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ON SUBSCHEMES OF 0-DIMENSIONAL SCHEMES WITH GIVEN GRADED BETTI NUMBERS

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ABSTRACT. We study the subschemes of a 0-dimensional scheme X for which either the Hilbert function or the graded Betti numbers are known. In the first case we find which kind of subscheme cannot stay in X and in the codimension 2 case what subschemes must be in X. In the case of graded Betti numbers we study the case of 2-codimensional partial intersection schemes or Artinian monomial ideals. More generally we give complete results for almost complete intersections and for other suitable Betti sequences.

Introduction. To understand the geometry of a 0-dimensional scheme X one should know which kind of subschemes it can contain. On this way we can see the Cayley-Bacharach property studied in [**3**] or the uniform position property stated in [**4**]. Of course, as much one knows about the 0-dimensional scheme X as much one can say about its subschemes. The first possible information regards the Hilbert functions of the subschemes. So one is interested in knowing the Hilbert functions of the subschemes of a scheme X having Hilbert function H. More precisely, one would like to know which Hilbert functions H' must necessarily live in X and which ones cannot stay in X. Then, if X is a 0-dimensional scheme of $\mathbf{P}^r$, H is a 0-dimensional O-sequence, $\psi = \Delta H$ one sets

$$\begin{aligned}\mathcal{H}_X &= \{\varphi \mid \exists Y \subseteq X \text{ such that } \Delta H_Y = \varphi\}\\ \mathcal{H}_H^{(\mathrm{gen})} &= \{\varphi \mid \forall X \text{ with } \Delta H_X = \psi \ \exists Y \subseteq X \text{ with } \Delta H_Y = \varphi\}\\ \mathcal{H}_H &= \{\varphi \mid \exists X \text{ and } \exists Y \subseteq X \text{ with } \Delta H_X = \psi \text{ and } \Delta H_Y = \varphi\}.\end{aligned}$$

Since $\mathcal{H}_H^{(\mathrm{gen})} \subseteq \mathcal{H}_X \subseteq \mathcal{H}_H$, where $H = H_X$, to have information about $\mathcal{H}_H^{(\mathrm{gen})}$ will mean to know which subschemes must necessarily

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be inside X and, on the other hand, information about $\mathcal{H}_H$ will say which subschemes cannot be in X. It is clear that every O-sequence $\Delta H'$ which is a kind of truncation of ΔH must belong to $\mathcal{H}_H^{(\mathrm{gen})}$. Nevertheless, in general, such O-sequences do not fill up $\mathcal{H}_H^{(\mathrm{gen})}$. Of course, if H is a possible Hilbert function for a UPP scheme then $\mathcal{H}_H^{(\mathrm{gen})}$ will consist only of those truncations of ΔH. Unfortunately, it is not yet known when an O-sequence is the Hilbert function of a scheme with UPP, except for the codimension 2 case (i.e. $\Delta H(1) = 2$) for which it is known that H is the Hilbert function of a UPP scheme if and only if it is of decreasing type (see [**6**]). Because of this result we can characterize $\mathcal{H}_H^{(\mathrm{gen})}$ (which is the same as in the reduced case) for every H of codimension 2 (Proposition 2.1). The description of the set $\mathcal{H}_H$ is quite easy. Indeed, one realizes very soon that if $\Delta H' \in \mathcal{H}_H$ then $\Delta H'(i) \le \Delta H(i)$ for all i. Now we show in Theorem 2.3 that this is also a sufficient condition for an O-sequence in order to stay in $\mathcal{H}_H$.

The situation appears more complicated when we have stronger information about the scheme X, e.g., when we know its graded Betti sequence $\beta = \{\beta_{ij}\}$. In this case, if β is an admissible graded Betti sequence, we define

$$\mathcal{H}_\beta = \{\varphi \mid \exists X \text{ and } Y \subseteq X \text{ with } B_X = \beta \text{ and } \Delta H_Y = \varphi\}.$$

Of course, if H is the Hilbert function corresponding to the Betti sequence β we have $\mathcal{H}_X \subseteq \mathcal{H}_\beta \subseteq \mathcal{H}_H$; hence, to know $\mathcal{H}_\beta$ means to have a stronger control on $\mathcal{H}_X$ than we have with $\mathcal{H}_H$.

A simple case in which we get the equality for $\mathcal{H}_\beta$ and $\mathcal{H}_H$ occurs when the Betti sequence β is maximal with respect to its Hilbert function H according to the results of [**1, 5, 9**] (cf. Theorem 3.1).

Now, in these kinds of problems, the codimension c plays a crucial role; indeed, many things that can be said in codimension 2 are no longer true when the codimension is bigger than 2. For instance, if β is the graded Betti sequence of a complete intersection, with Hilbert function H, it is easy to see that if $\Delta H' \in \mathcal{H}_\beta$ then $(\Delta H - \Delta H')^*$ must be an O-sequence, because of a liaison result. Now, this simple condition is sufficient to recover $\mathcal{H}_\beta$ in codimension 2 (Proposition 3.2) but simple examples show that it is not sufficient for $\Delta H'$ to stay in $\mathcal{H}_\beta$ when the codimension of H is ≥ 3. In codimension 2, if X is a partial intersection or I is an Artinian monomial ideal with graded Betti sequence β, we can characterize the elements in $\mathcal{H}_\beta$. More generally we are able to give a similar characterization for special graded Betti sequences, in particular for almost complete intersections.

1. Partial intersections: Definitions, properties and facts. Throughout this paper k will denote an algebraically closed field, $\mathbf{P}^r$ the r-dimensional projective space over k, $R = k[x_0, x_1, \ldots, x_r] = \bigoplus_{n\in\mathbf{Z}} H^0(\mathscr{O}_{\mathbf{P}^r}(n))$.

If $V \subset \mathbf{P}^r$ is a subscheme, I_V will denote its defining ideal, $H_V(n) = \dim_k R_n - \dim_k (I_V)_n$ its Hilbert function and $\Delta^{t+1} H_V(n) = \Delta^t H_V(n) - \Delta^t H_V(n-1)$ the $(t+1)$-th difference of the Hilbert function. Moreover, if $V \subset \mathbf{P}^r$ is a c-codimensional aCM scheme with minimal free resolution

$$0 \to \bigoplus_{j=1}^{n_{c-1}} R(-\beta_{c-1,j}) \to \cdots \to \bigoplus_{j=1}^{n_1} R(-\beta_{1j}) \\ \to \bigoplus_{j=0}^{n_0} R(-\beta_{0j}) \to I_V \to 0$$

with $0 < \beta_{ij} \leq \beta_{i,j+1}$ for all i, j and with n_i the i-th total Betti number, then the sequence

$$\beta(I_V) = \beta(V) := \big((\beta_{01}, \ldots \beta_{0n_0}); (\beta_{11}, \ldots \beta_{1n_1}); \\ \cdots ; (\beta_{c-1,1}, \ldots \beta_{c-1,n_{c-1}})\big)$$

will be called the *Betti sequence* of V or I_V.

In this section we recall the construction of the c-codimensional partial intersection schemes which were introduced in [**8**] and generalized in [**10**], and we collect from there the main facts that will be used in this paper.

If $(\mathcal{P}, \leq)$ is a poset, we denote, for every $H \in \mathcal{P}$,

$$\mathcal{S}_H = \{K \in \mathcal{P} \mid K < H\}, \quad \overline{\mathcal{S}}_H = \{K \in \mathcal{P} \mid K \leq H\}.$$

In the sequel by $\mathbf{N}$ we will mean the set of positive integers.

Definition 1.1. A subset $\mathcal{A}$ of the poset $\mathcal{P}$ is said to be a *left segment* if for every $H \in \mathcal{A}$, $\mathcal{S}_H \subseteq \mathcal{A}$. In particular, when $\mathcal{P} = \mathbf{N}^c$ with the ordering induced by the natural ordering on $\mathbf{N}$, a finite left segment will be mentioned as a *c-left segment.*

If $L \subset \mathbf{N}^c$, the set $\langle L \rangle := \{H \in \mathbf{N}^c \mid H \leq K \text{ for some } K \in L\}$ is the c-left segment generated by L. Note that every c-left segment $\mathcal{A}$ has a unique minimal set of generators consisting of the maximal elements of $\mathcal{A}$; we will denote it by $G(\mathcal{A})$. We use the notation $\mathcal{A} = \langle H_1, \dots, H_t \rangle$ to say that $\mathcal{A}$ is generated by $H_1, \dots, H_t$.

If $\pi_i : \mathbf{N}^c \to \mathbf{N}$ will denote the projection to the i-th component, and $\mathcal{A}$ is a c-left segment, we set $a_i = \max\{\pi_i(H) \mid H \in \mathcal{A}\}$, for $1 \leq i \leq c$. The c-tuple $T = T(\mathcal{A}) = (a_1, \dots, a_c)$ will be called the *size* of $\mathcal{A}$.

If $H = (h_1, \dots, h_c) \in \mathcal{A}$ we denote $v(H) = h_1 + \cdots + h_c$. If $\mathcal{A}$ is a c-left segment, $F(\mathcal{A})$ will denote the set of minimal elements of $\mathbf{N}^c \setminus \mathcal{A}$, i.e.,

$$F(\mathcal{A}) = \{H \in \mathbf{N}^c \setminus \mathcal{A} \mid \mathcal{S}_H \subseteq \mathcal{A}\}.$$

Note that, if $H = (m_1, \dots, m_c)$ is in $F(\mathcal{A})$ and $m_i > 1$, then each $H_i = (m_1, \dots, m_i - 1, \dots, m_c) \in \mathcal{A}$.

