neq i}^{n} x_j^2\right)\frac{\partial}{\partial x_i} + \sum_{j=1,\, j\neq i}^{n} x_i x_j \frac{\partial}{\partial x_j} + \sum_{j=1,\, j\neq i}^{n} (x_j X_i - x_i X_j)\frac{\partial}{\partial X_j} + (tX_i - x_i T)\frac{\partial}{\partial T}$$
$$- \left(\sum_{j=1}^{n} x_j X_j - tT + n - 1\right)\frac{\partial}{\partial X_i} - 2x_i U\frac{\partial}{\partial U}, \quad i = 1, 2, \ldots, n,$$

$$\Gamma_{5_{n+1}} = \frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 + t\right)\frac{\partial}{\partial t} + \sum_{i=1}^{n} t x_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n}(x_i T - tX_i)\frac{\partial}{\partial X_i} + \left(\sum_{i=1}^{n} x_i X_i - tT + n - 1\right)\frac{\partial}{\partial T} - 2tU\frac{\partial}{\partial U},$$

$$\Gamma_\alpha = \alpha u\frac{\partial}{\partial u} - 2\sum_{i=1}^{n} \alpha_{x_i}\frac{\partial}{\partial X_i} + 2\alpha_t\frac{\partial}{\partial T} + \left(\alpha_{tt} - \sum_{i=1}^{n} \alpha_{x_i x_i} - \sum_{i=1}^{n} \alpha_{x_i} X_i - \alpha_t T\right)\frac{\partial}{\partial U}$$

where $\alpha = \alpha(t, x_1, x_2, \ldots, x_n)$ is an arbitrary function.

Theorem 2. *Eqs.* (13) *admit the invariant of first order, namely,*

$$J = \frac{\sum_{i=1}^{n}(T_{x_i} + X_{i_t})^2 - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(X_{i_{x_j}} - X_{j_{x_i}})^2}{(\sum_{i=1}^{n} X_i^2 - T^2 + 2\sum_{i=1}^{n} X_{i_{x_i}} + 2T_t - 4U)^2}.$$

We point out that Theorem 2 refers to one differential invariant. We know that for $n = 3$, two differential invariants exist. A related conjecture is presented in the conclusions.

The invariant criterion $\Gamma_\alpha^{(1)}(J) = 0$ leads to $\frac{1}{2}n(n+1) + 1$ semi-invariants:

$$J_i = T_{x_i} + X_{i_t}, \quad i = 1, 2, \ldots, n,$$

$$J_{ij} = X_{i_{x_j}} - X_{j_{x_i}}, \quad i = 1, 2, \ldots, n-1, \ j = i+1, \ldots, n,$$

$$J_{\frac{1}{2}n(n+1)+1} = \sum_{i=1}^{n} X_i^2 - T^2 + 2\sum_{i=1}^{n} X_{i_{x_i}} + 2T_t - 4U.$$

Furthermore we point out that the $\frac{1}{2}n(n+1)$ equations

$$T_{x_i} + X_{i_t} = 0, \quad i = 1, 2, \ldots, n, \qquad X_{i_{x_j}} - X_{j_{x_i}} = 0, \quad i = 1, 2, \ldots, n-1, \ j = i+1, \ldots, n,$$

form an invariant system and

$$\sum_{i=1}^{n} X_i^2 - T^2 + 2\sum_{i=1}^{n} X_{i_{x_i}} + 2T_t - 4U = 0$$

is an invariant equation. Semi-invariants, invariant system and invariant equation generalize naturally with no exceptions.

6. Applications

Two given partial differential equations are called equivalent if one can be transformed into the other by a change of variables. The equivalence problem consists of two parts: deciding if there exists equivalence and then determining a transformation that connects the partial differential equations. The motivation for considering this problem is to translate a known solution of a partial differential equation to solutions of others which are equivalent to this one.

In general, the equivalence problem is considered to be solved when a complete set of invariants has been found. In practice, using invariants to solve the equivalence problem for a given class of partial differential equations may require substantial computational effort. However any set of invariants can provide necessary conditions for deriving equivalent equations.

Here we consider the problem of finding those forms of the class (3) that can be mapped to an equation of the same class with constant coefficients. That is, we determine the forms of the functions $X(t,x,y)$, $Y(t,x,y)$, $T(t,x,y)$ and $U(t,x,y)$ such that Eqs. (3) are mapped into

$$u_{tt} = u_{xx} + u_{yy} + c_1 u_x + c_2 u_y + c_3 u_t + c_4 u, \tag{14}$$

where $c_1, \ldots, c_4$ are constants. Firstly, we note that the mapping

$$t' = at, \qquad x' = \varepsilon_1 ax, \qquad y' = \varepsilon_2 ay, \qquad u' = e^{\frac{1}{2}(c_1 x + c_2 y - c_3 t)} u$$

where a is an arbitrary constant, $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$, transforms

$$u'_{t't'} = u'_{x'x'} + u'_{y'y'} + \frac{4c_4 - c_1^2 - c_2^2 + c_3^2}{4a^2} u'$$

into (14). Hence, choosing the appropriate value of the parameter a, Eq. (14) is equivalent with

$$u'_{t't'} = u'_{x'x'} + u'_{y'y'} + u'. \tag{15}$$

Therefore we can, equivalently, consider the problem of finding those forms of the class (3) that can be mapped into (15) instead of those forms that can be mapped into (14).