Fix a c-left segment $\mathcal{A}$ and consider c families of hyperplanes of $\mathbf{P}^r$, $c \leq r$,

$$\{A_{1j}\}_{1 \leq j \leq a_1}, \quad \{A_{2j}\}_{1 \leq j \leq a_2}, \dots, \{A_{cj}\}_{1 \leq j \leq a_c}$$

sufficiently generic, in the sense that $A_{1j_1} \cap \ldots \cap A_{cj_c}$ are $\prod_{i=1}^{c} a_i$ pairwise distinct linear varieties of codimension c.

For every $H = (j_1, \dots, j_c) \in \mathcal{A}$, we denote by

$$L_H = \bigcap_{h=1}^{c} A_{hj_h}.$$

With this notation we have the following

Definition 1.2. The subscheme of $\mathbf{P}^r$

$$V = \bigcup_{H \in \mathcal{A}} L_H$$

will be called a *c-partial intersection* with respect to the hyperplanes $\{A_{ij}\}$ and support on the c-left segment $\mathcal{A}$.

Theorem 1.3. *Every c-partial intersection X of $\mathbf{P}^r$ is a reduced* aCM *subscheme consisting of a union of c-codimensional linear varieties.*

Proof. See Theorem 1.9 in [**10**]. □

Here are the main results on c-codimensional partial intersections.

Theorem 1.4. *If $V \subset \mathbf{P}^r$ is a partial intersection of codimension c with support on $\mathcal{A}$, then the $(r-c+1)$-th difference of its Hilbert function is*

$$\Delta^{r-c+1} H_V(n) = \Big| \{H \in \mathcal{A} \mid v(H) = n + c\} \Big| .$$

Proof. See Theorem 2.1 in [**10**]. □

That function will be called the Hilbert function of the left segment $\mathcal{A}$.

Now, if X is a c-codimensional partial intersection with support on $\mathcal{A}$ and with respect to the families of hyperplanes A_{ij} whose defining forms are f_{ij}, to every $H = (m_1, \dots , m_c)$ such that $m_i \leq a_i + 1$ for all i, we associate the following form

$$P_H = \prod_{i=1}^{c} \prod_{j=1}^{m_i-1} f_{ij}.$$

Theorem 1.5. *Let $V \subset \mathbf{P}^r$ be a partial intersection of codimension c with support $\mathcal{A}$. Then a minimal set of generators for I_V is*

$$\{P_H \mid H \in F(\mathcal{A})\}.$$

Proof. See Theorem 3.1 in [**10**]. □

Corollary 1.6. *Let V be as above. Then its first graded Betti numbers depend only on $\mathcal{A}$, and they are the following integers*

$$d_H = v(H) - c \quad \forall H \in F(\mathcal{A}).$$

And finally,

Theorem 1.7. *Let $V \subset \mathbf{P}^r$ be a partial intersection of codimension c with support $\mathcal{A}$. Then the last graded Betti numbers of V are*

$$s_H = v(H) \quad \forall H \in G(\mathcal{A}).$$

Proof. See Theorem 3.4 in [**10**]. □

Since we are essentially interested in Hilbert functions and graded Betti numbers of c-codimensional aCM schemes we can deal with 0-dimensional schemes X of $\mathbf{P}^c$. In this case the Hilbert function $H_X = H$ of X is a differentiable 0-dimensional O-sequence, i.e., $\Delta H_X = \Delta H$ is an O-sequence too, with $\Delta H(n) = 0$ for $n \gg 0$. If $\Delta H(1) = c$ we say that the O-sequence ΔH is c-codimensional.

Remark 1.8. We used c-left segments $\mathcal{A}$ to produce partial intersection schemes. On the other hand, one can easily construct a one-to-one correspondence between c-left segments and monomial Artinian ideal in $R = k[x_1, \dots, x_c]$. Precisely, if we define for each $H = (h_1, \dots, h_c) \in \mathbf{N}^c$ the monomial $x^H = x_1^{h_1-1} \cdots x_c^{h_c-1} \in R$, we can associate to $\mathcal{A}$ the monomial ideal $I_{\mathcal{A}}$ of R generated by $\{x^H \mid H \in F(\mathcal{A})\}$. Note that, since $T_i = (1, \dots, a_i, \dots, 1)$ for every $i = 1, \dots, c$ belongs to $F(\mathcal{A})$, we have that $I_{\mathcal{A}}$ is Artinian. Vice versa, if J is an Artinian monomial ideal in R, generated by monomials $x^{K_1}, \dots x^{K_t}$, and define $U = (1, \dots, 1)$ and $H_i = K_i + U$ for $i = 1, \dots, t$, then $\mathcal{A}_J = \{H \in \mathbf{N}^c \mid \nexists\, x^K \in J \text{ with } H \geq K + U\}$ is a left segment. Moreover, by definition and Theorem 1.5, for every partial intersection X whose support is $\mathcal{A}$ we have $\beta_{1i}(X) = \beta_{1i}(I_{\mathcal{A}})$ for each i. On the other hand, since $x^H \notin I_{\mathcal{A}}$ just means that $H \in \mathcal{A}$, using Theorem 1.4 we deduce that $I_{\mathcal{A}}$ shares the Hilbert function of I_V for every partial intersection V with support on $\mathcal{A}$. This implies that in codimension 2, $\beta(I_{\mathcal{A}}) = \beta(I_V)$. Indeed, we believe that the same conclusion is true in any codimension.

2. Subschemes according to Hilbert function. In this section we work with a 0-dimensional scheme for which the Hilbert function is known. Since to understand its subschemes one possibility could be to know their Hilbert functions, we will use the following terminology. In the sequel X will denote a 0-dimensional scheme of $\mathbf{P}^r$, $X' \subseteq X$ will mean X' is a subscheme of X; H, H' will denote 0-dimensional O-sequences, $\Delta H = \varphi$, $\Delta H' = \varphi'$ their first difference, and $\sigma_H = \max\{j \in \mathbf{N} \mid \Delta H(j) \neq 0\}$. Moreover, we denote

$$\begin{aligned}\mathcal{H}_X &= \{\varphi' \mid \exists X' \subseteq X \text{ such that } \Delta H_{X'} = \varphi'\}\\ \mathcal{H}_H^{(\text{gen})} &= \{\varphi' \mid \forall X \text{ with } \Delta H_X = \varphi \exists X' \subseteq X \text{ with } \Delta H_{X'} = \varphi'\}\\ \mathcal{H}_H &= \{\varphi' \mid \exists X' \subseteq X \text{ with } \Delta H_X = \varphi \text{ and } \Delta H_{X'} = \varphi'\}.\end{aligned}$$

Since $\mathcal{H}_H^{(\text{gen})} \subseteq \mathcal{H}_X \subseteq \mathcal{H}_H$, where $H = H_X$, to have information on $\mathcal{H}_H^{(\text{gen})}$ means which subschemes must be necessarily inside X and, on the other hand, to know $\mathcal{H}_H$ will say which subschemes cannot be in X.

Note that the sets $\mathcal{H}_H^{(\text{gen})}$ and $\mathcal{H}_H$ can be defined from the algebraic point of view. Precisely, if $R = k[x_1, \dots, x_r]$, it is enough to substitute X with an Artinian algebra R/I and $X' \subseteq X$ with an Artinian algebra R/I' with $I \subseteq I'$.

Generalizing the classical notion of *truncation*, we say that an O-sequence $\Delta H'$ is a *generalized truncation* of $\Delta H \geq \Delta H'$ if $\Delta H'(i) = \Delta H(i)$ for all $i < \sigma_{H'}$. Since for any Artinian ideal I in R we can find an (Artinian) ideal $I' \supseteq I$ such that

$$\Delta H_{R/I'}(i) = \begin{cases} \Delta H_{R/I}(i) & \text{if } i < \sigma_{H_{R/I}} \\ \Delta H_{R/I}(i) - 1 & \text{if } i = \sigma_{H_{R/I}} \end{cases}$$

(just add a form of degree $\sigma_{H_{R/I}}$) it is clear that every O-sequence $\Delta H'$ which is a generalized truncation of ΔH belongs to $\mathcal{H}_H^{(\text{gen})}$. Nevertheless, in general, such O-sequences do not fill up $\mathcal{H}_H^{(\text{gen})}$. For instance, if $\Delta H :\ 1\ 2\ 1\ 1\ 0 \to$, then $\mathcal{H}_H^{(\text{gen})}$ contains the O-sequence $\Delta H' :\ 1\ 1\ 1\ 1\ 0 \to$ (which is not a generalized truncation of ΔH).

Indeed, any scheme X with Hilbert function H should contain a subscheme X' of degree 4 on a line!

Of course, when X is a reduced 0-dimensional scheme we can restrict our investigation to reduced subschemes. Indeed, if H is a possible Hilbert function for a UPP scheme (see [**4**] for definitions) then $\mathcal{H}_H^{(\text{gen})}$ will consist only of the generalized truncations of ΔH. Unfortunately, it is not yet known when an O-sequence is the Hilbert function of a scheme with UPP, except for the codimension 2 case (i.e. $\Delta H(1) = 2$) for which it is known that H is the Hilbert function of an UPP scheme if and only if it is of decreasing type (see **6**]). Because of this result we can characterize $\mathcal{H}_H^{(\text{gen})}$ for every H of codimension 2. To do this, since it is known that ΔH is not of increasing type, let us define the following integers

$$\begin{aligned}
&a = \min\{j \mid \Delta H(j) < j+1\} \\
&b = \min\{j \geq a \mid \Delta H(j) < a\} \\
&\text{and } a-1 \leq c_1 < c'_1 < \cdots < c_t < c'_t \leq c_{t+1} = \sigma_H \\
&\text{the integers such that for all } i = 1, \dots, t, \\
&\Delta H(c_i) = \Delta H(c'_i) = d_i \text{ and} \\
&\text{for every } c'_i \leq u < v \leq c_{i+1} \ \Delta H(u) > \Delta H(v).
\end{aligned}$$

Of course, if H is of decreasing type we have just $c'_1 = b-1$ and $c_2 = \sigma_H$; moreover, $d_i > d_{i+1}$ and we use $d_{t+1} = 0$. Note that, with this terminology, we can decompose ΔH into t O-sequences of decreasing type defined inductively by:

$$\Delta H_t(i) = \min\{\Delta H(i), d_t\}$$

and for $j = 1, \dots, t-1$,

$$\Delta H_j(i) = \min\left\{\Delta H(i+d_{j+1}) - \sum_{l=j+1}^{t} \Delta H_l(i+d_{j+1}-d_{l+1}), d_j - d_{j+1}\right\}$$

from which we get

$$\Delta H(i) = \sum_{j=1}^{t} \Delta H_j(i - d_{j+1}).$$

Now we are ready to state the characterization of $\mathcal{H}_H^{(\mathrm{gen})}$ for every H of codimension 2.

Proposition 2.1. *Let H be an O-sequence of codimension 2. Then $\Delta H' \in \mathcal{H}_H^{(\mathrm{gen})}$ if and only if, with the above terminology, $\Delta H'$ can be decomposed as*

$$\Delta H'(i) = \sum_{j=1}^{t} \Delta H_j'(i - d_{j+1})$$

where each $\Delta H_j'$ is a generalized truncation of ΔH_j.

Proof. One can deduce, by [**7**, Theorem 2.9], that every 0-dimensional scheme X of $\mathbf{P}^2$ with Hilbert function H must be of the form $X = \bigcup_{i=1}^t X_i$ where $H_{X_i} = H_i$. Now, since for every generalized truncation H_i' of H_i we can find a subscheme X_i' of X_i whose Hilbert function is H_i', one easily gets that $X' = \bigcup_{i=1}^t X_i'$ is a subscheme of X with Hilbert function H'. On the other hand, because of the decreasing type property of each H_i, one can construct for every $i = 1, \ldots, t$ a 0-dimensional scheme Y_i of $\mathbf{P}^2$ in UPP with Hilbert function H_i; moreover, one can construct such schemes so that they are generic to each other in the sense that the Hilbert function of $Y = \bigcup_{i=1}^t Y_i$ is given by $\sum_{j=1}^t \Delta H_j(i - d_{j+1}) = \Delta H(i)$. This choice can be done since in the irreducible scheme $\mathrm{Hilb}^{H_1} \times \ldots \times \mathrm{Hilb}^{H_t}$ the subset consisting of the points satisfying the above additive property is open and not empty (by the argument used in the first part of the proof) and, on the other hand, the UPP property is also open. Now, the subschemes of such a scheme Y must be $Y' = \bigcup_{i=1}^t Y_i'$ with $Y_i' \subseteq Y_i$ which, for the UPP property, have as Hilbert function a generalized truncation H_i' of H_i and consequently Y' has Hilbert function H' with $\Delta H'(i) = \sum_{j=1}^t \Delta H_j'(i - d_{j+1})$ as required. □

Example 2.2. As an application of the previous proposition we see that if

$$\Delta H: \; 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 10\ 10\ 9\ 7\ 7\ 7\ 6\ 4\ 3\ 3\ 1$$

we have $\Delta H': \; 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 10\ 8\ 4\ 3\ 3\ 3\ 3\ 3\ 2$ belongs to $\mathcal{H}_H^{(\mathrm{gen})}$ and $\Delta H'': \; 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 10\ 10\ 9\ 6\ 5$ does not belong to $\mathcal{H}_H^{(\mathrm{gen})}$.

Indeed, each scheme X with Hilbert function H should consist of the union of 3 subschemes X_1, X_2, X_3 with Hilbert functions, respectively, H_1, H_2, H_3 with

$$\Delta H_3: \ 1\ 2\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 1$$
$$\Delta H_2: \ 1\ 2\ 3\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 3\ 1$$
$$\Delta H_1: \ 1\ 2\ 3\ 3\ 3\ 2;$$

therefore, if we take the following generalized truncations

$$\text{Trunc}_3: \ 1\ 2\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 2$$
$$\text{Trunc}_2: \ 1\ 2\ 3\ 4\ 4\ 4\ 4\ 4\ 4\ 1$$
$$\text{Trunc}_1: \ 1\ 2\ 3\ 3\ 1$$

we can find 3 subschemes $X_1' \subseteq X_1$, $X_2' \subseteq X_2$, $X_3' \subseteq X_3$, with first difference Hilbert function, respectively, Trunc_1, Trunc_2, Trunc_3 such that $X' = X_1' \cup X_2' \cup X_3'$ has first difference Hilbert function $\Delta H'$, i.e., $\Delta H' \in \mathcal{H}_H^{(\text{gen})}$.

Now, if in the generic X with Hilbert function H there were a subscheme X'' with Hilbert function H'' we have that $X_1'' = X'' \cap X_1$, $X_2'' = X'' \cap X_2$ and $X_3'' = X'' \cap X_3$, should have first difference Hilbert functions some generalized truncation of ΔH_1, ΔH_2 and ΔH_3, respectively. Now, since $\Delta H''(13) = 6$ we have only two possibilities: either $\Delta H_1''(13) = 3$ and $\Delta H_2''(10) = 3$ ($\Delta H_3''(6) = 0$) or $\Delta H_1''(13) = 2$ and $\Delta H_2''(10) = 4$; in the first case $\Delta H_2''$ must be a generalized truncation at $n = 10$ hence $\Delta H''(14) \leq 3$ and in the second case $\Delta H_1''$ must be a generalized truncation at $n = 13$ hence $\Delta H''(14) \leq 4$; both are contradictions to $\Delta H''(14) = 5$.

The description of the set $\mathcal{H}_H$ is quite easy. Indeed, one realizes very soon that if $\Delta H' \in \mathcal{H}_H$, then $\Delta H'(i) \leq \Delta H(i)$ for all i. Now we will see that this is also a sufficient condition for an O-sequence in order to stay in $\mathcal{H}_H$.

Theorem 2.3. *Let H be an admissible Hilbert function for a 0-dimensional subscheme of $\mathbf{P}^c$; then an O-sequence $\Delta H'$ belongs to $\mathcal{H}_H$ if and only if $\Delta H'(i) \leq \Delta H(i)$ for all i.*

Proof. Using the maximal decomposition for $\Delta H'$ and ΔH used in [**2**], we can build two c-left segments $\mathcal{A}' \subseteq \mathcal{A}$ such that every pair of partial intersections $X' \subseteq X$ with support on $\mathcal{A}'$ and $\mathcal{A}$, respectively, have Hilbert functions $H_{X'} = H'$ and $H_X = H$. More precisely, one finds the maximal O-sequence ΔH_1 of codimension $\leq c-1$ such that $\Delta H - \Delta H_1 := \Delta \overline{H}_1$ is still an O-sequence. Repeating the argument one can define the O-sequence ΔH_i as the maximal O-sequence such that $\Delta \overline{H}_{i-1} - \Delta H_i := \Delta \overline{H}_i$ is again an O-sequence. We stop the procedure at the first integer t such that $\Delta \overline{H}_t$ is an O-sequence of codimension $< c$. With these definitions we have $\Delta H(n) = \sum_{i=1}^{t+1} \Delta H_i(n+1-i)$, where $\Delta H_{t+1} = \Delta \overline{H}_t$. Now, since $\Delta H' \leq \Delta H$, using the same decomposition for $\Delta H'$, we get $\Delta H'(n) = \sum_{i=1}^{s} \Delta H'_i(n+1-i)$, where $s \leq t$ and, for each j, $\Delta H'_j \leq \Delta H_j$ Now, by induction on c, we have $(c-1)$-left segments $\{\mathcal{A}'_i\}_{i=1}^{s}$ and $\{\mathcal{A}_i\}_{i=1}^{t}$, with $\mathcal{A}'_i \subseteq \mathcal{A}_i$ for $i = 1, \dots, s$, and whose Hilbert functions are, respectively, $\Delta H'_i$ and ΔH_i. Now, defining the c-left segments $\mathcal{A}' = \{(i, H) \mid \text{for all } i = 1, \dots, s; \text{ for all } H \in \mathcal{A}'_i\}$ and $\mathcal{A} = \{(i, H) \mid \text{for all } i = 1, \dots, t; \text{ for all } H \in \mathcal{A}_i\}$, we get partial intersections $X' \subseteq X$ with support $\mathcal{A}'$ and $\mathcal{A}$, and Hilbert functions H' and H, respectively. □

3. Subschemes according to Betti sequence in codimension 2. The situation appears more complicated when we have stronger information about the scheme X, that is, when we know its graded Betti sequence $B_X = \beta = \{\beta_{ij}\}$. For this, for every graded Betti sequence β, we define

$$\mathcal{H}_\beta = \{\varphi \mid \exists X \text{ and } \exists Y \subseteq X \text{ with } B_X = \beta \text{ and } \Delta H_Y = \varphi\}.$$

Of course, if H is the Hilbert function corresponding to the Betti sequence β we have $\mathcal{H}_X \subseteq \mathcal{H}_\beta \subseteq \mathcal{H}_H$; hence, to know $\mathcal{H}_\beta$ means to have a stronger control on $\mathcal{H}_X$ than $\mathcal{H}_H$.

A simple case in which we get the equality for $\mathcal{H}_\beta$ and $\mathcal{H}_H$ occurs when the Betti sequence β is maximal with respect to its Hilbert function H according to the results of [**1, 5, 10**].