In the special case $c_4 = \frac{1}{4}(c_1^2 + c_2^2 - c_3^2)$, Eq. (14) can be mapped into the two-dimensional linear wave equation

$$u'_{t't'} = u'_{x'x'} + u'_{y'y'}. \tag{16}$$

We point out that Eqs. (15) and (16) are inequivalent. Hence, there is merit to consider additionally the problem of finding those forms of the class (3) that can be mapped into (16).

Note 3. For equivalent Eqs. (14) and (15) the differential invariant J in Eq. (8) is equal to zero. Eqs. (14) and (16) satisfy the invariant system (9) and the invariant equation (10) only if the condition $c_4 = \frac{1}{4}(c_1^2 + c_2^2 - c_3^2)$ holds for Eqs. (14).

We state the results of this section in the following theorem. The proof can be carried out using first that equivalent equations have the same invariants or/and satisfy the invariant equations. This fact provides necessary conditions for connecting two equations. The second step is to find a point transformation that connects these equations (or special cases). Details of how such transformations are constructed can be found in [27,28].

Theorem 3. (i) *An equation of the class* (3) *can be mapped into the two-dimensional linear wave equation* (16) *by the point transformation which is a member of the equivalence transformations admitted by the class* (3)*, if and only if it is of the form*

$$\begin{aligned} u_{tt} &= u_{xx} + u_{yy} - F_x(t,x,y)u_x - F_y(t,x,y)u_y + F_t(t,x,y)u_t \\ &\quad + \frac{1}{4}\big[F_x^2 + F_y^2 - F_t^2 - 2(F_{xx} + F_{yy} - F_{tt})\big]u, \end{aligned} \tag{17}$$

where $F(t,x,y)$ is an arbitrary function. It can be shown that (17) *and* (16) *are connected by*

$$t' = t, \qquad x' = \varepsilon_1 x, \qquad y' = \varepsilon_2 y, \qquad u' = e^{-\frac{1}{2}F} u, \tag{18}$$

where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$.

(ii) *An equation of the class* (3) *can be mapped into the constant coefficient equation* (15) *by the point transformation* (18) *if and only if it is of the form*

$$\begin{aligned} u_{tt} &= u_{xx} + u_{yy} - F_x(t,x,y)u_x - F_y(t,x,y)u_y + F_t(t,x,y)u_t \\ &\quad + \frac{1}{4}\big[F_x^2 + F_y^2 - F_t^2 - 2(F_{xx} + F_{yy} - F_{tt}) + 4c^2\big]u. \end{aligned} \tag{19}$$

Note 4. Eqs. (16) and (17) satisfy the invariant system (9) and invariant equation (10). Eqs. (15) and (19) are such that the invariant (8) vanishes. This is the starting point for proving the above theorem.

Note 5. The results derived in this section can easily be generalized to n-dimensional equations of the class (13).

7. Conclusions

In the present paper we have derived differential invariants of an n-dimensional family of linear hyperbolic equations. To achieve this goal we had to classify the equivalence group for this family of equations. For the case $n = 2$ we obtained differential invariants of first order, but not of the second order. As it was expected, invariant differentiated operators cannot be formulated. Additionally, for the case $n = 2$ one invariant exists, but for the case $n = 3$ two invariants exist.

We conclude with the one conjuncture and one question.

Conjecture. *The family of n-dimensional linear hyperbolic equations* (13) *admits two differential invariants of first order if* $n = 3$. *For any other value of n it admits one differential invariant.*

The conjecture is supported by the fact that it agrees with $n = 2$, $n = 3$ and by the fact that it is valid for a number of other tested cases.

Question. For what values of n does the family of n-dimensional linear hyperbolic equations (13) admit differential invariants of second order?

Acknowledgments

C.T. would like to express her gratitude to the Cyprus Research Promotion Foundation for support through the grant CY-GR/0406/90. R.T. acknowledges the financial support from P.R.A. (ex 60%) of University of Catania and from M.I.U.R. through the PRIN 2005/2007: *Nonlinear Propagation and Stability in Thermodinamycal Processes of Continuous Media*. R.T. also expresses her gratitude to the hospitality shown by Prof. C. Sophocleous during her visit to the University of Cyprus. Finally, the authors would like to thank the referees for their suggestions for the improvement of this paper.

Appendix A

We consider the one-dimensional hyperbolic equation

$$u_{tt} = u_{xx} + X(t,x)u_x + T(t,x)u_t + U(t,x)u. \tag{A.1}$$

From the elementary study of partial differential equations, it is known that canonical variables connect the linear hyperbolic equations (2) and (A.1). Therefore the results of (2) [12,13] can be mapped into the results of (A.1) using the canonical variables. In fact, this procedure was carried out in [3]. However for completeness we present the results for the one-dimensional hyperbolic equation (A.1).

Equivalence transformations. The family of linear hyperbolic equations (A.1) has an infinite equivalence group $\mathcal{E}$. The corresponding Lie algebra $L_{\mathcal{E}}$ is spanned by the operators

$$\begin{aligned}
\Gamma_\phi &= -\phi\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial x} - \phi'(X+T)\frac{\partial}{\partial X} - \phi'(X+T)\frac{\partial}{\partial T} - 2\phi' U\frac{\partial}{\partial U},\\
\Gamma_\psi &= \psi\frac{\partial}{\partial t} + \psi\frac{\partial}{\partial x} - \psi'(X-T)\frac{\partial}{\partial X} + \psi'(X-T)\frac{\partial}{\partial T} - 2\psi' U\frac{\partial}{\partial U},\\
\Gamma_\alpha &= \alpha u\frac{\partial}{\partial u} - 2\alpha_x\frac{\partial}{\partial X} + 2\alpha_t\frac{\partial}{\partial T} + (\alpha_{tt} - \alpha_{xx} - \alpha_x X - \alpha_t T)\frac{\partial}{\partial U},
\end{aligned}$$

where $\phi = \phi(x-t)$, $\psi = \psi(x+t)$, $\alpha = \alpha(x,t)$ are arbitrary functions. We note that the above equivalence group is not a special form of the equivalence group of the family of n-dimensional linear hyperbolic equations (13).

First order semi-invariants for Γ_α. The invariant criterion $\Gamma_\alpha^{(1)}(J) = 0$ leads to two semi-invariants:

$$\begin{aligned}
J_1 &= X_t + T_x,\\
J_2 &= X^2 - T^2 + 2(X_x + T_t) - 4U.
\end{aligned}$$

These semi-invariants can be transformed into Laplace invariants, using canonical variables. We also point out that $J_1 = 0$ and $J_2 = 0$ are invariant equations.

Invariant of first order. We obtain one functionally independent differential invariant of first order

$$J = \frac{X_t + T_x}{X^2 - T^2 + 2(X_x + T_t) - 4U},$$

which is a function of the semi-invariants, as expected. Unlike the equivalence transformations of (A.1), the above differential invariant can be obtained from the general case by setting $n = 1$. However the family (A.1) admits differential invariants of higher order [13].

Appendix B

The invariance criterion

$$\Gamma_j^{(1)}(J) = 0, \quad j = 4, 5, \ldots, 10,$$

where $\Gamma_j^{(1)}$ is the first order extension of generators Γ_j admitted by the family of Eqs. (3), produces the system

$$E_4: \quad T J_X - X_x J_{X_t} + T_t J_{X_t} - X_t J_{X_x} + T_x J_{X_x} + T_y J_{X_y} - Y_x J_{Y_t} - Y_t J_{Y_x} + X J_T + X_t J_{T_t} - T_x J_{T_t} + X_x J_{T_x} - T_t J_{T_x} + X_y J_{T_y} = 0,$$

$$E_5: \quad -X_y J_{X_t} - X_t J_{X_y} + T J_Y - Y_y J_{Y_t} + T_t J_{Y_t} + T_x J_{Y_x} - Y_t J_{Y_y} + T_y J_{Y_y} + Y J_T + Y_t J_{T_t} - T_y J_{T_t} + Y_x J_{T_x} + Y_y J_{T_y} - T_t J_{T_y} = 0,$$

$$E_6: \quad -Y J_X - Y_t J_{X_t} - X_y J_{X_x} - Y_x J_{X_x} + X_x J_{X_y} - Y_y J_{X_y} + X J_Y + X_t J_{Y_T} + X_x J_{Y_x} - Y_y J_{Y_x} + X_y J_{Y_y} + Y_x J_{Y_y} - T_y J_{T_x} + T_x J_{T_y} = 0,$$

$$E_7: \quad -X J_X - 2X_t J_{X_t} - 2X_x J_{X_x} - 2X_y J_{X_y} - Y J_Y - 2Y_t J_{Y_t} - 2Y_x J_{Y_x} - 2Y_y J_{Y_y} - T J_T - 2T_t J_{T_t} - 2T_x J_{T_x} - 2T_y J_{T_y} - 2U J_U = 0,$$

$$E_8: \quad tE_7 + xE_4 + yE_5 - X J_{X_t} + T J_{X_x} - Y J_{Y_t} + T J_{Y_y} + J_T - T J_{T_t} + X J_{T_x} + Y J_{T_y} = 0,$$

$$E_9: \quad tE_4 + xE_7 + yE_6 - J_X + T J_{X_t} - X J_{X_x} - Y J_{X_y} - Y J_{Y_x} + X J_{Y_y} + X J_{T_t} - T J_{T_x} = 0,$$

$$E_{10}: \quad tE_5 - xE_6 + yE_7 + Y J_{X_x} - X J_{X_y} - J_Y + T J_{Y_t} - X J_{Y_x} - Y J_{Y_y} + Y J_{T_t} - T J_{T_Y} = 0,$$

where we have used that $J_t = J_x = J_y = J_u = J_{U_y} = J_{U_t} = J_{U_x} = 0$.

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