Theorem 3.1. *Let H be the Hilbert function of a* 0*-dimensional subscheme of* $\mathbf{P}^r$, *and let β be the Betti sequence maximal with respect to H. Then $\mathcal{H}_\beta = \mathcal{H}_H$.*

Proof. We need to show that every $\Delta H' \leq \Delta H$ is in $\mathcal{H}_\beta$. To do this it is enough to repeat the argument used in the proof of Theorem 2.3, just recalling that the maximal decomposition of any Hilbert function provides the maximal Betti sequence. □

In this setting, the codimension c plays a crucial role; indeed, many things that can be said in codimension 2 are no longer true when the codimension is bigger than 2. For instance, if β is the graded Betti sequence of a 0-dimensional complete intersection, with Hilbert function H, it is easy to see that if $\varphi \in \mathcal{H}_\beta$ then the sequence $(\Delta H-\varphi)^*$ defined by $(\Delta H - \varphi)^*(i) = \Delta H(\sigma_H - i) - \varphi(\sigma_H - i)$, must be an O-sequence, because of a liaison result. Now, this simple condition is not sufficient for φ to stay in $\mathcal{H}_\beta$ when the codimension of H is ≥ 3. In fact, for instance, if we take as β the Betti sequence of a complete intersection of codimension 3 given by $((2,2,2);(4,4,4);(6))$ we have ΔH : 1 3 3 1, so for $\Delta H'$: 1 1 1 we see that $(\Delta H - \Delta H')^*$ is an O-sequence, but no complete intersection $(2,2,2)$ can have 3 collinear points. Now, in codimension 2 we have that such a condition is indeed sufficient. This result seems to be known but we include here an easy proof.

Proposition 3.2. *Let $\beta = ((a,b);(a+b))$ be a Betti sequence of a complete intersection (a,b), $a \leq b$ and H its Hilbert function. Then $\Delta H' \in \mathcal{H}_\beta$ if and only if $(\Delta H - \Delta H')^*$ is an O-sequence.*

Proof. Of course, we have only to prove the sufficient condition. Since $(\Delta H - \Delta H')^*$ is an O-sequence, this will imply that $\Delta H'(n)$ must be decreasing for $n \geq b$. Now, if we define inductively

$$\Delta H'_1 := \Delta H';$$

and for $2 \leq i \leq a'$, where $a' = \min\{j \mid \Delta H'(j) < j+1\}$,

$$\Delta H'_i(n) = \begin{cases} \Delta H'_{i-1}(n+1) - 1 & 0 \leq n \leq \min\{\sigma_{H'_{i-1}}, b-1\} - 1 \\ \Delta H'_{i-1}(n+1) & \text{otherwise.} \end{cases}$$

By the above condition we see that each $\Delta H'_i$ is an O-sequence for all i. Therefore, if we set $b_i = \min\{\sigma_{H'_i}, b-1\} + 1$, the left segment $\mathcal{A}'$ generated by the elements $\{(i, b_i)\}_i$ for $i = 1, \dots, a'$ is a subset of $\mathcal{A} = \langle (a,b) \rangle$ and has Hilbert function $\Delta H'$, since $\mathcal{A}'$ realizes the above decomposition of $\Delta H'$. Hence we can construct partial intersections $X' \subseteq X$ with supports, respectively, $\mathcal{A}'$ and $\mathcal{A}$ and this says that $\Delta H' \in \mathcal{H}_\beta$. □

So now we restrict our attention to the codimension 2 case. In this situation, according to Remark 1.8, we will prove our results for Artinian monomial ideals $I_\mathcal{A}$ in $k[x, y]$ corresponding to 2-left segments $\mathcal{A}$; such results will hold for all partial intersections with support on $\mathcal{A}$. Now we can simplify our terminology. Let $I \subset k[x, y]$ be an Artinian ideal. The graded minimal free resolution of I will be denoted by

$$0 \longrightarrow \bigoplus_{i=1}^{n} R(-b_i) \longrightarrow \bigoplus_{i=1}^{n+1} R(-a_i) \longrightarrow I \longrightarrow 0.$$

We will denote by $\beta = (\{a_i\}, \{b_i\})$ the *Betti sequence* B_I of I. The integers a_i and b_j satisfy the following Gaeta conditions

$$\begin{array}{ccccccc} a_1 & \geq & \cdots & \geq & a_n & \geq & a_{n+1} \\ \wedge & & & & \wedge & & \\ b_1 & \geq & \cdots & \geq & b_n & & \end{array}$$

and

$$\sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n} b_i.$$

Now let us consider a couple of permutations $\sigma : \{1, \dots, n+1\} \to \{1, \dots, n+1\}$ and $\tau : \{1, \dots, n\} \to \{1, \dots, n\}$ such that $b_{\tau(i)} > a_{\sigma(i)}$ for $1 \leq i \leq n$ and $b_{\tau(i)} > a_{\sigma(i+1)}$ for $1 \leq i \leq n$; moreover, we say that a couple of such permutations (σ, τ) is equivalent to (σ', τ') if and only if $a_{\sigma(i)} = a_{\sigma'(i)}$ and $b_{\tau(i)} = b_{\tau'(i)}$ for all i. We will denote by Π_β the quotient set of such couples of permutations associated to the Betti sequence β.

Proposition 3.3. *There is a bijective correspondence between the set of left segments having Betti sequence β and Π_β.*

Proof. Let $\mathcal{A}$ be a left segment, whose Betti sequence is β, minimally generated by $(p_1, q_1), \dots, (p_n, q_n)$, with $p_i < p_{i+1}$ and $q_i > q_{i+1}$ for

$1 \le i \le n-1$. We also set $p_0 := 0, q_{n+1} := 0$. By Corollary 1.6

$$\begin{aligned}\{a_1, \dots, a_{n+1}\} &= \{v(H) - 2 \mid H \in F(\mathcal{A})\} \\ &= \{v(p_i + 1, q_{i+1} + 1) - 2 \mid 0 \le i \le n\} \\ &= \{p_0 + q_1, \dots, p_n + q_{n+1}\}\end{aligned}$$

and by Theorem 1.7, $\{b_1, \dots, b_n\} = \{p_1 + q_1, \dots, p_n + q_n\}$. So if we let $a_{\sigma(i)} = p_{i-1} + q_i$, $1 \le i \le n+1$ and $b_{\tau(i)} = p_i + q_i$, $1 \le i \le n$, the permutations σ and τ arise. It is now easy to show that $a_{\sigma(i)} < b_{\tau(i)}$ and $a_{\sigma(i+1)} < b_{\tau(i)}$, so that $(\sigma, \tau) \in \Pi_\beta$.

If $(\sigma, \tau) \in \Pi_\beta$ we set $k_i := \sum_{h=1}^{i}(b_{\tau(h)} - a_{\sigma(h)})$. Since $\sum_{i=1}^{n+1} a_{\sigma(i)} = \sum_{i=1}^{n} b_{\tau(i)}$ and $b_{\tau(i)} > a_{\sigma(i+1)}$ then $\sum_{h=1}^{i+1} a_{\sigma(h)} > \sum_{h=1}^{i} b_{\tau(h)}$ for $1 \le i \le n-1$; therefore, we have $a_{\sigma(i+1)} > k_i$ for $1 \le i \le n-1$. Then we can define the left segment $\mathcal{A} = \langle (k_1, a_{\sigma(1)}), (k_2, a_{\sigma(2)} - k_1), \dots, (k_n, a_{\sigma(n)} - k_{n-1}) \rangle$ whose graded Betti sequence is β. Note that this map is well defined and clearly it is the inverse of the previous one. □

Definition 3.4. We will call the left segment associated to the couple of fundamental permutations (i.e., the identity permutations) in Π_β the *β-maximal left segment* and we will denote it by $\mathcal{A}_\beta$. We will call also *β-maximal* the Artinian monomial ideal I_β corresponding to $\mathcal{A}_\beta$.

In the sequel of this section with β we will denote an admissible Betti sequence for an Artinian ideal of $R = k[x, y]$.

Now we define

$$\mathcal{P}_\beta = \{H \mid \exists \text{ Art. mon. id. } I, J \subset R, I \subseteq J, H_{R/J} = H, B_I = \beta\}.$$

Moreover, for every Artinian monomial ideal I in R, we define

$$\mathcal{P}_I = \{H \mid \exists \text{ Art. mon. id. } I \subseteq J, H_{R/J} = H\}.$$

The goal of this section is to prove that $\mathcal{P}_\beta = \mathcal{P}_{I_\beta}$. Since $\mathcal{P}_{I_\beta}$ is determined by $\mathcal{A}_\beta$ we will obtain in this way a complete description of $\mathcal{P}_\beta$.

If $\mathcal{A}$ is a 2-left segment and $K \in \mathcal{A}$ we will define

$$\delta(K) = \left|\left\{H \in \mathcal{A} \mid v(H) = v(K) \text{ and } \pi_1(H) \le \pi_1(K)\right\}\right|.$$

If i is a positive integer we will write briefly $[i]$ to denote the set $\{1, 2, \dots, i\}$. Let $\mathcal{B} \subseteq \mathcal{A}$ be two 2-left segments. Let φ and ψ be their Hilbert functions respectively. Of course $\varphi(t) \le \psi(t)$ for all t. There exists a standard way to decompose $\psi(t)$. In fact using the minimal free resolution of a graded Artinian algebra associated to $\mathcal{A}$ we obtain

$$\psi(t) = t + 1 - \sum_{i=1}^{n+1} \left(t + 1 - a_{\sigma(i)}\right)_+ + \sum_{i=1}^{n} \left(t + 1 - b_{\tau(i)}\right)_+.$$

Now we set

$$N := \{i \in [n] \mid t + 1 \ge b_{\tau(i)}\}, \qquad \widehat{M} := \{i \in [n] \mid t \ge a_{\sigma(i)}\};$$

obviously $N \subseteq \widehat{M}$ so we set $M := \widehat{M} \setminus N$ and $P^* := [n] \setminus \widehat{M}$. Then we can write

$$\begin{aligned}
\psi(t) &= t + 1 - \left(t + 1 - a_{\sigma(n+1)}\right)_+ - \sum_{i \in \widehat{M}} \left(t + 1 - a_{\sigma(i)}\right) \\
&\quad + \sum_{i \in N} \left(t + 1 - b_{\tau(i)}\right) \\
&= t + 1 - a_{\sigma(n+1)} - \left(t + 1 - a_{\sigma(n+1)}\right)_+ \\
&\quad + a_{\sigma(n+1)} - \sum_{i \in \widehat{M}} \left(b_{\tau(i)} - a_{\sigma(i)}\right) \\
&\quad + \sum_{i \in M} \left(b_{\tau(i)} - (t + 1)\right) \\
&= \min\left\{0, t + 1 - a_{\sigma(n+1)}\right\} \\
&\quad + \sum_{i=1}^{n} \left(b_{\tau(i)} - a_{\sigma(i)}\right) - \sum_{i \in \widehat{M}} \left(b_{\tau(i)} - a_{\sigma(i)}\right) \\
&\quad + \sum_{i \in M} \left(b_{\tau(i)} - (t + 1)\right) \\
&= \sum_{i \in P^*} \left(b_{\tau(i)} - a_{\sigma(i)}\right) + \sum_{i \in M} \left(b_{\tau(i)} - (t + 1)\right) \\
&\quad - \max\left\{0, a_{\sigma(n+1)} - (t + 1)\right\}.
\end{aligned}$$

Now we define $k_i := \sum_{j=1}^{i} \big(b_{\tau(j)} - a_{\sigma(j)}\big)$ and

$$m := \begin{cases} n & \text{if } t+1 > k_n = a_{\sigma(n+1)} \\ \min\{i \in \mathbf{N} \mid k_i \geq t+1\} & \text{if } t+1 \leq k_n = a_{\sigma(n+1)} \end{cases};$$

moreover, we set $P := [m] \cap P^*$. Note that if $t+1 > k_n = a_{\sigma(n+1)}$ then $P = P^*$ and, since $m = n$, $M \subseteq [m]$; if $t+1 \leq k_n = a_{\sigma(n+1)}$, then

$$i > m \Longrightarrow t+1 \leq k_{i-1} < a_{\sigma(i)} \Longrightarrow i \in P^* \Longrightarrow \big\{m+1, \dots, n\big\} \subseteq P^* \Longrightarrow M \subseteq [m].$$

Hence in any case $M \subseteq [m]$.

Therefore, if $t+1 > a_{\sigma(n+1)}$ then

$$\begin{aligned} \psi(t) &= \sum_{i \in P^*} \big(b_{\tau(i)} - a_{\sigma(i)}\big) + \sum_{i \in M} \big(b_{\tau(i)} - (t+1)\big) \\ &= \sum_{i \in P} \big(b_{\tau(i)} - a_{\sigma(i)}\big) + \sum_{i \in M} \big(b_{\tau(i)} - (t+1)\big). \end{aligned}$$

If $t+1 \leq a_{\sigma(n+1)}$, then $t+1 \leq k_m < b_{\tau(m)}$ so $m \notin N$. Consequently $m \in [n] \setminus (N \cup \{m+1, \dots, n\}) = M \cup P$.

If we set $P' := P \setminus \{m\}$ and $M' := M \setminus \{m\}$ we obtain

$$\begin{aligned} \max\big\{0, a_{\sigma(n+1)} - (t+1)\big\} &= a_{\sigma(n+1)} - (t+1) \\ &= k_n - k_m + k_m - (t+1) \\ &= \sum_{i=m+1}^{n} \big(b_{\tau(i)} - a_{\sigma(i)}\big) + k_m - (t+1). \end{aligned}$$

Therefore if $m \in P$, i.e., $t < a_{\sigma(m)}$ we have

$$\begin{aligned} \psi(t) &= \sum_{i \in P'} \big(b_{\tau(i)} - a_{\sigma(i)}\big) + \big(b_{\tau(m)} - a_{\sigma(m)}\big) \\ &\quad + \sum_{i=m+1}^{n} \big(b_{\tau(i)} - a_{\sigma(i)}\big) + \sum_{i \in M} \big(b_{\tau(i)} - (t+1)\big) \\ &\quad - \sum_{i=m+1}^{n} \big(b_{\tau(i)} - a_{\sigma(i)}\big) - k_m + (t+1) \\ &= \sum_{i \in P'} \big(b_{\tau(i)} - a_{\sigma(i)}\big) + \sum_{i \in M} \big(b_{\tau(i)} - (t+1)\big) \\ &\quad + \big(t+1 - k_{m-1}\big); \end{aligned}$$

if $m \in M$, i.e., $t \geq a_{\sigma(m)}$ (note that $P' = P$) we have

$$\begin{aligned}\psi(t) = &\sum_{i \in P} \big(b_{\tau(i)} - a_{\sigma(i)}\big) + \sum_{i=m+1}^{n} \big(b_{\tau(i)} - a_{\sigma(i)}\big) \\ &+ \sum_{i \in M'} \big(b_{\tau(i)} - (t+1)\big) + \big(b_{\tau(m)} - (t+1)\big) \\ &- \sum_{i=m+1}^{n} \big(b_{\tau(i)} - a_{\sigma(i)}\big) - k_m + (t+1) \\ = &\sum_{i \in P} \big(b_{\tau(i)} - a_{\sigma(i)}\big) + \sum_{i \in M'} \big(b_{\tau(i)} - (t+1)\big) + \big(a_{\sigma(m)} - k_{m-1}\big).\end{aligned}$$

Now we set for $i \in P$ and $i \neq m$

$$C_i := b_{\tau(i)} - a_{\sigma(i)};$$

similarly for $i \in M$ and $i \neq m$ we set

$$B_i := b_{\tau(i)} - (t+1).$$

If $m \in P$ we define

$$C_m := \begin{cases} b_{\tau(m)} - a_{\sigma(m)} & \text{if } t+1 > a_{\sigma(n+1)} \\ t+1-k_{m-1} & \text{otherwise (i.e. } t < a_{\sigma(m)} \text{ and } t < a_{\sigma(n+1)}); \end{cases}$$

if $m \in M$ we define

$$B_m := \begin{cases} b_{\tau(m)} - (t+1) & \text{if } t+1 > a_{\sigma(n+1)} \\ a_{\sigma(m)} - k_{m-1} & \text{otherwise (i.e. } a_{\sigma(m)} \leq t < a_{\sigma(n+1)}). \end{cases}$$

So in any case we can say that

$$\psi(t) = \sum_{i \in P} C_i + \sum_{i \in M} B_i.$$

Now if we set $B_0 := \sum_{i \in P} C_i$, we can write

$$\psi(t) = B_0 + \sum_{i \in M} B_i. \tag{1}$$

Remark 3.5. Let $R_i := \{H \in \mathbf{N}^2 \mid v(H) = i\}$. Recall that by Theorem 1.4 $\psi(t) = |R_{t+2} \cap \mathcal{A}|$. Note that in the previous decomposition of $\psi(t)$ (Equation (1))

$$B_i = \left|\{H \in \mathcal{A} \mid k_{i-1} + 1 \leq \pi_1(H) \leq k_i\} \cap R_{t+2}\right|$$

for $i \in M$; indeed, $i \in M$ implies that $a_{\sigma(i)} + 2 \leq t + 2 \leq b_{\tau(i)}$, and by the proof of Proposition 3.3, $\mathcal{A} = \langle (k_j, a_{\sigma(j)} - k_{j-1}) \mid 1 \leq j \leq n \rangle$, $k_0 := 0$; therefore, if $i \neq m$,

$$\begin{aligned}
&\{H \in \mathcal{A} \mid k_{i-1} + 1 \leq \pi_1(H) \leq k_i\} \cap R_{t+2} \\
&\qquad = \langle (k_i, a_{\sigma(i)} - k_{i-1}) \rangle \cap R_{t+2} \\
&\qquad = \{(t + 1 - a_{\sigma(i)} + k_{i-1} + j, a_{\sigma(i)} - k_{i-1} + 1 - j) \\
&\qquad\qquad\qquad\qquad \mid 1 \leq j \leq b_{\tau(i)} - (t + 1)\};
\end{aligned}$$

if $i = m$, then $k_m \geq t + 1$ and we have

$$\begin{aligned}
&\{H \in \mathcal{A} \mid k_{m-1} + 1 \leq \pi_1(H) \leq k_m\} \cap R_{t+2} \\
&\qquad = \langle (k_m, a_{\sigma(m)} - k_{m-1}) \rangle \cap R_{t+2} \\
&\qquad = \{(t + 1 - a_{\sigma(m)} + k_{m-1} + j, a_{\sigma(m)} - k_{m-1} + 1 - j) \\
&\qquad\qquad\qquad\qquad \mid 1 \leq j \leq a_{\sigma(m)} - k_{m-1}\}.
\end{aligned}$$

Analogously,

$$B_0 = \left|\{H \in \mathcal{A} \mid \exists i \in P \text{ such that } k_{i-1} + 1 \leq \pi_1(H) \leq k_i\} \cap R_{t+2}\right|.$$

Now, since $\mathcal{B} \subseteq \mathcal{A}$ we will construct a decomposition of φ (the Hilbert function of $\mathcal{B}$) with respect to $\mathcal{B}$ as a subset of $\mathcal{A}$. Let $t \geq 0$ be an integer. Then we can write

$$\varphi(t) = B'_0 + \sum_{i \in M} B'_i$$

where

$$B'_0 := \left|\{H \in \mathcal{B} \mid \exists i \in P \text{ such that } k_{i-1} + 1 \leq \pi_1(H) \leq k_i\} \cap R_{t+2}\right|$$

and, for $i \in M$,

$$B'_i := \left|\left\{H \in \mathcal{B} \mid k_{i-1}+1 \le \pi_1(H) \le k_i\right\} \cap R_{t+2}\right|.$$

Of course $B'_0 \le B_0$ and $B'_i \le B_i$ for every $i \in M$. Note that P and k_i depend on $\mathcal{A}$.

Definition 3.6. We will call this decomposition of $\varphi(t)$ depending on $\mathcal{B}$ as a subset of $\mathcal{A}$ the $(\mathcal{B}, \mathcal{A})$-*standard* decomposition of $\varphi(t)$.

Remark 3.7. Note that if $\mathcal{A} = \mathcal{A}_\beta$ is β-maximal then $a_{\sigma(n+1)} = a_{n+1}$ is the smallest among the a_i's; moreover, there exist $r, s \in [n]$ such that $P = \{1, \dots, r\}$ and $M = \{r+1, \dots, s\}$; so if $t \ge a_{n+1} - 1$ then

$$\begin{aligned}\psi(t) &= \sum_{i\in P}\big(b_i - a_i\big) + \sum_{i\in M}\big(b_i - (t+1)\big)\\ &= \sum_{i=1}^{r}\big(b_i - a_i\big) + \sum_{i=r+1}^{s}\big(b_i - (t+1)\big).\end{aligned}$$

Now we set $A_0 := \sum_{i=1}^{r}\big(b_i - a_i\big)$ and $A_j := b_{r+j} - (t+1)$ for $1 \le j \le s - r$, so we obtain the decomposition

$$\psi(t) = A_0 + A_1 + \cdots + A_{s-r}$$

and if $H \in \mathcal{A}_\beta$ then $\delta(H) \le \psi(t)$; so we can write it in the following way

$$\delta(H) = A_0 + A_1 + \cdots + A_{v-1} + A'_v$$

where $0 \le v \le s - r$ and $0 < A'_v \le A_v$.

Lemma 3.8. *Let $\mathcal{A}$ be a 2-left segment and let ψ be its Hilbert function. Let $t \ge 0$ be an integer. Let*

$$\psi(t) = B_0 + \sum_{i\in M} B_i = A_0 + \sum_{i=1}^{s-r} A_i$$

be the two standard decompositions of $\psi(t)$ as Hilbert function of $\mathcal{A}$ and as Hilbert function of $\mathcal{A}_\beta$. We order the B_i's for $i \neq 0$ in a non *increasing way*: $B_{i_1} \geq \cdots \geq B_{i_p}$. *Then* $\sum_{h=0}^{j} A_h \geq \sum_{h=0}^{j} B_{i_h}$ *for* $0 \leq j \leq \min\{s-r, p\}$ $(B_{i_0} := B_0)$.

Proof. The sets N, $\widehat{M}$, M and P previously defined depend on the left segment. So we denote by N_β, $\widehat{M}_\beta$, M_β and P_β those depending on $\mathcal{A}_\beta$. With the notation of Remark 3.7, we have

$$P_\beta = \{1, \dots, r\} \; M_\beta = \{r+1, \dots, s\}, \text{ and } N_\beta = \{s+1, \dots, n\}.$$

Of course,

$$N = \{\tau^{-1}(s+1), \dots, \tau^{-1}(n)\}$$

and

$$\widehat{M} \subseteq \{\sigma^{-1}(r+1), \dots, \sigma^{-1}(n+1)\}.$$

So if $1 \leq \sigma(n+1) \leq r$, then

$$\widehat{M} = \{\sigma^{-1}(r+1), \dots, \sigma^{-1}(n+1)\}$$

and

$$P = \{\sigma^{-1}(1), \dots, \sigma^{-1}(r)\} \setminus \{n+1\};$$

if $r+1 \leq \sigma(n+1) \leq n+1$, then

$$\widehat{M} = \{\sigma^{-1}(r+1), \dots, \sigma^{-1}(n+1)\} \setminus \{n+1\}$$

and

$$P = \{\sigma^{-1}(1), \dots, \sigma^{-1}(r)\}.$$

We distinguish three cases (we work in the first case since the other cases can be proved in a similar way):

1) $1 \leq \sigma(n+1) \leq r$ and $m \in P$, i.e., $1 \leq \sigma(m) \leq r$.

$$\begin{aligned} B_0 &= \sum_{\substack{i \in P \\ i \neq m}} \left(b_{\tau(i)} - a_{\sigma(i)}\right) + t + 1 - k_{m-1} \\ &\leq \sum_{i \in P} \left(b_{\tau(i)} - a_{\sigma(i)}\right) \leq \sum_{i=1}^{r} \left(b_i - a_i\right) = A_0, \end{aligned}$$

since the b_i's for $1 \le i \le r$ are the biggest ones and every $a_{\sigma(i)}$ is equal to a_j for some j, $1 \le j \le r$. Moreover $\sum_{h=0}^{j} A_h \ge \sum_{h=0}^{j} B_{i_h}$ for $1 \le j \le \min\{s-r, p\}$ again because of the A_i's take the b_i's in a non increasing order.

2) $1 \le \sigma(n+1) \le r$ and $m \in M$. Similar proof.

3) $r+1 \le \sigma(n+1) \le n+1$. Similar proof. □

Lemma 3.9. *Let $\mathcal{A}_\beta$ be a 2-left segment β-maximal, minimally generated by $(p_1, q_1), \dots, (p_n, q_n)$, with $p_1 < p_2 < \cdots < p_n$ and $q_1 > q_2 > \cdots > q_n$. Let $(x, y) \in \mathcal{A}_\beta$ such that $x + y > p_i + q_i$. Then $x < p_i$.*

Proof. Since $\mathcal{A}_\beta$ is β-maximal we have that $p_1 + q_1 > p_2 + q_2 > \cdots > p_n + q_n$, as they are the degrees of the syzygies. Since $(x, y) \in \mathcal{A}_\beta$, $(x, y) \le (p_j, q_j)$ for some j. Then $p_i + q_i < x + y \le p_j + q_j$ implies that $j < i$, i.e., $p_j < p_i$; hence, we obtain $x \le p_j < p_i$. □

Lemma 3.10. *Let $\mathcal{A}_\beta$ be a 2-left segment β-maximal. Let $H := (x, y) \in \mathcal{A}_\beta$ and $K := (x, y-1)$. If $\delta(H) = A_0 + A_1 + \cdots + A_{v-1} + A'_v$ is the standard decomposition, then $\delta(K) = \delta(H) + v$.*

Proof. We set $g := v(H) = x + y$, $t := g - 2$ and $k_i := \sum_{j=1}^{i} (b_j - a_j)$. We set also $k_0 = 0$. In correspondence of t, as we know, the following sets arise:

$$P_\beta = \{1, \dots, r\} \; M_\beta = \{r+1, \dots, s\}, \; N_\beta = \{s+1, \dots, n\}.$$

Furthermore by Remark 3.5, $A_i = |S_i|$ for $0 \le i \le v$, where

$$S_0 = \bigcup_{i=1}^{r} \{L \in \mathcal{A}_\beta \mid k_{i-1} + 1 \le \pi_1(L) \le k_i \text{ and } v(L) = g\}$$

and, for $1 \le i \le v$,

$$S_i = \{L \in \mathcal{A}_\beta \mid k_{r+i-1} + 1 \le \pi_1(L) \le k_{r+i} \text{ and } v(L) = g\};$$

moreover,

$$A'_v = \left|\{L \in S_v \mid \pi_1(L) \le x\}\right|.$$

We can write the elements of each S_i in the following way

$$S_i = \{(x_i+1, g-x_i-1), \dots, (x_i+A_i, g-x_i-A_i)\}$$

for $0 \le i \le v$, with $x_{i-1} + A_{i-1} < x_i + 1$.

Since $\mathcal{A}_\beta$ is a left segment if $(z,w) \in \mathcal{A}_\beta$, $w > 1$, then $(z, w-1) \in \mathcal{A}_\beta$; we can apply this observation to each element of S_i for $i = 1, \dots, v$. On the other hand, if $(z,w) \in \mathcal{A}_\beta$, $z > 1$, then $(z-1, w) \in \mathcal{A}_\beta$; hence, since $(x_i+1, g-1-x_i) \in \mathcal{A}_\beta$ then $(x_i, g-1-x_i) \in \mathcal{A}_\beta$, for $1 \le i \le v$. Therefore $\delta(K) \ge \delta(H) + v$.

By contradiction suppose that $\delta(K) > \delta(H) + v$; this implies that there is $Z := (z, g-1-z) \in \mathcal{A}_\beta$ different from the couples coming from some S_i. So $x_{i-1} + A_{i-1} < z < x_i$; consequently, $(z+1, g-1-z)$ and $(z, g-z)$ do not belong to $\mathcal{A}_\beta$, i.e., Z should be a maximal element in $\mathcal{A}_\beta$. But $Q = (x_i+1, g-x_i-1) \in \mathcal{A}_\beta$, $v(Q) = g$ and $x_i > z$, a contradiction by Lemma 3.9. □

Lemma 3.11. *Let $\mathcal{A}$ be a 2-left segment with Hilbert function ψ, and let $\psi(t) = B_0 + \sum_{i \in M} B_i$ be the standard decomposition of $\psi(t)$ where $t \ge 0$ is an integer. Then $\psi(t-1) \ge \psi(t) + |M|$.*

Proof. Using Remark 3.5 if $i \in M$, $i \ne m$, the set

$$\begin{aligned}&\{H \in \mathcal{A} \mid k_{i-1}+1 \le \pi_1(H) \le k_i\} \cap R_{t+2}\\ &= \{(t+1-a_{\sigma(i)}+k_{i-1}+j, a_{\sigma(i)}-k_{i-1}+1-j) \mid 1 \le j \le b_{\tau(i)}-(t+1)\}\end{aligned}$$

has B_i elements. Since $\mathcal{A}$ is a left segment, the set

$$X_i = \{(t-a_{\sigma(i)}+k_{i-1}+j, a_{\sigma(i)}-k_{i-1}+1-j) \mid 1 \le j \le b_{\tau(i)}-t\}$$

is contained in $\mathcal{A}$ and it has $B_i + 1$ elements (analogously one can construct X_m when $m \in M$). Note that $X_i \cap X_j = \varnothing$ for all $i, j \in M$, $i \ne j$; thus, the assertion is proved. □

Remark 3.12. When $\mathcal{B} \subseteq \mathcal{A}$ are 2-left segments with Hilbert functions φ and ψ, respectively, and $\varphi(t) = B'_0 + \sum_{i \in M} B'_i = B'_0 + \sum_{i \in U} B'_i$ is the $(\mathcal{B}, \mathcal{A})$-standard decomposition of $\varphi(t)$, where $U = \{i \in M \mid B'_i \ne 0\}$, then $\varphi(t-1) \ge \varphi(t) + |U|$.

Theorem 3.13. *Let β be an admissible Betti sequence for an Artinian ideal in $R = k[x, y]$. Then the β-maximal ideal I_β has the following property: for all $J \supseteq I$ Artinian monomial ideals in R with $B_I = \beta$ and $H_J = \varphi$ there is a $J' \supseteq I_\beta$ such that $H_{R/J'} = \varphi$.*

Proof. Let $\mathcal{A} = \mathcal{A}_I$ and $\mathcal{B} = \mathcal{A}_J$ be the corresponding left segments, and let $\mathcal{A}_\beta$ be the left segment corresponding to I_β. We set $\psi := H_{R/I}$ and

$$\mathcal{B}' = \{K \in \mathcal{A}_\beta \mid \delta(K) \leq \varphi(v(K) - 2)\}.$$

By definition of $\mathcal{B}'$ it is enough to prove that $\mathcal{B}'$ is a left segment since in this case the Hilbert function $H_{R/I_{\mathcal{B}'}} = \varphi$. So, we have to prove that if $K \in \mathcal{B}'$ and $L \leq K$, then $L \in \mathcal{B}'$. Let us fix a $t \geq 0$; again, by definition of $\mathcal{B}'$, it is enough to prove that if $K := (x, t + 2 - x) \in \mathcal{B}'$, $\delta(K) = \varphi(t)$ then $L := (x, t+1-x) \in \mathcal{B}'$, i.e., $\delta(L) \leq \varphi(t-1)$. Indeed, once this is proved, we get that even each $L' = (y, t+1-y) \in \mathcal{A}_\beta$, with $\delta(L') < \varphi(t - 1)$ belongs to $\mathcal{B}'$ by definition.

Since $\varphi = \Delta H$, $\varphi(t) \leq \psi(t) = B_0 + \sum_{i \in M} B_i$, by formula (1); now we use the $(\mathcal{B}, \mathcal{A})$-standard decomposition of $\varphi(t)$ (see Definition 3.6): $\varphi(t) = B'_0 + \sum_{i \in U} B'_i$, where $0 \leq B'_0 \leq B_0$, $U = \{i \in M \mid B'_i \neq 0\}$ and $B'_i \leq B_i$ for every $i \in U$. We set $u := |U|$. By Remark 3.12 $\varphi(t - 1) \geq \varphi(t) + u$, so it is enough to show that $\delta(L) \leq \varphi(t) + u$.

We can write $\delta(K) = \sum_{i=0}^{v-1} A_i + A'_v$. If $v > u$, then by Lemma 3.8 $\varphi(t) = B'_0 + \sum_{i \in U} B'_i \leq \sum_{i=0}^{u} A_i < \varphi(t)$, a contradiction. Therefore $v \leq u$, so, by Lemma 3.10, $\delta(L) = \varphi(t) + v \leq \varphi(t) + u$, and we are done. □

This result permits a combinatorial characterization of $\mathcal{P}_\beta$. To do this, let us recall our setting. Let β be an admissible Betti sequence for an Artinian ideal in $k[x, y]$. Let ψ be the Hilbert function of $\mathcal{A}_\beta$. Let $t \geq 0$ be an integer. Using the same notation on the graded Betti numbers of the previous section we have

$$a_1 \geq \cdots \geq a_r > t \geq a_{r+1} \geq \cdots a_n \geq a_{n+1}$$

and

$$b_1 \geq \cdots \geq b_s > t + 1 \geq b_{s+1} \geq \cdots \geq b_n$$

for suitable r and s (obviously $r \leq s$); as we saw in Remark 3.7

$$\psi(t) = A_0 + A_1 + \cdots + A_{s-r}.$$

If $\varphi(t) \leq \psi(t)$ we define $v = \max\{j \mid \sum_{h=0}^{j} A_h \leq \varphi(t)\}$. Note that v depends on t and on $\varphi(t)$ and that $0 \leq v \leq s - r$. Finally, define $\varphi(t)^{(t)} := \varphi(t) - v$. The next proposition gives an explicit description of the set $\mathcal{P}_\beta$.

Proposition 3.14. *Let φ be an O-sequence. $\varphi \in \mathcal{P}_\beta$ if and only if $\varphi(t+1) \leq \varphi(t)^{(t)}$, for all $t \geq a_{n+1}$. Therefore, if $\varphi = H_{R/J}$ where J is an Artinian monomial ideal containing a monomial Artinian ideal I with $B_I = \beta$ then $\varphi(t+1) \leq \varphi(t)^{(t)}$, for all t greater than or equal to the smallest degree of a minimal generator of I.*

Proof. It is a trivial consequence of Theorem 3.13. □

Remark 3.15. Note that there are no restrictions on φ when $0 \leq t \leq a_{n+1} - 1$.

In the next remark we illustrate how to build the β-maximal ideal I_β once one knows the Betti sequence β.

Remark 3.16. Take the admissible Betti sequence

$$\beta = \big((a_1, \dots, a_{n+1}); (b_1, \dots, b_n)\big),$$

with $a_1 \geq \cdots \geq a_{n+1}$ and $b_1 \geq \cdots \geq b_n$. Define $k_0 := 0$ and $k_i := \sum_{h=1}^{i}(b_h - a_h)$. Then the β-maximal ideal I_β in $k[x,y]$ is generated by the monomials $x^{k_i} y^{a_{i+1}-k_i}$ for $0 \leq i \leq n$. Note that for $i = 0$ we get that $y^{a_1} \in I_\beta$; on the other hand, since $k_n = a_{n+1}$ we get that $x^{k_n} \in I_\beta$. Hence, I_β is Artinian.

The previous results apply to partial intersection schemes.

Corollary 3.17. *Let $Y \subseteq X \subset \mathbf{P}^2$ be 2-codimensional partial intersections. Let $H := H_Y$ and $\beta := B_X$. Let X_β be a β-maximal partial intersection. Then there exists a partial intersection $Y' \subset X_\beta$ such that $H_{Y'} = H$.*

Proof. It follows immediately by Remark 1.8, Theorem 3.13 and Proposition 3.14. □

4. Special Betti sequences. To describe $\mathcal{H}_\beta$ is in general much more complicated than $\mathcal{P}_\beta$. Indeed, we have seen that it is possible to give a complete description of $\mathcal{H}_\beta$ when β is the Betti sequence of a 0-dimensional complete intersection in $\mathbf{P}^2$. This section is devoted to describing $\mathcal{H}_\beta$ when $\beta = \big((a_1, a_2, a_3); (b_1, b_2)\big)$, $a_1 \geq a_2 \geq a_3$, $b_1 \geq b_2$, is the Betti sequence of a 0-dimensional subscheme of $\mathbf{P}^2$ whose syzygy module has only two minimal generators (an almost complete intersection).

Here we will use the same terminology as at the end of the last section.

Lemma 4.1. *Let β be a Betti sequence admissible for a 0-dimensional subscheme X of $\mathbf{P}^r$, and let $\psi := \Delta H_X$. Let $\varphi \in \mathcal{H}_\beta$ and $t \geq 0$ an integer. Now we define the sequence*

$$\chi(i) = \begin{cases} \psi(i) & \text{if } i \leq t-1 \\ \varphi(i) & \text{if } i \geq t. \end{cases}$$

Then $\chi \in \mathcal{H}_\beta$.

Proof. Let R be a polynomial ring with r indeterminates. The hypotheses tell us that there exist two homogeneous ideals $I \subseteq J \subset R$, defining two Artinian algebras R/I and R/J, such that the Hilbert functions of R/I and R/J are respectively ψ and ϕ and the Betti sequence of R/I is β. Now we define the following ideal of R

$$J' := \bigoplus_{s=0}^{t-1} I_s \oplus \bigoplus_{s \geq t} J_s$$

where by I_s and J_s we denote the homogeneous pieces of I and J. The algebra R/J' is trivially Artinian, its Hilbert function is χ and $I \subseteq J' \subseteq J$, so it is enough to lift up J' to see that $\chi \in \mathcal{H}_\beta$. □

The following lemma gives us our first information about $\mathcal{H}_\beta$ for any admissible Betti sequence β of a 0-dimensional subscheme of $\mathbf{P}^2$ with first difference Hilbert function ψ.

Lemma 4.2. *With the above notation let $\varphi \in \mathcal{H}_\beta$, and let $t \geq a_{n+1}$ be an integer. Let $\varphi(t+1) = \varphi(t) - z$. If $v = s - r$, then $z \geq v$.*

Proof. Since $\varphi \in \mathcal{H}_\beta$, there exist $Y \subseteq X \subset \mathbf{P}^2$ 0-dimensional schemes such that the Betti sequence of X is β and the first difference of the Hilbert function of Y is φ; consequently, by Lemma 4.1, there exists a $Y' \subseteq X$ such that

$$\Delta H_{Y'}(i) = \begin{cases} \psi(i) & \text{if } i \leq t-1 \\ \varphi(i) & \text{if } i \geq t. \end{cases}$$

We set $\chi := \Delta H'_Y$. Denote by G the number of minimal generators of $I_{Y'}$ of degree less than or equal to $t+1$ and by S the number of minimal syzygies of $I_{Y'}$ of degree less than or equal to $t+1$. Since $\Delta\chi(t+1) = -z$ we get that $1 - G + S = -z$, i.e., $S = G - z - 1$.

By contradiction let us suppose

$$\begin{aligned} z < v &\iff z \leq s - r - 1 \iff G - z - 1 \geq -s + r + G \\ &\iff S > G - (s + 1 - r). \end{aligned}$$

Let f_i be the minimal generator of I_X of degree a_i; since $I_{X,i} = I_{Y',i}$ for $i \leq t-1$, f_i is a minimal generator also for $I_{Y'}$ for $r+1 \leq i \leq n+1$. $I_{Y'}$ has further $m := \psi(t) - \varphi(t)$ minimal generators in degree t, and we denote them by $g_1, \dots, g_m$. Of course, the syzygies of degrees $b_{s+1}, \dots, b_n$ of I_X are also minimal syzygies for $I_{Y'}$, so $I_{Y'}$ has further $S - (n-s)$ syzygies all in degree $t+1$.

A syzygy of degree $t+1$ has the form

$$l_1 g_1 + \cdots + l_m g_m + h_{r+1} f_{r+1} + \cdots + h_{n+1} f_{n+1} = 0,$$

where the l_i's are linear forms. Moreover, $S > G - s - 1 + r$ implies that $S - (n-s) > G - s - 1 + r - n + s = G - (n - r + 1)$, i.e., the number of "new" syzygies of degree $t+1$ is greater than the number of "new" generators of degree t; so we can eliminate using these relations the generators $g_1 \dots, g_m$ and we obtain a new syzygy of degree $d \leq t+1+m$ involving only the f_i's. But

$$m = \psi(t) - \varphi(t) = \sum_{i=0}^{s-r} A_i - \sum_{i=0}^{s-r-1} A_i - A'_{s-r} = A_{s-r} - A'_{s-r},$$

so

$$\begin{aligned} d \leq t + 1 + A_{s-r} - A'_{s-r} &= t + 1 + b_s - (t+1) - A'_{s-r} \\ &= b_s - A'_{s-r} < b_s, \end{aligned}$$

a contradiction because d is the degree of a syzygy involving only the f_i's independent of the syzygies of I_X of degrees $b_{s+1}, \dots, b_n$. □

Lemma 4.3. *Let $X \subset \mathbf{P}^2$ be a 0-dimensional scheme with defining ideal I_X, and let $f_1, f_2, \dots, f_{n+1}$ be a minimal set of generators for I_X, of degrees $a_1 \geq a_2 \geq \cdots \geq a_{n+1}$, respectively. Moreover, denote by $b_1 \geq b_2 \geq \cdots \geq b_n$ the degrees of a minimal set of generators for the first syzygies module of I_X. Let $1 \leq h \leq n$ and $g := \mathrm{GCD}\,(f_{h+1}, \dots, f_{n+1})$. Then*

$$\deg g \leq k_h := \sum_{i=1}^{h} (b_i - a_i).$$

Proof. Let us consider the first h largest syzygies

$$\sum_{j=1}^{n+1} c_{ij} f_j = 0, \quad 1 \leq i \leq h$$

and the matrix

$$M = \begin{pmatrix} c_{12} & c_{13} & \cdots & c_{1h} \\ c_{22} & c_{23} & \cdots & c_{2h} \\ \vdots & \vdots & & \vdots \\ c_{h2} & c_{h3} & \cdots & c_{hh} \end{pmatrix}.$$

Now by multiplying the above relations by $(-1)^i M_i$, where M_i is the minor obtained from M by deleting the ith row, and summing up we get a new relation of the type

$$d_1 f_1 + d_{h+1} f_{h+1} + \cdots + d_{n+1} f_{n+1} = 0;$$

note that, since the previous syzygies form a basis for the syzygies module, we can suppose that, up to change of bases,

$$d_1 = \begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1h} \\ c_{21} & c_{22} & \cdots & c_{2h} \\ \vdots & \vdots & & \vdots \\ c_{h1} & c_{h2} & \cdots & c_{hh} \end{vmatrix} \neq 0,$$

with $\deg d_1 = k_h$; therefore, g divides the product $d_1 f_1$. Since f_1 is the generator having the biggest degree, we can suppose that GCD $(g, f_1) = 1$, so g divides d_1, i.e., $\deg g \leq k_h$. □

Theorem 4.4. *Let $\beta = \big((a_1, a_2, a_3), (b_1, b_2)\big)$, and let φ be an O-sequence admissible for a 2-codimensional Artinian graded algebra. Then $\varphi \in \mathcal{H}_\beta$ if and only if $\varphi(t+1) \leq \varphi(t)^{(t)}$, for all $t \geq a_3$.*

Proof. If φ satisfies the condition $\varphi(t+1) \leq \varphi(t)^{(t)}$ for all t, then, by Proposition 3.14, $\varphi \in \mathcal{P}_\beta \subseteq \mathcal{H}_\beta$.

If $\varphi \in \mathcal{H}_\beta$, then we distinguish two cases:

1) $a_3 \leq a_2 \leq a_1 \leq b_2 \leq b_1$;

2) $a_3 \leq a_2 < b_2 \leq a_1 < b_1$.

Recall by definition that $\varphi(t)^{(t)} = \varphi(t) - v$. Now, in the first case, for $a_3 \leq t \leq a_2 - 1$, $v = 0$; for $a_2 \leq t \leq a_1 - 1$, $0 \leq v \leq 1$; for $a_1 \leq t \leq b_2 - 1$, $1 \leq v \leq 2$; for $b_2 \leq t \leq b_1 - 1$, $v = 1$; for $t \geq b_1$, $v = 0$.

When $v = 0$ then the maximal growth is allowed, so the conclusion is trivial. So we have to examine only the cases $v = 1$ and $v = 2$.

By contradiction let us suppose that $v = 1$ and $\varphi(t+1) > \varphi(t)^{(t)} = \varphi(t) - 1$, i.e., $\varphi(t) = \varphi(t+1)$; by hypotheses there exist two 0-dimensional schemes $Y \subseteq X \subset \mathbf{P}^2$ such that the Betti sequence of X is β and $\Delta H_Y = \varphi$. So the curves through Y of degree less than or equal to $t+1$ have a fixed component of degree $\varphi(t)$; consequently, the curves of degrees a_2 and a_3 through X have a fixed component of degree $\varphi(t)$, i.e., the generators of I_X of degrees a_2 and a_3 have a greatest common divisor of degree $\varphi(t)$. Now, $v = 1$ can happen if $a_2 \leq t$. If $a_2 \leq t \leq a_1 - 1$ we can use Lemma 4.3 to say that $\varphi(t) \leq b_1 - a_1$, but this should imply that $v = 0$, a contradiction. If $t \geq a_1$ the curves through X of degree less than or equal to a_1 should have a fixed component, a contradiction since X is a 0-dimensional scheme. The case $v = 2$ may happen only if $a_1 \leq t \leq b_2 - 1$, and the conclusion follows from Lemma 4.2.

In the second case v may assume only the values 0 and 1, so we get the same conclusion using analogous arguments. □

Remark 4.5. When we know the Betti sequence β of a 0-dimensional scheme X it seems much more natural to try to understand the Betti sequences of its subschemes, i.e., the set

$$\mathcal{B}_X = \{\gamma \mid \exists Y \subseteq X \text{ with } B_Y = \gamma\}.$$

So, in the same fashion we used for Hilbert functions, one can define, for every Betti sequence β

$$\mathcal{B}_\beta = \{\gamma \mid \exists Z \text{ and } \exists Y \subseteq Z \text{ with } B_Y = \gamma, B_Z = \beta\}.$$

Thus, if $B_X = \beta$, then $\mathcal{B}_X \subseteq \mathcal{B}_\beta$.

Now, despite the fact that this setting seems similar to the case of the Hilbert functions of the subschemes, in this situation, even for partial intersection schemes (or Artinian monomial ideals) the β-maximal scheme does not necessarily contain all possible Betti sequences for subschemes of any scheme whose Betti sequence is β, as we show in the following example.

Example 4.6. Take the Betti sequence $\beta = \big((3,7,8);(9,9)\big)$ (in codimension 2). The β-maximal left segment is given by $\mathcal{A} = \langle(1,8),(3,6)\rangle$. Now, any partial intersection X with support on $\mathcal{A}$ cannot contain a complete intersection $(2,7)$, i.e., the Betti sequence $\gamma = \big((2,7);(9)\big) \notin \mathcal{B}_X$. This is easy to see since any subscheme $Y \subseteq X$ consisting of 14 points should stay either on a conic split in 2 lines, but there are no 2 lines each containing 7 points of X, or on a cubic consisting of 3 lines with 2 of them with at least 3 points of Y, so no conic could contain Y.

On the other hand, take the left segment $\mathcal{B} = \langle(2,7),(3,6)\rangle$, for which the Betti sequence is again β, and one easily sees that every partial intersection with support on $\mathcal{B}$ contains a complete intersection $(2,7)$.

Of course, in any case, $\varphi = \Delta H$, where H is the Hilbert function of the complete intersection $(2,7)$ stays on $\mathcal{H}_\beta$: just take $\mathcal{A}' \subseteq \mathcal{A}$ with $\mathcal{A}' = \langle(1,8),(2,6)\rangle$.

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